Martin Schmidt
Ross Ogilvie

Introduction to Partial Differential Equations
Exercise sheet 3

14th October, 2020

6. Linear Partial Differential Equations

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ be a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogeneous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(2 point(s))

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$.
.
(4 point(s))

(c) With the help of the previous part, re-derive the solution of the inhomogeneous transport equation. *(2 point(s))*

7. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_1\partial_1 u + x_2\partial_2 u = 2u$ on the domain $x_1 \in \mathbb{R}, x_2 > 0$, with initial condition $u(x_1, 1) = g(x_1)$.
(4 point(s))

(b) $x_1\partial_2 u - x_2\partial_1 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$.
(4 point(s))

(c) $x_1\partial_1 u + 2x_2\partial_2 u + \partial_3 u = 3u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$.
(4 point(s))

(d) $u\partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2}x_1$.
(5 point(s))

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.