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Introduction to Partial Differential Equations
Exercise sheet 2

30th September, 2020

3. Royale with Cheese

Recall Burgers' equation from Example 1.6 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $u_0(x) = x$.

(a) According to Theorem 1.5, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point(s))*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 point(s))*

(c) On an (x, t)-plane, draw the characteristics and describe the behaviour of this solution. *(2 point(s))*

(d) Finally, derive the following solution to the initial value problem:

$$u(x, t) = \frac{x}{1+t}.$$

(2 point(s))

(e) Is this solution well-defined? Check by substitution that actually solves the initial value problem. *(2 point(s))*

(f) Why is the method of characteristics well-suited to solving first order PDEs that are linear in the derivatives? *(1 point(s))*

4. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.7. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x, t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x, t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x, t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x, t)-plane for this function. *(2 point(s))*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(3 point(s))*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(3 point(s))*

5. You're not in traffic, you are traffic

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) Argue that $f(u) = s_m u(1 - u/u_m)$ is a reasonable model. Hence write down a PDE to describe the traffic flow. *(2 point(s))*

(b) Consider the situation of a traffic light at $x = 0$: to the left of the traffic light, the cars are queued up at maximum density. To the right of the traffic light, the road is empty. Now, at time $t = 0$, the traffic light turns green. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(6 point(s))*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.