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Introduction to Partial Differential Equations
Exercise sheet 1

25th September, 2020

1. **Multiindices and the Generalised Leibniz rule.** In this question we introduce multiindex notation. A *multiindex* of n variables is a vector $\gamma \in \mathbb{N}_0^n$.

   (a) Let $x = (x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$. Write out the full expression for the derivative $\partial^{(0,2,1)}$. *(1 point)*

   (b) Why do we need to assume that partial derivatives commute for multiindex notation to be useful? *(1 point)*

   (c) Which multiindices satisfy $|\gamma| \leq 2$ and which satisfy $\gamma \leq (0,2,1)$? *(2 points)*

   (d) The generalised binomial coefficient for multiindices is defined to be

   $$\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\binom{\gamma_2}{\delta_2}\cdots\binom{\gamma_n}{\delta_n}.$$

   One justification for calling these binomial coefficients is the following property. Let $e_j = (0, \ldots, 1, \ldots, 0)$ be the multiindex with 1 is the j-th position and 0 in all other positions. Then for any j

   $$\binom{\gamma}{\delta} = \binom{\gamma - e_j}{\delta - e_j} + \binom{\gamma - e_j}{\delta}.$$

   Prove this property. *(2 points)*

   (e) Let $u, v : \Omega \to \mathbb{R}$ be smooth enough functions on an open subset $\Omega \subset \mathbb{R}^n$. Show for all multiindices $\gamma \in \mathbb{N}_0^n$ the following product rule:

   $$\partial^\gamma(uv) = \sum_{0 \leq \delta \leq \gamma} \binom{\gamma}{\delta} \partial^\delta u \, \partial^{\gamma-\delta} v$$

   (6 points)

2. **Inhomogeneous Transport Equation.** First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous equation

   $$\frac{du}{dt} = f(t).$$

   (a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

   (b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

   We consider now the inhomogeneous transport equation

   $$\partial_t u + b \cdot \nabla u = f$$

   with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

   $$u(x,t) = g(x - tb) + \int_0^t f(x + (s-t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique, if you haven't seen the lecture/read the script.) *(2 points)*

Solutions are due on Tuesday 12 noon, the day before the tutorial. Please email to `r.ogilvie@math.uni-mannheim.de`. One possibility is to write your solutions neatly by hand and then scan them with your phone to make a pdf. There are many apps that do this; two examples on Android are 'Tiny Scanner' and 'Simple Scanner'.