# Introduction into Partial Differential Equations

## HS 20

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# Chapter 1

# First Order PDEs

In this introductory chapter we first introduce partial differential equations and then consider first order partial differential equations. We shall see that they are simpler than higher order partial differential equations. In contrast to higher order partial differential equations these first order partial differential equations are similar to ordinary differential equations and can be solved by using the theory of ordinary differential equations. After this introductory chapter we shall focus on second order partial differential equations. Before we consider the three main examples of second order differential equations we introduce some general concepts in the next chapter. These general concepts are partially motivated by observations contained in the first chapter.

A partial differential equation is an equation on the partial derivatives of a function depending on at least two variables.

**Definition 1.1.** *A possibly vector valued equation of the following form*

$$F\left(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x\right) = 0$$

*is called partial differential equation of order $k$. Here $F$ is a given function and $u$ an unknown function. The expressions $D^k u$ denotes the vector of all partial derivatives of the function $u$ of order $k$. The function $u$ is called a solution of the differential equation, if $u$ is $k$ times differentiable and obeys the partial differential equation.*

On open subsets $\Omega \subset \mathbb{R}^n$ we denote the partial derivatives of higher order by $\partial^\gamma = \prod_i \partial_i^{\gamma_i} = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}$ with multiindices $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$. The multiindices are ordered by $\delta \leq \gamma \iff \delta_i \leq \gamma_i$ for $i = 1, \dots, n$. The partial derivative acts only on the immediately following function; they only act on a product of functions if the product is grouped together in brackets.

**Exercise 1.2.** *Show for all $\gamma \in \mathbb{N}_0^n$ the generalised Leibniz rule*

$$\partial^\gamma(uv) = \sum_{0\le\delta\le\gamma} \binom{\gamma}{\delta} \partial^\delta u \partial^{\gamma-\delta} v := \sum_{\delta_1=0}^{\gamma_1} \binom{\gamma_1}{\delta_1} \dots \sum_{\delta_n=0}^{\gamma_n} \binom{\gamma_n}{\delta_n} \partial^\delta u \partial^{\gamma-\delta} v.$$

## 1.1 Homogeneous Transport Equation

One of the simplest partial differential equations is the transport equation:

$$\dot{u} + b \cdot \nabla u = 0.$$

Here $\dot{u}$ denotes the partial derivative $\frac{\partial u}{\partial t}$ of the unknown function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $b \in \mathbb{R}^n$ is a vector, and the product $b \cdot \nabla u$ denotes the scalar product of the vector $b$ with the vector of the first partial derivatives of $u$ with respect to $x$:

$$b \cdot \nabla u(x,t) = b_1 \frac{\partial u(x,t)}{\partial x_1} + \ldots + b_n \frac{\partial u(x,t)}{\partial x_n}.$$

Let us first assume that $u(x,t)$ is a differentiable solution of the transport equation. For all fixed $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ the function

$$z(s) = u(x + s \cdot b, t + s)$$

is a differentiable function on $s \in \mathbb{R}$, whose first derivative vanishes:

$$z'(s) = b\nabla u(x + s \cdot b, t + s) + \dot{u}(x + s \cdot b, t + s) = 0.$$

Therefore $u$ is constant along all parallel straight lines in direction of $(b, 1)$. Furthermore, $u$ is completely determined by the values on all these parallel straight lines.

**Initial Value Problem 1.3.** *We are looking for a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the transport equation with given $b$, which at $t = 0$ is equal to some given function $g : \mathbb{R}^n \to \mathbb{R}$.*

All parallel straight lines in direction of $(b, 1)$ intersect $\mathbb{R}^n \times \{0\}$ exactly once:

$$(x + sb, t + s) \in \mathbb{R}^n \times \{0\} \iff s = -t$$

Hence the solution has to be equal to $u(x,t) = g(x - tb)$. If $g$ is differentiable on $\mathbb{R}^n$, then this function indeed solves the transport equation. In this case the initial value problem has a unique solution. Otherwise, if $g$ is not differentiable on $\mathbb{R}^n$, then the initial value problem does not have a solution. As we have seen above, whenever the initial value problem has a solution, then the function $u(x,t) = g(x - bt)$ is the unique solution. So it might be that this candidate is a solution in a more general sense. In fact in the next chapter we shall see in Exercise 2.10 that in case of generalised differentiable functions $g$ which include all continuous functions, the function $u(x,t) = g(x - bt)$ is the unique solution.

## 1.2 Inhomogeneous Transport Equation

Now we consider the corresponding inhomogeneous transport equation:

$$\dot{u} + b \cdot \nabla u = f.$$

Again $b \in \mathbb{R}^n$ is a given vector, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a given function and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the unknown function.

**Initial Value Problem 1.4.** *Given a vector $b \in \mathbb{R}$, a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and an initial value $g : \mathbb{R}^n \to \mathbb{R}$, we are looking for a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the inhomogeneous transport equation which is at $t = 0$ equal to $g$.*

We define for each $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ the function $z(s) = u(x + sb, t + s)$ which solves

$$z'(s) = b \cdot \nabla u(x + sb, t + s) + \dot{u}(x + sb, t + s) = f(x + sb, t + s).$$

This function obeys

$$\begin{aligned} u(x,t) - g(x - bt) = z(0) - z(-t) & & = \int_{-t}^{0} z'(s)ds \\ = \int_{-t}^{0} f(x + sb, t + s)ds & & = \int_{0}^{t} f(x + (s - t)b, s)ds. \end{aligned}$$

Hence the solution $u$ is equal to $\quad u(x,t) = g(x - bt) + \int_{0}^{t} f(x + (s - t)b, s)ds.$

We observe that this formula is analogous to the formula for solutions of inhomogeneous initial value problems of linear ODEs. The unique solution is the sum of the unique solution of the corresponding homogeneous initial value problem and the integral over solution of the homogeneous equation with the inhomogeneity as initial values. Again one can show that the initial value problem has a unique solution in a generalised sense if the initial value is a generalised differentiable function. We obtained these solutions of the first order homogeneous and inhomogeneous transport equation by solving an ODE. We shall generalise this method in Section 1.5 and solve more general first order PDEs by solving an appropriate chosen system of first order ODEs.

## 1.3 Scalar Conservation Laws

In this section we consider the non-linear first order differential equation

$$\dot{u}(x,t) + \frac{\partial f(u(x,t))}{\partial x} = \dot{u}(x,t) + f'(u(x,t)) \cdot \frac{\partial u(x,t)}{\partial x} = 0$$

with a smooth function $f : \mathbb{R} \to \mathbb{R}$. Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the unknown function. This equation is a non-linear first order PDE, and the method of characteristic applies. We impose the initial condition $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$ with some given function $u_0 : \mathbb{R} \to \mathbb{R}$. For any compact interval $[a,b]$ we calculate

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \dot{u}(x,t)dx = -\int_a^b \frac{\partial f(u(x,t))}{\partial x}dx = f(u(a,t)) - f(u(b,t)).$$

This is the meaning of a conservation law: the change of the integral of $u(\cdot,t)$ over $[a,b]$ is equal to the 'flux' of $f(u(x,t))$ through the 'boundary' $\partial[a,b] = \{a,b\}$.

For any $x \in \mathbb{R}$ we solve the ordinary differential equation $x'(s) = f'(u(x(s),s))$ with initial value $x(0) = x$. Consequently the derivative of the function $z(s) = u(x(s),s)$ is

$$z'(s) = \frac{\partial u(x(s),s)}{\partial x}x'(s) + \dot{u}(x(s),s) = \frac{\partial u(x(s),s)}{\partial x}f'(u(x(s),s)) + \dot{u}(x(s),s) = 0.$$

Hence $z$ is constant and equal to $z(s) = u(x(0),0) = u(x,0) = u_0(x)$ and therefore the derivative of $x(s)$ is constant equal to $x'(s) = f'(u(x,0)) = f'(u_0(x))$. The unique solution of the corresponding initial value problem is $x(s) = x + sf'(u_0(x))$. This implies

$$u(x + tf'(u_0(x)),t) = u_0(x) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2.$$

The solutions for initial values $x_1, x_2 \in \mathbb{R}^n$ with $u_0(x_1) \neq u_0(x_2)$ might intersect at $t \in \mathbb{R}^+$. In this case the method of characteristic implies $u_0(x_1) = u(x_1 + tf'(u_0(x_1)),t) = u(x_2 + tf'(u_0(x_2)),t) = u_0(x_2)$, which is impossible. This intersection of solutions of the characteristic equations is called crossing characteristics. There is a crossing of characteristics for $f'(u_0(x_2)) < f'(u_0(x_1))$ with $x_2 > x_1$.

**Theorem 1.5.** *If $f \in C^2(\mathbb{R},\mathbb{R})$ and $u_0 \in C^1(\mathbb{R},\mathbb{R})$ with $f''(u_0(x))u_0'(x) > -\alpha$ for all $x \in \mathbb{R}$ and some $\alpha \geq 0$, then there is a unique $C^1$-solution of the initial value problem*

$$\frac{\partial u(x,t)}{\partial t} + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0 \quad \textit{with} \quad u(x,0) = u_0(x)$$

*on $(x,t) \in \mathbb{R} \times [0,\alpha^{-1})$ for $\alpha > 0$ and on $(x,t) \in \mathbb{R} \times [0,\infty)$ for $\alpha = 0$.*

*Proof.* By the method of characteristic the solution $u(x,t)$ is on the lines $x+tf'(u_0(x))$ equal to $u_0(x)$. For all $t \geq 0$ with $1+t\alpha > 0$ the derivative of $x \mapsto x+tf'(u_0(x))$ obeys

$$1 + tf''(u_0(x))u_0'(x) \geq 1 + t\alpha > 0.$$

This implies $\lim_{x\to\pm\infty} x+tf'(u_0(x)) = \pm\infty$. So $x \mapsto x+tf'(u_0(x))$ is $C^1$-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Therefore there exists for any $y \in \mathbb{R}$ a unique $x$ with $x+tf'(u_0(x)) = y$. Then $u(y,t) = u_0(x)$ solves the initial value problem. **q.e.d.**

**Example 1.6.** *For $n = 1$ and $f(u) = \frac{1}{2}u^2$ we obtain Burgers equation:*

$$\dot{u}(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0.$$

*The solutions of the corresponding characteristic equations are $x(t) = x_0 + u_0(x_0)t$. Therefore the solutions of the corresponding initial value problem obey*

$$u(x + tu_0(x), t) = u_0(x).$$

*If $u_0$ is continuously differentiable and monotonic increasing, then for all $t \in [0,\infty)$ the map $x \mapsto x + tu_0(x)$ is a $C^1$-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there is a unique $C^1$-solution on $\mathbb{R} \times [0,\infty)$. More generally, if $u_0'(x) > -\alpha$ with $\alpha \geq 0$, then there is a unique $C^1$-solution on $\mathbb{R} \times [0,\alpha^{-1})$ for $\alpha > 0$ and $(x,t) \in \mathbb{R} \times [0,\infty)$ for $\alpha = 0$.*

## 1.4 Weak Solutions

In this section we look for more general notions of solutions which allow us to extend solutions across the crossing characteristics. For this purpose we use the conserved integrals. Since we will restrict ourselves to the one-dimensional situation for the moment, the natural domains are intervals $\Omega = [a,b]$ with $a < b \in \mathbb{R}$. In this case the conservation law implies

$$\frac{d}{dt}\int\limits_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

Now we look for functions $u$ with discontinuities along the graph $\{(x,t) \mid x = y(t)\}$ of a $C^1$–function $y$. In the case that $y(t)$ belongs to $[a,b]$, we split the integral over $[a,b]$ into the integrals over $[a,b] = [a,y(t)] \cup [y(t),b]$. In such a case let us calculate the derivative of the integral over $[a,b]$:

$$\frac{d}{dt}\int_a^b (u(x,t)dx = \frac{d}{dt}\int_a^{y(t)} u(x,t)dx + \frac{d}{dt}\int_{y(t)}^b u(x,t)dx =$$
$$= \dot{y}(t)\lim_{x\uparrow y(t)} u(x,t) + \int_a^{y(t)} \dot{u}(x,t)dx - \dot{y}(t)\lim_{x\downarrow y(t)} u(x,t) + \int_{y(t)}^b \dot{u}(x,t)dx.$$

We abbreviate $\lim_{x\uparrow y(t)} u(x,t)$ as $u^l(y(t),t)$ and $\lim_{x\downarrow y(t)} u(x,t)$ as $u^r(y(t),t)$ and assume that on both sides of the graph of $y$ the function $u$ is a classical solution of the conservation law:

$$\frac{d}{dt}\int_a^b (u(x,t)dx = \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) - \int_a^{y(t)} \frac{d}{dx} f(u(x,t))dx - \int_{y(t)}^b \frac{d}{dx} f(u(x,t))dx$$
$$= \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) + f(u(a,t)) - f(u(b,t)) + f(u^r(y(t),t) - f(u^l(y(t),t).$$

Hence the integrated version of the conservation law still holds, if the following Rankine-Hugonoit condition is fulfilled:

$$\dot{y}(t) = \frac{f(u^r(y,t)) - f(u^l(y,t)}{u^r(y,t) - u^l(y,t)}.$$

**Example 1.7.** *We consider Burgers equation $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ with the following continuous initial values $u(x,0) = u_0(x)$ and*

$$u_0(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 - x & \text{for } 0 \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

*The first crossing of characteristics happens for $t = 1$:*

$$x + tu_0(x) = \begin{cases} x + t & \text{for } x \leq 0, \\ x + t(1 - x) & \text{for } 0 < x < 1, \\ x & \text{for } 1 \leq x. \end{cases}$$

*For $t < 1$ the evaluation at $t$ is a homeomorphism from $\mathbb{R}$ onto itself with inverse*

$$x \mapsto \begin{cases} x - t & \text{for } x \leq t, \\ \frac{x-t}{1-t} & \text{for } t < x < 1, \\ x & \text{for } 1 \leq x. \end{cases}$$

*Therefore the solution is for $0 < t < 1$ equal to*

$$u(x,t) = \begin{cases} 1 & \text{for } x < t, \\ \frac{x-1}{t-1} & \text{for } t < x < 1, \\ 0 & \text{for } 1 \le x. \end{cases}$$

*At $t = 1$ the solutions of the characteristic equations starting at $x \in [0,1]$ all meet at $x = 1$. For $t > 1$ there exists a unique discontinuous solution satisfying the Rankine-Hugonoit condition. For small $x$ this solution is $1$ and for large $x$ it is $0$. The corresponding regions has to be separated by a path with velocity $\frac{1}{2}$ which starts at $(x,t) = (1,1)$. For $t \ge 1$ this discontinuous solution is equal to*

$$u(x,t) = \begin{cases} 1 & \text{for } x < 1 + \frac{t-1}{2}, \\ 0 & \text{for } 1 + \frac{t-1}{2} < x. \end{cases}$$

The second initial value problem is not continuous but monotonic increasing. For continuous monotonic increasing functions $u_0$ the evaluation at $t$ of the solutions of the characteristic equation would be a homeomorphism for all $t > 0$. Therefore in such cases there exists a unique continuous solution for all $t > 0$. But for non-continuous initial values this is not the case.

**Example 1.8.** *We again consider Burgers equation $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ with the following non-continuous initial values $u(x,0) = u_0(x)$ and*

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 < x. \end{cases}$$

*Again there is a unique discontinuous solution which is for small $x$ equal to $0$ and for large $x$ equal to $1$. By the Rankine-Hugonoit condition both regions are separated by a path with velocity $\frac{1}{2}$. This solution is equal to*

$$u(x,t) = \begin{cases} 0 & \text{for } x < \frac{t}{2}, \\ 1 & \text{for } \frac{1}{2} < x. \end{cases}$$

*But there exists another continuous solution, which clearly also satisfies the Rankine-Hugonoit condition:*

$$u(x,t) = \begin{cases} 0 & \text{for } x \le 0, \\ \frac{x}{t} & \text{for } 0 < x < t, \\ 1 & \text{for } 1 \le x. \end{cases}$$

*These solutions are constant along the lines $x = ct$ for $c \in [0, 1]$. These lines all intersect in the discontinuity at $(x, t) = (0, 0)$. Besides these two extreme cases there exists infinitely many other solutions with several regions of discontinuity, which all satisfy the Rankine-Hugonoit condition.*

These examples show that such weak solutions exists for all $t \geq 0$ but are not unique. Therefore we want to restrict the space of weak solutions such that they have a unique solutions for all $t \geq 0$. Since we want to maximise the regularity we only accept discontinuities if there are no continuous solutions. In the last example we prefer the continuous solution. So for Burgers equation this means we only accept discontinuous solutions, which take larger values for smaller $x$ and smaller values for larger $x$.

**Definition 1.9** (Lax Entropy condition)**.** *A discontinuity of a weak solution along a $C^1$–path $t \mapsto y(t)$ satisfies the Lax entropy condition, if along the path the following inequality is fulfilled:*

$$f'(u^l(y,t),t) > \dot{y}(t) > f'(u^r(y,t)).$$

*A weak solutions with discontinuities along $C^1$–paths is called an admissible solution, if along the path both the Rankine–Hugonoit condition and the Lax Entropy condition are satisfied.*

For continuous $u_0$ there is a crossing of characteristics if $f'(u_0(x_1)) > f'(u_0(x_2))$ for $x_1 < x_2$. So this condition ensures that discontinuities can only show up if we cannot avoid a crossing of characteristics.

**Theorem 1.10.** *Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be convex and $u$ and $v$ two admissible solutions of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0.$$

*in $L^1(\mathbb{R})$. Then $t \mapsto \|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})}$ is monotonically decreasing.*

*Proof.* We divide $\mathbb{R}$ into maximal intervals $I = [a(t), b(t)]$ with the property that either $u(x,t) > v(x,t)$ or $v(x,t) > u(x,t)$ for all $x \in (a(t), b(t))$. This means that either $x \mapsto u(x,t) - v(x,t)$ vanishes at the boundary, or is discontinuous and changes sign at the boundary. We claim that the boundaries $a(t)$ and $b(t)$ of these maximal intervals are differentiable. We prove this only for $a(t)$. For $b(t)$ the proof is analogous. If either $u(\cdot,t)$ or $v(\cdot,t)$ is discontinuous at $a(t)$, then by definition of an admissible solution the locus of the discontinuity $a(t)$ is differentiable with respect to $t$. If $u(\cdot,t)$ and $v(\cdot,t)$ are both continuously differentiable at $a(t)$ with $u(a(t),t) = v(a(t),t)$, then by the method of characteristic for sufficiently small $\epsilon > 0$ all $x \in (a(t)-\epsilon, a(t)+\epsilon)$ with $u(x,t) = v(x,t)$

preserve this property along characteristic lines $x+tf'(u(x(t),t) = x+tf'(v((x(t),t)$. So along these lines also the properties $u(x,t) \neq v(x,t)$ and $u(x,t) > v(x,t)$ are preserved. This implies that $a(t)$ is differentiable with $\dot{a}(t) = f(u(a(t),t)) = f(v(a(t),t))$. Let us only consider intervals on whose interior $u(\cdot,t) - v(\cdot,t)$ is positive. For the other intervals we apply the same arguments with interchanged $u$ and $v$. Now we calculate

$$\begin{aligned}
\frac{d}{dt}\int\limits_{a(t)}^{b(t)} (u(x,t)-v(x,t))dx &= \int\limits_{a(t)}^{b(t)} (\dot{u}(x,t)-\dot{v}(x,t))dx + \\
&\quad + \dot{b}(t)(u(b(t),t)-v(b(t),t)) - \dot{a}(t)(u((a(t),t)-v(a(t),t)) \\
&= \int\limits_{a(t)}^{b(t)} \frac{d}{dx}(f(v(x,t)-f(u(x,t))dx \\
&\quad + \dot{b}(t)(u(b(t),t)-v(b(t),t)) - \dot{a}(t)(u((a(t),t)-v(a(t),t)) \\
&= f(v(b(t),t) - f(u(b(t),t) + \dot{b}(t)(u(b(t),t)-v(b(t),t)) \\
&\quad + f(u(a(t),t) - f(v(a(t),t) + \dot{a}(t)(v(a(t),t)-u(a(t),t)).
\end{aligned}$$

If $u(\cdot,t)$ and $v(\cdot,t)$ are both differentiable at $a(t)$, then they take the same values at $a(t)$ and the last line vanishes. Analogously, if $u(\cdot,t)$ and $v(\cdot,t)$ are both differentiable at $b(t)$, then the second last line vanishes. For convex $f$ the derivative $f'$ is monotonically increasing and the Lax-Entropy condition implies

$$u^l(y,t) > u^r(y,t), \qquad\qquad v^l(y,t) > v^r(y,t)$$

at all discontinuities $y$ of $u(\cdot,t)$ and $v(\cdot,t)$, respectively. If one of the two solutions $u$ and $v$ is at the boundary of $I$ continuous and the other is non-continuous, then the value of the continuous solution has to lie in between the limits of the non-continuous solution, because at the boundary either $u-v$ becomes zero or changes sign. Since $u > v$ on $(a(t),b(t))$ either $u(\cdot,t)$ is continuous and differentiable at $a(t)$ and $v(\cdot,t)$ is discontinuous at $a(t)$ or $u$ is discontinuous at $b(t)$ and $v$ is continuous and differentiable at $b(t)$. In the first case we use the Rankine Hugonoit condition to determine $\dot{a}(t)$ and $\dot{b}(t)$. To simplify notation we write $a$ and $b$ instead of $a(t)$ and $b(t)$. The corresponding contribution to the derivative of $\|u(\cdot,t)-v(\cdot,t)\|_1$ is

$$\begin{aligned}
&f(u(a,t)) - f(v^r(a,t)) + \dot{a}(t)\left(v^r(a,t)-u(a,t)\right) = \\
&\qquad = f(u(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t)) - f(v^l(a,t))}{v^r(a,t)-v^l(a,t)}\left(v^r(a,t)-u(a,t)\right) \\
&\qquad = f(u(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t)-u(a,t)}{v^l(a,t)-v^r(a,t)} + f(v^l(a,t))\frac{u(a,t)-v^r(a,t)}{v^l(a,t)-v^r(a,t)}\right).
\end{aligned}$$

Since $f$ is convex the secant lies above the graph of $f$. Hence due to $u(a,t) \in [v^r(a,t), v^l(a,t)]$ this expression is non-positive. In the second case the contribution to the derivative of $\|u(\cdot,t) - v(\cdot,t)\|_1$ is

$$
\begin{aligned}
&f(v(b,t)) - f(u^l(b,t)) + \dot{b}(t)\left(u^l(b,t) - v(b,t)\right) = \\
&\qquad = f(v(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t)) - f(u^l(b,t))}{u^r(b,t) - u^l(b,t)}\left(u^l(b,t) - v(b,t)\right) \\
&\qquad = f(v(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t) - v(b,t)}{u^l(b,t) - u^r(b,t)} + f(u^l(b,t))\frac{v(b,t) - u^r(b,t)}{u^l(b,t) - u^r(b,t)}\right).
\end{aligned}
$$

Again due to $v(b,t) \in [u^r(b,t), u^l(b,t)]$ this expression is non-positive.

If finally both solutions are discontinuous at $a(t)$ or $b(t)$. Since $u(\cdot,t) - v(\cdot,t)$ is positive on $I$, the Lax Entropy condition implies $u^r(a,t) \in [v^r(a,t), v^l(a,t)]$ and $v^l(b,t) \in [u^r(b,t), u^l(b,t)]$, respectively. The corresponding contributions to the derivative of $\|u(\cdot,t) - v(\cdot,t)\|_1$ are again non-positive:

$$
\begin{aligned}
&f(u^r(a,t)) - f(v^r(a,t)) + \dot{a}(t)\left(v^r(a,t) - u^r(a,t)\right) = \\
&\qquad = f(u^r(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t)) - f(v^l(a,t))}{v^r(a,t) - v^l(a,t)}\left(v^r(a,t) - u^r(a,t)\right) \\
&\qquad = f(u^r(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t) - u^r(a,t)}{v^l(a,t) - v^r(a,t)} + f(v^l(a,t))\frac{u^r(a,t) - v^r(a,t)}{v^l(a,t) - v^r(a,t)}\right).
\end{aligned}
$$

$$
\begin{aligned}
&f(v^l(b,t)) - f(u^l(b,t)) + \dot{b}(t)\left(u^l(b,t) - v^l(b,t)\right) = \\
&\qquad = f(v^l(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t)) - f(u^l(b,t))}{u^r(b,t) - u^l(b,t)}\left(u^l(b,t) - v^l(b,t)\right) \\
&\qquad = f(v^l(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t) - v^l(b,t)}{u^l(b,t) - u^r(b,t)} + f(u^l(b,t))\frac{v^l(b,t) - u^r(b,t)}{u^l(b,t) - u^r(b,t)}\right).
\end{aligned}
$$

Hence the contributions to $\frac{d}{dt}\|u(\cdot,t) - v(\cdot,t)\|_1$ of all intervals are non-positive. **q.e.d.**

This implies that admissible solutions are unique, if they exist. By utilising an explicit formula for admissible solutions one can also prove the existence of admissible solutions. The following theorem is Theorem 10.3 in the lecture notes "Hyperbolic Partial Differential Equations" by Peter Lax, Courant Lecture Notes in Mathematics **14**, American Mathematical Society (2006), which also supplies a proof.

**Theorem 1.11.** *For $f \in C^2(\mathbb{R},\mathbb{R})$ is strictly convex and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there exists an unique admissible solution $u(x,t)$ of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0 \quad \textit{and} \quad u(x,0) = u_0(x) \quad \textit{for all } x \in \mathbb{R}.$$

## 1.5 Method of Characteristics

In this section we shall solve the following first order PDE:

$$F(\nabla u(x), u(x), x) = 0.$$

Here $u$ is a real unknown function on an open domain $\Omega \subset \mathbb{R}^n$ and $F$ is a real function on an open subset of $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We try to obtain the solution by solving an appropriate system of first order ODEs for the values of the function $u$ along some integral curves along some vector fields. So let $x(s)$ be such such an integral curve and $p(s) = \nabla u(x(s))$ the gradient of the unknown function along this curve. We want to determine the curve $s \mapsto x(s)$ in such a way, that the triple $s \mapsto (p(s), z(s), x(s))$ with $z(s) = u(x(s))$ solves an ODE. For this purpose we differentiate

$$\frac{dp_i(s)}{ds} = \frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{dx_j(s)}{ds}.$$

The total derivative of $F(\nabla u(x), u(x), x) = 0$ with respect to $x_i$ gives

$$\begin{aligned} 0 &= \frac{dF(\nabla u(x), u(x), x)}{dx_i} = \\ &= \sum_{j=1}^{n} \frac{\partial F(\nabla u(x), u(x), x)}{\partial p_j}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial z}\frac{\partial u(x)}{\partial x_i} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial x_i}. \end{aligned}$$

Due to the commutativity $\partial_i \partial_j u = \partial_j \partial_i u$ of the second partial derivatives we obtain

$$\sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}\frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} + \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) + \frac{\partial F(p(s), z(s), x(s))}{\partial x_i} = 0.$$

Now we choose the vector field for the integral curves $s \mapsto x(s)$ as

$$\frac{dx_j}{ds} = \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

This choice allows us to rewrite the differential equation

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{dx_j}{ds}(s)$$

as

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} =$$
$$= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s).$$

Finally we differentiate

$$\frac{dz(s)}{ds} = \frac{du(x(s))}{ds} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x(s)) \frac{dx_j(s)}{ds} = \sum_{j=1}^{h} p_j(s) \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

In this way we indeed obtain the following system of first order ODEs:

$$\begin{aligned} x_i'(s) &= \frac{\partial F(p(s), z(s), x(s))}{\partial p_i} \\ p_i'(s) &= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \\ z'(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s). \end{aligned}$$

This is a system of first order ODEs with $2n+1$ unknown real functions. Let us summarise these calculations in the following theorem:

**Theorem 1.12.** *Let $F$ be a real differentiable function on an open subset $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a twice differentiable solution on an open subset $\Omega \subset \mathbb{R}^n$ of the first order PDE $F(\nabla u(x), u(x), x) = 0$. For every solution $s \mapsto x(s)$ of the ODE*

$$x_i'(s) = \frac{\partial F}{\partial p_i}(\nabla u(x(s)), u(x(s)), x(s))$$

*the functions $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$ solve the ODEs*

$$\begin{aligned} p_i'(s) &= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \text{ and} \\ z'(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s). \end{aligned}$$

**q.e.d.**

Now we want to introduce a boundary value problem of the following form:

$$u(y) = g(y) \text{ for all } y \in \Omega \cap H \text{ with } H = \{y \in \mathbb{R}^n \mid y \cdot e_n = x_0 \cdot e_n\}.$$

Here $e_n = (0, \ldots, 0, 1)$ denotes the $n$–th element of the canonical basis and $H$ the unique hyperplane through $x_0 \in \Omega$ orthogonal to $e_n$. By the implicit function theorem general continuously differentiable hyper surfaces may be brought into this form by a continuously differentiable coordinate transformation with continuously differentiable inverse: Let $\Phi : \Omega \to \Omega'$ be a continuously differentiable homeomorphism with continuously differentiable inverse $\Phi^{-1}$. Then by the chain rule the composition $u = v \circ \Phi$ of a function $v : \Omega' \to \mathbb{R}$ with $\Phi$ obeys for $y = \Phi(x)$ i.e. $x = \Phi^{-1}(y)$

$$\nabla u(x) = \nabla v(\Phi(x)) \cdot \Phi'(x) = \nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right).$$

Here $\nabla v$ and $\nabla u$ are row vectors and $\Phi'(x)$ the Jacobi matrix. Hence $u$ solves the PDE

$$F(\nabla u(x), u(x), x) = 0$$

if and only if $v$ solves the PDE

$$F\left(\nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right), v(y), \Phi^{-1}(y)\right) = 0.$$

Therefore the PDE for the function $v$ is the zero set of the function

$$G(\nabla v(y), v(y), y) = F\left(\nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right), v(y), \Phi^{-1}(y)\right)$$

In the sequel we assume that the hyperplane $H$ has the following form:

$$H = \{y \in \mathbb{R}^n \mid y \cdot e_n = x_0 \cdot e_n\}.$$

If the hyper surface $H' \subset \Omega'$ is the zero set of a continuously differentiable function $\Lambda : \Omega' \to \mathbb{R}$ whose gradient $\nabla\Lambda$ does not vanish on $H'$, then the implicit function theorem shows that in a neighbourhood of $y_0 \in H'$ there exists such a $\Phi$. Furthermore, $\Phi$ is as often differentiable as $\Lambda$. In the foregoing theorem the functions $u$ and $v$ has to be twice differentiable. We assume that $\Phi$ and $\Phi^{-1}$ are twice differentiable. Consequently $\Lambda$ should be twice differentiable. On $\Omega \cap H$ there must hold

$$F(\nabla u(y), u(y), y) = 0.$$

On order to define initial conditions at $y \in \Omega \cap H$

$$z(0) = g(y), \qquad p(0) = q(y) \quad \text{and} \qquad x(0) = y$$

we have to find a solution $q : \Omega \cap H \to \mathbb{R}^n, \quad y \mapsto q(y)$ of the following equation:

$$F(q(y), g(y), y) = 0 \quad \text{and} \qquad \frac{\partial g(y)}{\partial y_i} = q_i(y) \text{ for } i = 1, \ldots, n-1.$$

The second equations uniquely determine $q_1(y), \ldots, q_{n-1}(y)$ as

$$q_1(y) = \tfrac{\partial g(y)}{\partial y_1}, \ldots, q_{n-1}(y) = \tfrac{\partial g(y)}{\partial y_{n-1}}.$$

It remains to determine the component $q_n(y)$ in such a way, that

$$F(q(y), g(y), y) = 0$$

holds for all $y \in \Omega \cap H$. Now the implicit function theorem implies that this equation implicitly defines a continuously differentiable function $y \mapsto q_n(y)$, if

$$\frac{\partial F(p_0, z_0, x_0)}{\partial p_n} \neq 0.$$

This proves the following lemma:

**Lemma 1.13.** *Let $F : W \to \mathbb{R}$ and $g : H \to \mathbb{R}$ be continuously differentiable, $x_0 \in \Omega \cap H$, $z_0 = g(x_0)$ and $p_{0,1} = \tfrac{\partial g(x_0)}{y_1}, \ldots, p_{0,n-1} = \tfrac{\partial g(x_0)}{y_n}$. If there exists $p_{0,n}$ with*

$$(p_0, z_0, x_0) \in W, \qquad F(p_0, z_0, x_0) = 0 \quad \text{and} \qquad \tfrac{\partial F(p_0,z_0,x_0)}{\partial p_n} \neq 0,$$

*then on an open neighbourhood of $x_0 \in \Omega \cap H$ there exists a unique solution $q$ of*

$$F(q(y), g(y), y) = 0, \quad q_i(y) = \tfrac{\partial g(y)}{\partial y_i} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \quad q(y_0) = p_0. \quad \textbf{q.e.d.}$$

**Theorem 1.14.** *Let $F : X \to \mathbb{R}$ and $g : \Omega \cap H \to \mathbb{R}$ be three times differentiable functions on open subsets. Furthermore, let $(p_0, z_0, x_0) \in W$ and $g$ satisfy*

$$F(p_0, z_0, x_0) = 0, \quad g(x_0) = z_0, \quad p_{0,1} = \tfrac{\partial g(x_0)}{y_1}, \ldots, \quad p_{0,n-1} = \tfrac{\partial g(x_0)}{y_n}, \quad \tfrac{\partial F}{\partial p_n}(p_0, z_0, x_0) \neq 0.$$

*Then there exists on a neighbourhood $\Omega$ of $x_0$ a solution of the boundary value problem*

$$F(\nabla u(x), u(x), x) = 0 \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u(y) = g(y) \quad \text{for} \quad y \in \Omega \cap H.$$

*Proof.* By the foregoing Lemma there exists a solution $q$ on an open neighbourhood of $x_0$ in $H$ of the following equations

$$F(q(y), g(y), y) = 0, \qquad q_i(y) = \tfrac{\partial g(y)}{\partial y_i} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \qquad q(y_0) = p_0.$$

If $F$ is twice and $g$ are three times differentiable then the implicit function theorem yields a twice differentiable solution. The theorem of Picard Lidenlöff shows that the following initial value problem has for all $y$ in the intersection of an open neighbourhood of $x_0$ with $H$ a unique solution:

$$\begin{aligned}
x_i'(s) &= \frac{\partial F}{\partial p_i}(p(s), z(s), x(s)) && \text{with} \quad x(0) = y \\
p_i'(s) &= -\frac{\partial F}{\partial x_i}(p(s), z(s), x(s)) - \frac{\partial F}{\partial z}(p(s), z(s), x(s))p_i(s) && \text{with} \quad p(0) = q(y) \\
z'(s) &= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(p(s), z(s), x(s))p_j(s) && \text{with} \quad z(0) = g(y).
\end{aligned}$$

We denote the family of solutions by $(x(y,s), p(y,s), z(y,s))$. For small $\Omega \ni x_0$ there exists an $\epsilon > 0$ such that these solutions are uniquely defined on $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$. Since $F$ and $g$ are three times differentiable all coefficients and initial values are twice differentiable. The theorem on the dependence of solutions of ODEs on the initial values gives that $(y,s) \mapsto (x(y,s), p(y,s), z(y,s))$ is on $(\Omega \cap H) \times (-\epsilon, \epsilon)$ twice differentiable. Due to the choice of the initial values at $s = 0$, the function

$$(\Omega \cap H) \times (-\epsilon, \epsilon) \to \Omega, \quad (y,s) \mapsto x(y,s)$$

has at $(y,s) = (x_0, 0)$ the Jacobi matrix

$$\begin{pmatrix}
1 & 0 & \dots & 0 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_1} \\
 & & \vdots & & \vdots \\
0 & 0 & \dots & 1 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_{n-1}} \\
0 & 0 & \dots & 0 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_n}
\end{pmatrix}.$$

Since $\frac{\partial F(p_0,z_0,x_0)}{\partial p_n} \neq 0$ this matrix is invertible, and the inverse function theorem implies that on an possibly diminished neighbourhood $\Omega$ of $x_0$ and an appropriately chosen $\epsilon > 0$ this map is a twice differentiable homeomorphism with twice differentiable inverse mapping. Now we define the function $u : \Omega \to \mathbb{R}$ by

$$u(x(y,s)) = z(y,s) \text{ for all } (y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Our next task is to show that this function solves the PDE $F(\nabla u(x), u(x), x) = 0$.

In a first step we observe that the ODE implies

$$\frac{\partial}{\partial s} F(p(y,s), z(y,s), x(y,s)) = 0.$$

Since $F(q(y), g(y), y)$ vanishes for all $y \in \Omega \cap H$ we conclude

$$F(p(y,s), z(y,s), x(y,s)) = 0 \text{ for all } (y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Hence it suffices to show $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$.

In a second step we show

$$\frac{\partial z(y,s)}{\partial s} = \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial s} \quad \text{and} \quad \frac{\partial z(y,s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}$$

for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$ and all $i = 1, \ldots, n-1$. The first equation follows from the ODE for $x(y,s)$ and $z(y,s)$. For $s = 0$ the second equation follows from the initial conditions for $z(y,s)$, $p(y,s)$ and $x(y,s)$. The derivative of the first equation with respect to $y_i$ yields

$$\frac{\partial^2 z(y,s)}{\partial y_i \partial s} = \sum_{j=1}^{n} \left( \frac{\partial p_j(y,s)}{\partial y_i}\frac{\partial x_j(y,s)}{\partial s} + p_j(y,s)\frac{\partial^2 x_j(y,s)}{\partial y_i \partial s} \right).$$

By the commutativity of the second partial derivatives we obtain

$$\begin{aligned}
\frac{\partial}{\partial s}&\left( \frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&= \frac{\partial^2 z(y,s)}{\partial s \partial y_i} - \sum_{j=1}^{n} \frac{\partial p_j(y,s)}{\partial s}\frac{\partial x_j(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s)\frac{\partial^2 x_j(y,s)}{\partial s \partial y_i} \\
&= \sum_{j=1}^{n} \left( \frac{\partial p_j(y,s)}{\partial y_i}\frac{\partial x_j(y,s)}{\partial s} - \frac{\partial p_j(y,s)}{\partial s}\frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&= \sum_{j=1}^{n} \frac{\partial p_j(y,s)}{\partial y_i}\frac{\partial F(p(y,s), z(y,s), x(y,s))}{\partial p_j} + \\
+ \sum_{j=1}^{n} &\left( \frac{\partial F(p(y,s), z(y,s), x(y,s))}{\partial x_j} + \frac{\partial F(p(y,s), z(y,s), x(y,s)) p_j(y,s)}{\partial z} \right) \frac{\partial x_j(y,s)}{\partial y_i} \\
&= \frac{\partial}{\partial y_i} F(p(y,s), z(y,s), x(y,s)) - \\
&- \frac{\partial F(p(y,s), z(y,s), x(y,s))}{\partial z} \left( \frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i} \right).
\end{aligned}$$

We insert the result $F(p(y,s),z(y,s),x(y,s)) = 0$ of the first step and obtain

$$\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y_i}(y,s) - \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}\right) =$$
$$= -\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial z}\left(\frac{\partial z}{\partial y_i}(y,s) - \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}\right).$$

This is a linear homogeneous ODE with initial value 0 at $s = 0$. The unique solution vanishes identically. This implies the second equation and finishes the second step:

$$\frac{\partial z(y,s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}.$$

Finally in a third step we show $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$. Locally the derivative of the map $(y,s) \mapsto x$ is invertible. Altogether we obtain

$$\frac{\partial u}{\partial x_j} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} \quad = \left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial s}\right)\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\left(\sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial y_i}\right)\frac{\partial y_i}{\partial x_j}$$
$$= \sum_{k=1}^{n} p_k\left(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right) \quad = \sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial x_j} \quad = p_j.$$

Due to the initial values $z(y,0)$ we have $u(y) = g(y)$ for all $y \in \Omega \cap H$. The uniqueness of the solution follows from the Theorem 1.12 and the theorem of Picard-Lindelöf. **q.e.d.**

We solved the boundary value problem by solving a family of ODEs. As in the case of the inhomogeneous transport equation, we combine the coordinates $x$ and $t$ to one coordinate $(x,t)$. Consequently we write

$$F(p,z,(x,t)) = \tilde{F}(p,x,t) = b_1p_1 + \ldots + b_np_n + p_{n+1} - f(x,t).$$

We use the equation $F(p,z,(x,t)) = 0$ and rewrite the ODE for $z$. Then the ODE becomes independent of $p$ and we can solve $x(s)$, $t(s)$ and $z(s)$ separately:

$$x' = b \qquad t' = 1 \qquad p' = (\nabla f(x,t), \dot{f}(x,t)) \qquad z' = \tilde{F}(p,x,t) + f(x,t) = f(x,t).$$

Whenever the function $F$ is a first order polynomial with respect to $p$, then the functions

$$\frac{\partial F(p(s),z(s),x(s))}{\partial p_i} \text{ for } i = 1,\ldots,n \text{ and}$$
$$F(p(s),z(s),x(s)) - \sum_{j=1}^{n}\frac{\partial F(p(s),z(s),x(s))}{\partial p_j}p_j(s)$$

do not depend on $p$. Therefore the ODE system becomes independent of $p(s)$, and the components $x(s)$ and $z(s)$ can be solved independently of $p(s)$. This situation describes the transport equation with vector $b$ depending on $z$, $x$ and $t$. For the solution of this equation we do not need to introduce the function $p(s) = \nabla u(x(s))$. Another example is the scalar conservation law in the general form for unknown function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$:

$$\dot{u}(x,t) + \nabla f(u(x,t)) = \dot{u}(x,t) + f'(u(x,t)) \cdot \nabla u(x,t) = 0$$

with a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}^n$. Again we impose the initial values $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}^n$ and some given function $u_0 : \mathbb{R}^n \to \mathbb{R}$. If $x_{n+1} = t$ then the corresponding function $F$ is indeed linear in $p$:

$$F(p, z, (x,t) = f'(z) \cdot (p_1, \ldots, p_m) + p_{n+1}.$$

So the corresponding ODE is independent of $p$

$$\dot{x} = f'(u(x(s), t(s) \qquad \dot{t} = 1 \qquad \dot{z} = F(p, z, (x,t)) = 0.$$

For any $x \in \mathbb{R}^n$ the unique solution is $x(s) = x + sf'(u_0(x))$, $t(s) = s$ and $z(s) = u_0(x)$. So we recover in this more general situation the implicit equation from Section 1.3:

$$u(x + tf'(u_0(x)), t) = u_0(x) \quad \text{for all} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

# Chapter 2

# General Concepts

In this chapter we prepare for our investigation of the three main examples of linear second order partial differential equations in the subsequent three chapters.

## 2.1 Divergence Theorem

In this section we present the divergence theorem, which is a generalisation of the fundamental theorem of calculus to higher dimensions. This will have many important consequences, but let us just mention two of them here: First we can generalise partial integration to higher dimensions. Second it allows us to understand the sense in which the higher dimensional scalar conservation law describes a conserved quantity. In order to state the theorem we have to describe how to integrate over sub manifolds of $\mathbb{R}^n$. We start with a definition of such sub manifolds.

**Definition 2.1.** *A subset $A \subset \mathbb{R}^n$ is called a $k$-dimensional sub manifold if $A$ is covered by the images $O$ of homeomorphisms $\Phi : U \to O$ from open subsets $U \subset \mathbb{R}^k$ onto open subset of $A$, such that $\Phi$ considered as maps into $\mathbb{R}^n$ are continuously differentiable and have on $U$ a Jacobean of full rank $k$.*

The Jacobean of $\Phi$ is a $n \times k$ matrix, whose rank cannot be greater than $n$, so $1 \leq k \leq n$. If $\Phi : U \to O$ has the properties in the definition, then choose for any $x \in O$ a $k$-dimensional linear subspace $V \subset \mathbb{R}^n$, such that the composition $P_V \circ \Phi'(\Phi^{-1}(x))$ with the orthogonal projection $P_V$ onto $V$ is bijective onto $V$. By the inverse function theorem we may diminish $U$ and $O$ such that $P_V \circ \Phi$ is a $C^1$-diffeomorphism from $U$ onto an open subset $W$ of $V$. The image $O$ of $\Phi$ is the zero set of the function $W \times V^\perp \to V^\perp$, $(y, z) \mapsto z - (1 - P_V) \circ \Phi \circ (P_V \circ \Phi)^{-1}(y)$, which has rank $n - k$. We consider $W \times V^\perp \subset V \times V^\perp \simeq \mathbb{R}^n$ as an open subset of $\mathbb{R}^n$ which contains $O$. So the set $A$ satisfies for any $x \in A$ the following condition: There exists on an open neighbourhood of $x$ in $\mathbb{R}^n$ a continuously differentiable function to $\mathbb{R}^{n-k}$ whose Jacobean has at $x$ rank $n-k$, such that the intersection of $A$ with this neighbourhood is the level set of this function through $x$. Conversely, by the implicit function theorem, a subset $A \subset \mathbb{R}^n$ which satisfies the latter condition for all $x \in A$ is a $k$-dimensional submanifold. So we may alternatively characterise submanifolds by this latter condition.

The definition of an integral over submanifolds uses so called *partitions of unity.*

**Definition 2.2.** *(Partition of Unity) For a given family $(U_\alpha)_{\alpha \in A}$ of open subsets of $\mathbb{R}^n$ with union $\bigcup_{\alpha\in A} U_\alpha = \Omega \subset \mathbb{R}^n$ a smooth partition of unity is a countable family $(h_l)_{l\in\mathbb{N}}$ of smooth functions $h_l : \Omega \to [0,1]$ with the following properties:*

**(i)** *Each $x \in \Omega$ has a neighbourhood where all but finitely $h_l$ vanish identically.*

**(ii)** *For all $x \in \Omega$ we have $\sum_{l=1}^\infty h_l(x) = 1$.*

**(iii)** *Each $h_l$ vanishes outside a compact subset of $U_\alpha$ for some $\alpha \in A$.*

For every family of open subsets of $\mathbb{R}^n$ there exists a smooth partition of unity. A proof you can find in many textbooks and in my script of the lecture Analysis II.

**Definition 2.3.** *Let $A \subset \mathbb{R}^d$ be a $k$-dimensional submanifold of $\mathbb{R}^n$ and let $f \in C(A, \mathbb{R})$ vanish outside of a compact subset $K \subset A$. We cover $K$ by finitely many open subsets $O \subset \mathbb{R}^d$ with $A \cap O = \Phi[U]$ for a map $\Phi$ as in Definition 2.1 and choose a corresponding partition of unity $(h_l)_{l\in\mathbb{N}}$. The integral of $f$ over $A$ is defined as*

$$\int_A f d\sigma = \sum\nolimits_{l\in\mathbb{N}} \int_U (h_l f) \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu.$$

Note that the volume of the $k$-dimensional parallelotope spanned by the column vectors of a $n \times k$ matrix $A$ is equal to $\sqrt{\det(A^T A)}$. Here $A^T A$ is the matrix of all scalar products between the column vectors of $A$.

**Lemma 2.4.** *The integral $\int_A f d\sigma$ neither depends on the choice of the parametrizations $\Phi : U \to O$ in definition 2.1 nor on the choice of the partition of unity.*

*Proof.* Due to condition (i) on the partition of unity the sum in the definition of $\int_A f d\sigma$ is finite. For two covers of $K$ by sets of the form $\Phi[U]$ and $\Psi[V]$ as in Definition 2.1 with corresponding partitions of unity, the intersections of two such sets (one from each cover) and the products of two functions (one from each partition of unity) build another cover of $K$ with a corresponding partition of unity. The linearity of the integral and condition (ii) on the partition of unities together ensure that it suffices to consider the subcase that $K$ is contained the images $\Phi[U]$ and $\Psi[V]$ of two continuously differentiable homeomorphisms as described in Definition 2.1. The restrictions of $\Phi$ to

$\Phi^{-1}[\Phi[U]\cap\Psi[V]]$ and of $\Psi$ to $\Psi^{-1}[\Phi[U]\cap\Psi[V]]$ are both homeomorphisms onto the open subset $\Phi[U]\cap\Psi[V]$ of $A$. The composition of the second with the inverse of the first yields a homeomorphism $\Upsilon:\Psi^{-1}[\Phi[U]\cap\Psi[V]]\to\Phi^{-1}[\Phi[U]\cap\Psi[V]]$, such that $\Psi(x)=\Phi(\Upsilon(x))$ holds for all $x\in\Psi^{-1}[\Phi[U]\cap\Psi[V]]$.

Now we claim that $\Upsilon$ is continuously differentiable. For any $x\in\Psi^{-1}[\Phi[U]\cap\Psi[V]]$ there exists a $k$-dimensional linear subspace of $\mathbb{R}^n$ such that the composition $P\circ\Phi'(\Upsilon(x))$ with the orthogonal projection $P$ onto this subspace is bijective onto this subspace. By the inverse function theorem an open neighbourhood of $\Upsilon(x)$ is mapped by $P\circ\Phi$ homeomorphically onto an open neighbourhood of $P(\Psi(x))$. The inverse mapping is together with $P\circ\Phi$ continuously differentiable. The map $\Upsilon$ is on this neighbourhood of $x$ equal to the composition of $P\circ\Psi$ with the inverse map of $P\circ\Phi$, since $P\circ\Psi$ and $P\circ\Phi\circ\Upsilon$ coincide there. This shows that $\Upsilon$ is on this neighbourhood continuously differentiable. Because this is true for all $x\in\Psi^{-1}[\Phi[U]\cap\Psi[V]]$ the claim follows. We conclude

$$\int\limits_{\Psi^{-1}[\Phi[U]\cap\Psi[V]]} f\circ\Psi\sqrt{\det((\Psi')^T\Psi')}d\sigma=\int\limits_{\Psi^{-1}[\Phi[U]\cap\Psi[V]]} f\circ\Phi\circ\Upsilon\sqrt{\det(((\Phi\circ\Upsilon)')^T(\Phi\circ\Upsilon)'}d\sigma=$$
$$=\int\limits_{\Psi^{-1}[\Phi[U]\cap\Psi[V]]}\left(f\circ\Phi\sqrt{\det((\Phi')^T\Phi')}\right)\circ\Upsilon|\det\Upsilon'|d\sigma=\int\limits_{\Phi^{-1}[\Phi[U]\cap\Psi[V]]} f\circ\Phi\sqrt{\det((\Phi')^T\Phi')}d\sigma.$$

In the last step we applied the transformation formula of Jacobi. **q.e.d.**

In the divergence theorem we consider open subsets $\Omega\subset\mathbb{R}^n$ whose boundary are $n-1$-dimensional submanifolds. After the Definition 2.1 we explained how the implicit function theorem applies to these submanifolds. Since any $n-1$-dimensional subspace is the image of $\mathbb{R}^{n-1}\simeq\mathbb{R}^{n-1}\times\{0\}\subset\mathbb{R}^n$ with respect to some linear rotation $\mathsf{O}$ of $\mathbb{R}^n$ these arguments show that the homoemorphisms in Definition 2.1 are of the form

$$\Phi:U\subset\mathbb{R}^{n-1}\to\mathbb{R}^n,\quad y\mapsto\mathsf{O}(x,g(x))\text{ for some }C^1\text{-function }g:U\to(a,b)\subset\mathbb{R},$$
$$\text{with }\det((\Phi'(x))^T\Phi'(x))=\det\left(\begin{pmatrix}\mathbb{1} & \nabla g(x)\end{pmatrix}\mathsf{O}^T\mathsf{O}\begin{pmatrix}\mathbb{1}\\ \nabla^T g(x)\end{pmatrix}\right)=1+(\nabla g(x))^2.$$

The plane tangent to $\partial\Omega$ in $\mathsf{O}(x,g(x))$ is image of the kernel of the derivative of $U\times(a,b)\to\mathbb{R}$, $(x,z)\mapsto z-g(x)$ with respect to $\mathsf{O}$. So

$$N(x,g(x))=\frac{\mathsf{O}^T(-\nabla^T g(x),1)}{\sqrt{1+(\nabla g(x))^2}}$$

is up to sign unique normalised vector orthogonal the tangent plane which is called normal. Since the last component is positive this normal points outwards of $\mathsf{O}[\{(y,z)\in U\times(a,b)\mid z<g(y)\}]$, which for an appropriate $\mathsf{O}$ is equal to $\Omega\cap\mathsf{O}[U\times(a,b)]$.

**Theorem 2.5.** *(Divergence Theorem)* *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Let $f : \bar{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on $\Omega$ such that $\nabla f$ continuously extends to $\partial\Omega$. Then we have*

$$\int_{\Omega} \nabla \cdot f \,\mathrm{d}\mu = \int_{\partial\Omega} f \cdot N \,\mathrm{d}\sigma$$

*Here $N$ is the outward-pointing normal and $N\,\mathrm{d}\sigma$ the corresponding measure on $\partial\Omega$.*

*Proof.* We cover $\bar{\Omega}$ by open subsets $U \times (a,b) \subset \mathbb{R}^n$ as described above and $\Omega$. We choose an compatible partition of unity. Due to the compactness of $\bar{\Omega}$ and due to condition (iii) on the partition of unity this partition has only finitely many members. By linearity it suffices to show the statement for any term individually.

First we consider a continuously differentiable function $f : \Omega \to \mathbb{R}^n$ with compact support in $\Omega$. By setting it zero outside of $\Omega$ it extends continuously differentiable to $\mathbb{R}^n$. Choose a Cartesian product of finite intervals which contains $\Omega$. The continued function vanishes on the boundary of this box. By Fubini we may integrate the $i$-th term of $\nabla \cdot f = \partial_1 f_1 + \ldots, +\partial_n f_n$ first $dx_i$. Due to the fundamental theorem of calculus this integral is the difference of the values of $f$ at two boundary points and vanishes. This shows that in this case both sides of the divergence theorem vanish.

Now we consider a function $f$ on $\Omega \cap O[U \times (a,b)] = \{O(x,z) \mid z \le g(x)\}$ which vanishes outside a compact subset. We replace $x$ by $Ox$, $x \mapsto f(x)$ by $x \mapsto O^T f(Ox)$, $x \mapsto N(x)$ by $x \mapsto O^T N(Ox)$ and $\Omega \ni Ox \Leftrightarrow O^{-1}[\Omega] \ni x$. Consequently $O^T O = \mathbb{1}$, $\det O = \pm 1$ and $\nabla \cdot O^T f(Ox) = \mathrm{trace}(O^T \circ f \circ O)'(x) = \mathrm{trace}(OO^T f'(Ox) = \nabla \cdot f(Ox)$. By Jacobi's transformation formula both sides of the divergence theorem do not change, and we may omit $O$. Again we extend $f$ to $\mathbb{R}^{d-1} \times (a,b)$ by setting it zero outside of $U \times (a,b)$. For any $(x,y) \in \mathbb{R}^{d-1} \times (a,b)$, $1 \le i < n$ we have

$$\int_a^y \int_{-\infty}^0 \frac{\partial}{\partial x_i} f(x + te_i, z) dt dz = \int_a^y f(x,z) dz$$

By Fubini this function is continuously differentiable with

$$\frac{\partial}{\partial x_i} \int_a^y f(x,z) dz = \int_a^y \frac{\partial f(x,z)}{\partial x_i} dz \text{ for } 1 \le i < d, \quad \frac{\partial}{\partial y} \int_a^y f(x,z) dz = f(x,y).$$

The following function vanishes outside a compact subset of $U$:

$$x \mapsto \int_a^{g(x)} f(x,z) dz \quad \text{with} \quad \frac{\partial}{\partial x_i} \int_a^{g(x)} f(x,z) dz = \int_a^{g(x)} \frac{\partial f(x,z)}{\partial x_i} dz + \frac{\partial g(x)}{\partial x_i} f(x, g(x)).$$

So the arguments of the first case apply and show that the integral over $U$ on the right hand side vanishes. This proves for $1 \leq i < n$ the divergence theorem:

$$\int_U \int_a^{g(x)} \frac{\partial f_i(x,z)}{\partial x_i} dz d^{d-1}x = -\int_U f_i(x,g(x)) \frac{\partial g(x)}{\partial x_i} d^{d-1}x = \int_U f_i(x,g(x)) N_i(x,g(x)) d\sigma.$$

The fundamental theorem of calculus finishes the proof, since $f$ vanishes on $U \times \{a\}$:

$$\int_U \int_a^{g(x)} \frac{\partial f_n(x,z)}{\partial x_n} dz d^{d-1}x = \int_U f_n(x,g(x)) d^{d-1}x = \int_U f_n(x,g(x)) N_n(x,g(x)) d\sigma. \textbf{ q.e.d.}$$

The divergence theorem implies for all $i = 1, \ldots, n$

$$\int_\Omega \partial_i f \, \mathrm{d}\mu = \int_{\partial\Omega} f N_i \, \mathrm{d}\sigma$$

For two functions $f$ and $g$ whose product vanishes on the boundary $\partial\Omega$ and satisfies the corresponding assumptions of the divergence theorem we obtain by the Leibniz rule

$$\int_\Omega f \partial_i g \, \mathrm{d}\mu = -\int_\Omega g \partial_i f \, \mathrm{d}\sigma \quad \text{for all } i = 1, \ldots, n.$$

This is called integration by parts. Inductively we get for any multiindex $\gamma$

$$\int_\Omega f \partial^\gamma g \, \mathrm{d}\mu = (-1)^{|\gamma|} \int_\Omega g \partial^\gamma f \, \mathrm{d}\sigma.$$

As a second application of the divergence theorem we present conserved quantities for any continuously differentiable function $f : \mathbb{R} \to \mathbb{R}^n$ and any solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the general scalar conservation law introduced in the last chapter

$$\dot{u}(x,t) + \nabla f(u(x,t)) = \dot{u}(x,t) + f'(u(x,t)) \cdot \nabla u(x,t) = 0.$$

For open $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ being a $n-1$-dimensional submanifolds of $\mathbb{R}^n$ we obtain

$$\frac{d}{dt} \int_\Omega u(x,t) d^n x = \int_\Omega \dot{u}(x,t) d^n x = -\int_\Omega \nabla f(u(x,t)) d^n x = -\int_{\partial\Omega} f(u(x,t)) \cdot N(x) d\sigma(x).$$

This is the meaning of a conservation law: the change of the integral of $u(\cdot,t)$ over $\Omega \subset \mathbb{R}^n$ is equal to the integral of the flux $-f(u(\cdot,t)) \cdot N$ through the boundary $\partial\Omega$.

## 2.2 Classification of Second order PDEs

For PDEs of order greater then one, there does not exists a general theory. We shall present in Section 2.3 an example of a PDE with smooth coefficients, which has in a neighbourhood of some point no solutions at all. Over the time there have been discovered different methods to solve several PDEs, in particular those PDEs which show up in physics. Afterwards these methods were extended to larger and larger classes of PDEs. It turned out that the successful methods of solving PDEs differ from each other substantially. As a result there does not exists one unified theory of PDEs, but there exist several islands of well understood families of PDEs inside the large set of all PDEs. It was Jacobi who formulated in his lectures on Dynamics in the years 1842-43 the following general recipe:

"The main obstacle for the integration of a given differential equations lies in the definition of adapted variables, for which there is no general rule. For this reason we should reverse the direction of our investigation and should endeavour to find, for a successful substitution, other problems which might be solved by the same."

The strategy is to determine for any successful method all PDEs which can be solved by this method. We already presented for the first order PDEs a more or less general method. Now we investigate the second order PDEs. In this lecture we consider only second order linear PDEs. A general second order linear PDE has the following form

$$Lu(x) = \sum\nolimits_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u + \sum\nolimits_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x) = 0.$$

By Schwarz's Theorem for twice differentiable $u$ this expression does not change if we replace $a_{ij}$ by $\frac{1}{2}(a_{ij} + a_{ji})$. So we may assume that $a_{ij}$ is symmetric and diagonalizable.

**Elliptic PDEs.** If the matrix $a_{ij}$ is the unity matrix and $b = 0 = c$, then this is the

**Laplace equation.** $$\triangle u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Solutions of the Laplace equation are called harmonic functions. In Chapter 3 we present several tools which establish many properties of these harmonic functions. It turns out that many properties of the harmonic functions also apply to general solutions of $Lu = 0$, if the matrix $a_{ij}$ is positive (or negative) definite. These are the main examples of the so called elliptic PDEs. There has been done a lot of work to extend these tools to larger and larger classes of elliptic PDEs. One of the results is that the influence of the higher order derivatives on the properties of solutions is much more important than the influence of the lower order derivatives. An important tool are so called a priori estimates. Such estimates show that the lower order derivatives can be estimated in terms of the second order derivatives. We offer another lecture which presents many of these tools for such elliptic second order PDEs.

Beside the linear elliptic PDEs there are also non-linear PDEs, to which these methods of elliptic PDEs apply. An important example whose investigation played a major role in the development of the elliptic theory is the

**Minimal surface equation.** $\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0, \quad u : \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^n$ open.

The graphs of solutions describe so called minimal surfaces. The area of such hypersurfaces in $\mathbb{R}^{n+1}$ does not change with respect to infinitesimal variations. Soap bubbles are examples of such minimal surfaces. The boundary value problem of the minimal surface equation is called Plateau's problem. For the first proof of the existence of solutions of this Plateau problem in the 1930s, Jesse Douglas received the first Field's Medal. In this non-linear second order PDE the coefficients of the second derivatives also depend on the solution. A lot of work has been done to extend the tools of elliptic theory to elliptic PDEs whose coefficients belong to larger and larger functions spaces. This development induced the introduction of many new function spaces. In Section 2.4 we shall introduce the so called space of distributions. Many of the more advanced functions spaces are build on the base of these spaces.

**Parabolic PDEs.** For these linear PDEs the matrix $a_{ij}$ considered as a symmetric bilinear form is only semi-definite and they belong to the boundary of the class of elliptic PDEs. Most of the methods of elliptic PDEs have an extension to this limiting case. So these limiting cases together with the class of elliptic PDEs form some extended class of elliptic PDEs. Of particular importance is the subclass of linear PDEs with semi-definite matrices $a_{ij}$ which have a one-dimensional kernel. Since symmetric matrices are always diagonalizable this means that one eigenvalue of $a_{ij}$ vanishes and all other eigenvalues have the same sign. In spite of the deep relationship to the elliptic PDEs these equations have their own label: parabolic PDEs. The simplest example is the

**Heat equation.** $\dot{u} - \triangle u = 0.$

These parabolic PDEs describe diffusion processes. These are processes which level inhomogeneities of some quantity by some flow along the negative gradient of the quantity. A typical example for this quantity is the temperature from which the name for the heat equation originates. Many stochastic processes have this property. So the theory of parabolic PDEs has a deep relationship to the theory of stochastic processes. In this lecture we present in Chapter 4 this simplest example of linear parabolic PDE. We shall see how the tools for the Laplace equation can be applied in modified form to this heat equation. In case of the parabolic PDEs there too exists a non-linear example from the geometric analysis, whose investigation played a major role for the development of the elliptic theory (the tensor fields $g$ and $R$ are defined below):

**Ricci Flow.** $\dot{g}_{ij} = -2R_{ij}.$

This PDE describes a diffusion-like process on Riemannian manifolds. It levels the inhomogeneities of the metric, namely the Riemannian metric $g$. In the long run the corresponding Riemannian manifolds converge to metric spaces with large symmetry groups. Richard Hamilton proposed (in the 1970s) a program that aims to prove the geometrization conjecture of Thurston with the help of these PDEs. It states that every three-dimensional manifold can be split into parts, which can be endowed with an Riemannian metric such that the isometry group acts transitively. This conjecture implies the Poincare conjecture, which states that every simply connected compact manifold is the 3-sphere. Hamilton tries to control the long time limit of the Ricci flow on a general 3-dimensional Riemannian manifold. In 2003 the Russian mathematician Grisha Perelman published on the internet three articles which overcome the last obstacle of this program. This lead to the first proof of one of the Millennium Problems of the American Mathematical Society and was a great success of geometric analysis.

**Hyperbolic PDEs.** Besides the elliptic PDEs (including the limiting cases) the second important class of linear PDEs are called hyperbolic. In this case the matrix $a_{ij}$ has one eigenvalue of opposite sign than all other eigenvalues. The simplest example is the

**Wave equation.** $$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0.$$

In Chapter 5 we present the methods how to solve this equation. We shall see that it describes the propagation of waves with constant finite speed. The solutions of general hyperbolic equations are similar to the solutions of this case, and many tools can be generalised to all hyperbolic PDEs. The investigation of these PDEs depend on the understanding of all trajectories, which propagate by the given speed. It was motivated by theory of the electrodynamic fields, whose main system of PDEs are the

**Maxwell equations.** $$\begin{aligned} \dot{E} - \nabla \times B &= -4\pi j & \dot{B} + \nabla \times E &= 0 \\ \nabla \cdot E &= 4\pi\rho & \nabla \cdot B &= 0. \end{aligned}$$

In this theory there is given a distribution of charges $\rho$ and currents $j$ on space time $\mathbb{R}\times\mathbb{R}^3$. The unknown functions are the electric magnetic fields $E$ and $B$, which describe the electrodynamic forces induced by the given distributions of charges and currents $\rho$ and $j$. The conservation of charge is formulated in the same way as in the scalar conservation law. So the change of the total charge contained in a spatial domain is described by the flux of the current through the boundary of the domain. By the divergence theorem this means that distributions of charge $\rho$ and currents $j$ obey

$$\dot{\rho} + \nabla \cdot j = 0.$$

Again there exists a non-linear version which stimulated the development of the theory:

**Einsteins field equations of general relativity.** $$R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij}.$$

Here for a given distribution of masses the energy stress tensor and the space time metric $g_{ij}$ are the unknown functions. This metric is a symmetric bilinear form with one positive and three negative eigenvalues on the tangent space of space time. The corresponding Ricci curvature is denoted by $R_{ij}$ and the scalar curvature by $R$:

$$\Gamma^k_{ij} := \frac{1}{2}\sum\nolimits_{l=0}^{3} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right), \quad \left(g^{ij}\right) := \left(g_{ij}\right)^{-1} \text{ inverse metric}$$

$$R_{ij} := \sum\nolimits_{k=0}^{3} g^{kl}\left(\frac{\partial \Gamma^k_{ij}}{\partial x^k}-\frac{\partial \Gamma^k_{ik}}{\partial x^j}+\sum\nolimits_{l=0}^{3}\left(\Gamma^k_{lk}\Gamma^l_{ij}-\Gamma^k_{lj}\Gamma^l_{ik}\right)\right), \quad R := \sum\nolimits_{i,j=0}^{3} g^{ij}R_{ij}.$$

**Integrable Systems with Lax operators.** Finally I want to mention a smaller class of PDEs, which are the main objects of my research. They are non-linear PDEs which describe an evolution with respect to time which is very stable. This means that the solutions have in a specific sense a maximal number of conserved quantities. The theory of integrable systems belongs to the field of Hamiltonian mechanics, which originated from Newtons description of the motion of the planets. The Scottish Lord John Scott Russell got very excited in 1934 about the observation of an solitary wave in a Scottish channel and published a “Report on Waves”. This report was quite influential. The two Dutch mathematicians Korteweg and De Vries translated his observation into a PDE describing the profile of water waves travelling along the channel:

**Korteweg-de-Vries equation.** $$4\dot{u} - 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0.$$

First by numerical experiments in the 1950s with the first computers and latter in the 1970s by mathematical theory, the solutions of this PDE were shown to have exactly the properties which made Lord Russell so exited: they describe waves which propagate through each other without changing their shape. This lead to the discovery of an hidden relation of the theory of integrable systems with the theory of Riemann surfaces, which is another field with a long history. A major step towards the discovery of this relation was the observation of Peter Lax that this equation can be written as

$$\dot{L} = [A, L] \qquad \text{with} \qquad L := \frac{\partial^2}{\partial x^2} + u \qquad A := \frac{\partial^3}{\partial x^3} + \frac{3u}{2}\frac{\partial}{\partial x} + \frac{3}{4}\frac{\partial u}{\partial x}.$$

## 2.3 Existence of Solutions

In order to demonstrate the difference between ODEs and PDEs we shall present an example of a partial differential equation with smooth coefficients without solutions. This example is a simplification by Nirenberg of an example of H. Levy.

For a given complex-valued function $f$ on a domain $(x, y) \in \mathbb{R}^2$ we look for a complex valued solution $u$ on the same domain of the following differential equations:

$$\frac{\partial u}{\partial x} + \imath x\frac{\partial u}{\partial y} = f(x, y).$$

We impose the following two conditions on the smooth function $f$:

**(i)** $f(-x,y) = f(x,y)$

**(ii)** there exists a sequence of positive numbers $\varrho_n \downarrow 0$ converging to zero, such that $f$ vanishes on a neighbourhood of the circles $\partial B(0,\varrho_n)$ in contrast to non-vanishing integrals $\int_{B(0,\varrho_n)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \neq 0$.

If $h : \mathbb{R} \to [0,\infty)$ is a smooth periodic function vanishing on an interval but not on $\mathbb{R}$, then $f(x) := \exp(-1/|x|)h(1/|x|)$ has these two properties.

Now we shall prove by contradiction that there exists no continuously differentiable solution $u$ in a neighbourhood of $(0,0) \in \mathbb{R}^2$.

**Step 1:** If the function $u(x,y)$ is a solution, then due to (i) $-u(-x,y)$ is also a solution. Hence we may replace $u(x,y)$ by $\frac{1}{2}(u(x,y)-u(-x,y))$ and assume $u(-x,y) = -u(x,y)$.

**Step 2:** We claim that every solution $u$ vanishes on the circles $\partial B(0,\varrho_n)$. In fact, we transform small annuli $A$ onto domains $\tilde{A}$ in $\mathbb{R}^2$:

$$A \to \tilde{A}, \qquad\qquad (x,y) \mapsto \begin{cases} (x^2/2, y) & \text{for } x \geq 0 \\ (-x^2/2, y) & \text{for } x < 0. \end{cases}$$

These transformations are homeomorphisms from $A$ onto $\tilde{A}$. On the sub domains $\tilde{A}_+ = \left\{(s,y) \in \tilde{A} \mid s > 0\right\}$ the function $\tilde{u}(s,y) = u(x^2/2, y)$ is holomorphic:

$$2\bar{\partial}\tilde{u} = \frac{\partial\tilde{u}(s,y)}{\partial s} + \imath\frac{\tilde{u}(s,y)}{\partial y} = \frac{dx}{ds}\frac{\partial u(x,y)}{\partial x} + \imath\frac{\partial u(x,y)}{\partial y} = \frac{1}{x}\left(\frac{\partial u(x,y)}{\partial x} + \imath x\frac{\partial u(x,y)}{\partial y}\right) = 0.$$

Due to step 1. the function $\tilde{u}$ vanishes on the line $s = 0$. This implies that $\tilde{u}$ together with the Taylor series vanishes identically on $\tilde{A}_+$ and due to step 1 on $\tilde{A}$.

**Step 3:** The Divergence Theorem yields a contradiction to the assumption (ii):

$$\begin{aligned} \int_{B(0,\varrho_n)} f\,\mathrm{d}x\,\mathrm{d}y &= \int_{B(0,\varrho_n)} \left(\frac{\partial u}{\partial x} + \imath x\frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{B(0,\varrho_n)} \nabla\cdot\begin{pmatrix} u \\ \imath x u\end{pmatrix}\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\partial B(0,\varrho_n)} \begin{pmatrix} u \\ \imath x u\end{pmatrix}\cdot N(x,y)\,\mathrm{d}\sigma(x,y) = 0, \end{aligned}$$

Therefore the given differential equation has no continuously differentiable solution.

This example also implies that the following partial differential equation with smooth real coefficients has no four times differentiable real solution:

$$\left(\frac{\partial}{\partial x} + \imath x\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \imath x\frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x\frac{\partial}{\partial y}\right)\tilde{u} = \left(\left(\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial y^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right)\tilde{u} = f.$$

Here $f$ is a real smooth function with the properties (i) and (ii). For any real solution $\tilde{u}$, the following complex function would be a solution of the complex PDE:

$$u = \left(\frac{\partial}{\partial x} - \imath x\frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x\frac{\partial}{\partial y}\right)\tilde{u}.$$

## 2.4 Distributions

Our investigation of partial differential equations aims to find as many solutions as possible and, in addition, conditions which uniquely determines the solutions. The existence and uniqueness of solutions depends on the notion of solution we use. Clearly all partial derivatives of a solution which occur in the partial differential equation have to exist. We might use several possible generalisations of derivatives in order to define such solutions. In this section we introduce generalised functions which can always be differentiated infinitely many times. For this achievement we have to pay a price: these generalised functions cannot be multiplied with each other. Linear partial differential equations extend to well defined equations on such generalised functions. Generalised functions solving the linear partial differential equations are called *weak solutions* or solutions in the sense of distributions. There exist other notions of weak solutions which also apply to non-linear partial differential equations. An example of more general functions with finitely many derivatives are so called Sobolev spaces. These Sobolev spaces are introduced in more advanced lectures on partial differential equations. The elements of the Sobolev spaces are distributions. So the distributions which we introduce now are the most general functions with derivatives.

The support $\operatorname{supp} f$ of a function $f$ is the closure of $\{x \mid f(x) \neq 0\}$. On an open set $\Omega \subseteq \mathbb{R}^n$ let $C_0^\infty(\Omega)$ denote the algebra of smooth functions whose support is a compact subset of $\Omega$. We say such functions have compact support in $\Omega$ and we use the notation $\operatorname{supp} f \Subset \Omega$. Each $f \in L^1(\Omega)$ defines a linear map

$$F : C_0^\infty(\Omega) \to \mathbb{R}, \qquad\qquad \phi \mapsto \int_\Omega f\phi \, \mathrm{d}\mu.$$

Generalised functions on $\Omega$ are such linear forms $F$ on $C_0^\infty(\Omega)$. When considering the elements of $C_0^\infty(\Omega)$ as the domain of the linear form $F$, we call them test functions. If $f$ has a derivative, then by integration by parts we obtain

$$\int_\Omega \partial_i f \phi \, \mathrm{d}^\mathrm{n} x = -\int_\Omega f \partial_i \phi \, \mathrm{d}^\mathrm{n} x.$$

For any linear form $F$ on $C_0^\infty(\Omega)$ we define the partial derivatives as

$$\partial_i F : C_0^\infty(\Omega) \to \mathbb{R}, \qquad\qquad \phi \mapsto -F(\partial_i \phi).$$

Therefore such generalised functions have infinitely many derivatives. The vector space of test functions is infinite dimensional. In order to avoid abstract nonsense we should impose some continuity on the linear forms $F$. The derivative of a continuous functional $F$ is again continuous, if the derivatives are linear continuous maps on the space $C_0^\infty(\Omega)$.

For $f \in L^1(\Omega))$ the corresponding linear functionals $F$ are continuous with respect to the supremum norm on compact subsets of $\Omega$. We define for any compact subset $K \subset \Omega$ and every multiindex $\alpha$ the following semi-norm:

$$\|\cdot\|_{K,\alpha} : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \|\phi\|_{K,\alpha} := \sup_{x \in K} |\partial^\alpha \phi(x)| .$$

**Definition 2.6.** *On an open subset $\Omega \subseteq \mathbb{R}^n$ the space of distributions $\mathcal{D}'(\Omega)$ is defined as the vector space space of all linear maps $F : C_0^\infty(\Omega) \to \mathbb{R}$ which are continuous with respect to the semi norms $\|\cdot\|_{K,\alpha}$; i.e. for each compact $K \subset \Omega$ there exist finitely many multi indices $\alpha_1, \ldots, \alpha_M$ and constants $C_1 > 0, \ldots, C_M > 0$ such that the following inequality holds for all test functions $\phi \in C_0^\infty(\Omega)$ with compact support in $K$:*

$$|F(\phi)| \leq C_1 \|\phi\|_{K,\alpha_1} + \ldots + C_M \|\phi\|_{K,\alpha_M}.$$

The support $\operatorname{supp} F$ of a distribution $F \in \mathcal{D}'(\Omega)$ is defined as the complement of the union of all open subsets $O \subset \Omega$, such that $F$ vanishes on all test functions $\phi$ whose support is contained in $O$. We denote the Euclidean length of $x \in \mathbb{R}^n$ by $|x|$. The test function

$$\phi(x) := \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

has support $\overline{B(0,1)}$ and is non-negative. By rescaling of $x$ and $\phi$ and by translations we obtain for each ball $B(x_0, \epsilon)$ a non-negative test function $\phi_{B(x_0,\epsilon)}$ with $\operatorname{supp} \phi_{B(x_0,\epsilon)} = \overline{B(x_0, \epsilon)}$ with $\int \phi_{B(x_0,\epsilon)} \, \mathrm{d}\mu = 1$. In particular, there exists for every open subset $O \subset \Omega$ a non-negative test function with support contained $O$. Every continuous function $f$ on $\Omega$ which does not vanish identically takes values in $(-\infty, \epsilon)$ or $(\epsilon, \infty)$ for some $\epsilon > 0$ on some properly chosen open ball. Therefore there exists $\phi \in C_0^\infty(\Omega)$ with $\int_\Omega f\phi \, \mathrm{d}\mu \neq 0$.

The following distribution does not correspond to a usual function:

$$\delta : C_0^\infty(\Omega) \to \mathbb{R} \qquad \phi \mapsto \phi(0).$$

A corresponding function would vanish on $\mathbb{R}^n \setminus \{0\}$ and would have a total integral one. Since $\{0\}$ has measure zero such a function does not exist. This generalised function is called Dirac's $\delta$-function. We shall see that the family of distributions which corresponds to the functions $\phi_{B(0,\epsilon)}$ converge in the limit $\epsilon \downarrow 0$ to this distribution. The support of all derivatives of this distribution contains only the point $0 \in \Omega$.

The product of a distribution with a function $g \in C^\infty(\Omega)$ is defined as

$$gF : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(g\phi).$$

This product makes the embedding $C^\infty(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ to a homomorphism of modules over the algebra $C^\infty(\Omega)$. However, even the product of a distribution with a continuous non-smooth functions is not defined. The convolution is another product on $C_0^\infty(\mathbb{R}^n)$:

$$(g * f)(x) := \int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^\mathrm{n}y = \int_{\mathbb{R}^n} g(y)f(x-y)\,\mathrm{d}^\mathrm{n}y.$$

This product is commutative and associative (Exercise). In order to extend this product to a product between a smooth function and a distribution we calculate:

$$\int\limits_{\mathbb{R}^n} \phi(g*f)\,\mathrm{d}^\mathrm{n}x = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \phi(x)g(x-y)f(y)\,\mathrm{d}^\mathrm{n}y\,\mathrm{d}^\mathrm{n}x = \int\limits_{\mathbb{R}^n} \phi(x)\int\limits_{\mathbb{R}^n} (\mathsf{T}_x\mathsf{P}g)(y)f(y)\,\mathrm{d}^\mathrm{n}y\,\mathrm{d}^\mathrm{n}x$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(x)g(x-y)f(y)\,\mathrm{d}^\mathrm{n}x\,\mathrm{d}^\mathrm{n}y = \int_{\mathbb{R}^n} (\phi * \mathsf{P}g)f\,\mathrm{d}^\mathrm{n}y$$
$$\text{with } \mathsf{T}_x : C_0^\infty(\Omega) \to C_0^\infty(x+\Omega), \quad \phi \mapsto \mathsf{T}_x\phi, \text{ and } (\mathsf{T}_x\phi)(y) = \phi(y-x)$$
$$\text{and } \mathsf{P} : C_0^\infty(\Omega) \to C_0^\infty(-\Omega), \quad \phi \mapsto \mathsf{P}\phi, \text{ with } (\mathsf{P}\phi)(y) = \phi(-y).$$

Therefore we define for $g \in C_0^\infty(\mathbb{R}^n)$ and $F \in \mathcal{D}'(\mathbb{R}^n)$

$$g * F : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto F(\mathsf{T}_x\mathsf{P}g) \text{ or equivalently } g * F : C_0^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad \phi \mapsto F(\phi * \mathsf{P}g).$$

**Lemma 2.7.** *The convolutions of a distribution $F$ with a test function $g \in C_0^\infty(\Omega)$ is a distribution which corresponds to a smooth function. The support of this distribution is contained in the point-wise sum of the supports of the functions and the distribution.*

*Proof.* For each $F \in \mathcal{D}'(\Omega)$ the linearity and continuity imply

$$g * F(\phi) = F(\mathsf{P}g * \phi) = \int_{\mathbb{R}^n} F(\mathsf{T}_x\mathsf{P}g)\phi(x)\,\mathrm{d}^\mathrm{n}x.$$

Due to the continuity of $F$ with respect to the semi norms $\|\cdot\|_{K,0}$ the functions $x \mapsto F(\mathsf{T}_x\mathsf{P}g)$ are continuous. Furthermore, these functions are smooth since $\frac{\mathsf{T}_{y+\epsilon h} - \mathsf{T}_y}{\epsilon}\phi = \mathsf{T}_y\frac{\mathsf{T}_{\epsilon h} - \mathbb{1}}{\epsilon}\phi$ converges for all $\phi \in C^\infty(\mathbb{R}^n)$ in the limit $\epsilon \to 0$ on the space $C^\infty(\mathbb{R}^n)$ with respect to the topology induced by the semi norms $\|\cdot\|_{K,\alpha}$ to $\mathsf{T}_y\left(\sum_{i=1}^n h_i\partial_i\phi\right)$.

If $x \mapsto F(T_xPg)$ does not vanish on a neighbourhood of a point $x$, then $g(x-y) \neq 0$ for an element $y \in \operatorname{supp} F$. Hence $x = y + (x-y)$ is the sum of an element of $\operatorname{supp} F$ with an element of $\operatorname{supp} g$. **q.e.d.**

This Lemma implies that even the convolution of a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ with a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ with compact support $\operatorname{supp} G$ is a well defined distribution:

$$F * G : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(\phi * \mathsf{P}G) \text{ with } \qquad \mathsf{P}G(\phi) := G(\mathsf{P}\phi).$$

In particular, the $\delta$-distribution is the neutral element of the product defined by the convolution, i.e. the convolution with the $\delta$-distribution maps each distribution to itself. We introduced a family of test functions $(\phi_{B(0,\epsilon)})_{\epsilon>0}$ which converge in the limit $\epsilon \downarrow 0$ to the $\delta$-distribution. For each $F \in \mathcal{D}'(\Omega)$ the family $f_\epsilon := \phi_{B(0,\epsilon)} * F$ converge in the limit $\epsilon \downarrow 0$ in a specific sense to $F$. Such a family $(\lambda_\epsilon)_{\epsilon>0}$ in $C_0^\infty(\mathbb{R}^n)$ with

$$\lambda_\epsilon \geq 0 \qquad \operatorname{supp}\lambda_\epsilon \subset \overline{B(0,\epsilon)} \qquad \int_{\mathbb{R}^n} \lambda_\epsilon \,\mathrm{d}^{\mathrm{n}}x = 1,$$

which converges in the limit $\epsilon \downarrow 0$ to the $\delta$-distribution, is called mollifier. Now we can show that all distributions can be approximated by smooth functions.

**Lemma 2.8.** *Let $f \in C(\Omega)$ and $(\lambda_\epsilon)_{\epsilon>0}$ be a mollifier. In the limit $\epsilon \downarrow 0$ the family of smooth functions $\lambda_\epsilon * f$ converges uniformly on compact subsets of $\Omega$ to $f$. For smooth functions the same holds for all derivatives of $f$.*

*Proof.* On compact sets continuous functions are uniformly continuous. Any $x \in \Omega$ is contained in an open ball $B(x,\epsilon) \subset \Omega$. For sufficiently small $\epsilon$ we have

$$|(\lambda_\epsilon * f)(x) - f(x)| = \left| \int_{B(x,\epsilon)} \lambda_\epsilon(x-y)(f(y)-f(x))\,\mathrm{d}^{\mathrm{n}}y \right| \leq \sup_{y \in B(x,\epsilon)} |f(y)-f(x)|.$$

This shows the uniform convergence $\lim_{\epsilon\downarrow 0} \lambda_\epsilon * f = f$. By definition of the convolution two smooth functions $f$ and $g$ obey

$$\partial_i(f * g) = f * \partial_i g = \partial_i f * g.$$

Hence for $f \in C^\infty(\Omega)$ these arguments carry over to all partial derivatives of $f$. **q.e.d.**

As previously mentioned, any $f \in L^1_{\mathrm{loc}}(\Omega)$ defines in a canonical way a distribution

$$F_f : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \int_\Omega f\phi \,\mathrm{d}\mu.$$

For $\phi \in C_0^\infty(\Omega)$ with support in a compact subset $K \subset \Omega$ and $f \in L^1(\Omega)$ we have

$$|F_f(\phi)| \leq \sup_{x \in K} |\phi(x)| \|f\|_{L^1(\Omega)}.$$

For $f \in L^1_{\mathrm{loc}}(\Omega)$ every compact subset $K \subset \Omega$ has a cover of open subsets $O_1, \ldots, O_L$ of $\Omega$ such that $f|_{O_l} \in L^1(O_l)$ for $l = 1, \ldots, L$. This shows $F_f \in \mathcal{D}'(\Omega)$:

$$|F_f(\phi)| \leq \sup_{x \in K} |\phi(x)| \sum_{l=1}^{L} \|f|_{O_l}\|_{L^1(O_l)} \quad \text{for} \quad \operatorname{supp}\phi \subset K.$$

**Lemma 2.9.** *(Fundamental Lemma of the Calculus of Variations) If $f \in L^1_{\mathrm{loc}}(\Omega)$ obeys $F_f(\phi) \geq 0$ for all non-negative test functions $\phi \in C_0^\infty(\Omega)$, then $f$ is non-negative almost everywhere. In particular the map $L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto F_f$ is injective.*

*Proof.* It suffices to prove the local statement for $f \in L^1(\Omega)$. We extend $f$ to $\mathbb{R}^n$ by setting $f$ on $\mathbb{R}^n \setminus \Omega$ equal to zero. The extended function is also denoted by $f$ and belongs to $f \in L^1(\mathbb{R}^n)$. For a mollifier $(\lambda_\epsilon)_{\epsilon>0}$ we have

$$\begin{aligned}\|\lambda_\epsilon * f - f\|_1 &= \int_{\mathbb{R}^n} \left| \int_{B(0,\epsilon)} \lambda_\epsilon(y) f(x-y)\, \mathrm{d}^{\mathrm{n}} y - f(x) \right| \mathrm{d}^{\mathrm{n}} x \leq \\ &\leq \int_{B(0,\epsilon)} \int_{\mathbb{R}^n} \lambda_\epsilon(y) |f(x-y) - f(x)|\, \mathrm{d}^{\mathrm{n}} x\, \mathrm{d}^{\mathrm{n}} y \leq \sup_{y \in B(0,\epsilon)} \|f(\cdot - y) - f\|_1.\end{aligned}$$

If $f$ is the characteristic functions of a rectangle, then the supremum on the right hand side converges to zero for $\epsilon \downarrow 0$. Due to the triangle inequality the same holds for step functions, i.e. finite linear combinations of such functions. Since step functions are dense in $L^1(\mathbb{R}^n)$ for each $f \in L^1(\mathbb{R}^n)$ this supremum becomes arbitrary small for sufficiently small $\epsilon$. Hence the family of functions $(\lambda_\epsilon * f)_{\epsilon>0}$ converges in $L^1(\mathbb{R}^n)$ in the limit $\epsilon \downarrow 0$ to $f$. Hence there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ which converges to zero, with $\|f_{n+1} - f_n\|_1 \leq 2^{-n}$ for all $n \in \mathbb{N}$ and $f_n = \lambda_{\epsilon_n} * f$. This ensures that the series $|f_1| + \sum_{n\in\mathbb{N}} |f_{n+1} - f_n|$ converges in $L^1(\mathbb{R}^n)$. Furthermore, due to Lebesgue's bounded convergence theorem the sequence $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to $f$. The non-negativity of the mollifiers together with the assumption on $F_f$ implies $(\lambda_\epsilon * f)(x) = F_f(\lambda_\epsilon(x - \cdot) \geq 0$. This indeed shows that $f$ is a.e. non-negative.

In particular, if $f$ belongs to the kernel of $f \mapsto F_f$, then $f$ is almost everywhere non-negative and non-positive. So $f$ vanishes almost everywhere. **q.e.d.**

**Exercise 2.10.** *In this exercise we show that for distributions there is a one-to-one correspondence between solutions of the linear transport equation and initial values.*

1. *Show that for any distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$, the following distribution solves the equation $\partial_t \tilde{F} = 0$:*

   $\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ *with* $\tilde{F}(\phi) = F(\tilde{\phi})$ *and* $\tilde{\phi}(y,t) = \phi(y - bt, t)$ *for all* $(y,t) \in \mathbb{R}^n \times \mathbb{R}$.

2. *Show that the following formula defines a linear continuous map*

   $$I : C^\infty(\mathbb{R}^n \times \mathbb{R}) \to C_0^\infty(\mathbb{R}^n) \quad \text{with} \quad I(\phi)(x) = \int_{\mathbb{R}} \phi(x,t)\, \mathrm{d}t.$$

3. *Let $\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ solve $\partial_t \tilde{F} = 0$. Show $\tilde{F}(\phi) = G(I(\phi))$ for some $G \in \mathcal{D}'(\mathbb{R}^n)$.*

4. *Show that for any mollifier $(\lambda_\epsilon)_{\epsilon>0}$ on $\mathbb{R}$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$ the functions*
$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \text{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$
*belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ and that $\tilde{F}(\phi \times \lambda_\epsilon)$ does not depend on $\epsilon > 0$.*

5. *Show that for any $G \in \mathcal{D}'(\mathbb{R}^n)$ the following $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ solves $(\partial_t + b\nabla)F = 0$:*
$$F : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}, \qquad \phi \mapsto G\left(\int_{\mathbb{R}} \mathsf{T}_{-tb}\phi(\cdot,t)\,\mathrm{d}t\right).$$

6. *Show that $G \to F$ is bijective onto $\{F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \mid (\partial_t + b\nabla)F = 0\}$.*

A short and lucid introduction into the theory of distributions is contained in the first chapter of the book of Lars Hörmander: “Linear Partial Differential Operators”.

## 2.5 Regularity of Solutions

The regularity of a solution of a differential equation refers to the local properties of the corresponding functions. The most general functions we shall consider are distributions, which we say have the lowest regularity. They contain the measurable functions with the next highest regularity. The elements of $L^p_{\text{loc}}$ describe ever smaller families of functions, whose regularity increase with $p \in [1,\infty]$. The next smallest class are Sobolev functions whose $k$-th order partial derivatives belong to $L^p_{\text{loc}}$. The regularity further increases for the functions in $C^k$. Finally we end with the smooth functions and the analytic functions with the highest regularity.

## 2.6 Boundary Value Problems

Our investigations of solutions of partial differential equations aims for a complete characterisations of all solutions. In general partial differential equations have an infinite dimensional space of solutions. A solution of an ordinary differential equations of $m$-th order is in many cases uniquely determined by fixing the values of the first $m$ derivatives at some initial value of the parameter. For partial differential equations we search a similar characterisation. The solutions are functions on higher dimensional domains $\Omega \subset \mathbb{R}^n$. A natural condition is the specification of the values of the solution and some of its derivatives on the boundary of the domain. The search for solutions which obey this further specification are called boundary value problems. So an important objective in the investigation of partial differential equations is to find boundary value problems that have unique solutions. If we determine in addition all possible boundary values that have solutions, then the space of solutions is completely parameterised.

# Chapter 3

# Laplace Equation

One of the most important PDEs is the Laplace equation

$$\triangle u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

The corresponding inhomogeneous PDE is Poisson's equation

$$-\triangle u = f.$$

Both equations are linear PDEs of second order with the unknown function $u : \mathbb{R}^n \to \mathbb{R}$. They show up in many situations. In physics they describe for example the potential of an electric field in the vacuum with some distribution of charges $f$.

## 3.1 Fundamental Solution

The Laplace equation is invariant with respect to all rotations and translations of the Euclidean space $\mathbb{R}^n$. Therefore we first look for solutions which are invariant with respect to all rotations. These solutions depend only on the length $r = |x| = \sqrt{x \cdot x}$ of the position vector $x$. For such functions $u(x) = v(r) = v(\sqrt{x \cdot x})$ we calculate:

$$\nabla_x u(x) = v'\left(\sqrt{x \cdot x}\right) \nabla_x r = v'\left(\sqrt{x \cdot x}\right) \frac{2x}{2r}.$$

Hence the Laplace equation simplifies to an ODE

$$\triangle_x u(x) = \nabla_x \cdot \nabla_x u = v''(r)\frac{x^2}{r^2} + v'(r)\frac{n}{r} - v'(r)\frac{x^2}{r^2 r} = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

Let us solve this ODE:

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \Rightarrow \ln(v'(r)) = (1-n)\ln(r) + C \Rightarrow v(r) = \begin{cases} C' \ln(r) + C'' & \text{for } n = 2 \\ \frac{C'}{r^{n-2}} + C'' & \text{for } n \geq 3. \end{cases}$$

**Definition 3.1.** *Let $\Phi(x)$ be the following solutions of the Laplace equation:*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & \textit{for } n = 2 \\ \frac{1}{n(n-2)\omega_n|x|^{n-2}} & \textit{for } n \geq 3. \end{cases}$$

*Here $\omega_n$ denotes the volume of the unit ball $B(0,1)$ in Euclidean space $\mathbb{R}^n$.*

This solution has a singularity at the origin $x = 0$. We have chosen the constants in such a way that the following theorem holds:

**Theorem 3.2.** *For $f \in C_0^2(\mathbb{R}^n)$ a solution of Poisson's equations $-\triangle u = f$ is given by*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)d^n y = \int_{\mathbb{R}^n} \Phi(z)f(x-z)d^n z.$$

*Proof.* The equality of both integrals in the definition of $u(x)$ follows from the substitution $z = x - y$. The second integral is twice continuously differentiable, since $f$ is twice continuously differentiable and has compact support. We calculate

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)d^n y.$$

In particular, $\triangle u(x) = \int_{\mathbb{R}^n} \Phi(y)\triangle_x f(x-y)d^n y$. We decompose this integral in the sum of an integral nearby the singularity of $\Phi$ and an integral away from this singularity:

$$\begin{aligned} \triangle u(x) &= \int_{B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)d^n y & &+ \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)d^n y \\ &= I_\epsilon & &+ J_\epsilon. \end{aligned}$$

We use $\int r \ln r dr = \frac{r^2}{2}(\ln r - \frac{1}{2})$ and $\int r dr = \frac{r^2}{2}$ and estimate the first integral for $\epsilon \downarrow 0$:

$$|I_\epsilon| \leq C\|\triangle_x f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)|d^n y \leq \begin{cases} C\epsilon^2(|\ln \epsilon| + 1) & (n = 2) \\ C\epsilon^2 & (n \geq 3). \end{cases}$$

Integration by parts of the second integral yields

$$\begin{aligned} J_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y)\triangle_y f(x-y)d^n y \\ &= -\int_{\mathbb{R}^n \setminus B(0,\epsilon)} \nabla_y \Phi(y) \cdot \nabla_y f(x-y)d^n y & &+ \int_{\partial B(0,\epsilon)} \Phi(y)\nabla_y f(x-y) \cdot N d\sigma(y) \\ &= K_\epsilon & &+ L_\epsilon. \end{aligned}$$

Here $N$ is the outer normal and $d\sigma$ the measure on the boundary of $\mathbb{R}^n \setminus B(0,\epsilon)$. The second term converges in the limit $\epsilon \downarrow 0$ to zero:

$$|L_\epsilon| \le |\nabla f|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)| d\sigma(y) \le \begin{cases} C\epsilon|\ln \epsilon| & (n=2) \\ C\epsilon & (n \ge 3). \end{cases}$$

Another integration by parts of the first term yields

$$\begin{aligned} K_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \triangle_y \Phi(y) f(x-y) d^n y - \int_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x-y) \cdot N d\sigma(y) \\ &= -\int_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x-y) \cdot N d\sigma(y). \end{aligned}$$

Here we used that $\phi$ is harmonic for $y \neq 0$. The gradient of $\Phi$ is equal to $\nabla\Phi(y) = -\frac{1}{n\omega_n}\frac{y}{|y|^n}$. The outer normal points towards the origin and is equal to $-\frac{y}{|y|}$. Let $\sigma_n(r)$ denote the area of $\partial B(0,r) \subset \mathbb{R}^n$. By the divergence theorem for $x \mapsto x$ we have

$$n\omega_n r^n = \int_{B(0,r)} \nabla \cdot x d\mu = \int_{\partial B(0,r)} x \cdot N(x) d\sigma(x) = \int_{\partial B(0,r)} x \cdot \tfrac{x}{|x|} d\sigma(x) = r\sigma_n(r), \quad \sigma_n(r) = n\omega_n r^{n-1}.$$

Now $K_\epsilon$ is the mean value of $-f$ on $\partial B(0,\epsilon)$, since $\sigma_n(\epsilon) = n\omega_n \epsilon^{n-1}$ is the area of $\partial B(0,\epsilon)$. By continuity of $f$ this mean value converges for $\epsilon \downarrow 0$ to $-f(x)$. **q.e.d.**

By this theorem we have $-\triangle\Phi(x) = \delta(x)$ in the sense of distributions. This relation justifies the choice of the constant in the definition of $\phi$. The convolution of $f$ with $\Phi$ is defined for continuous functions $f \in L^1(\mathbb{R}^n)$. On can show that as a measurable function the convolution is defined even for $f \in L^1(\mathbb{R}^n)$ and belongs to $L^1(\mathbb{R}^n)$. In this case the convolution is in the sense of distributions a solution of Poisson's equation. In general, for a continuous $f \in L^1(\mathbb{R}^n)$ the corresponding $u$ is not twice differentiable, and the theorem is not valid for such $f$. But it is valid for all Lipschitz continuous functions $f$ in $L^1(\mathbb{R}^n)$. Since Poisson's equation is an inhomogeneous linear PDE, all solutions are defined up to adding a solution of the homogeneous equation which is Laplace equation.

## 3.2 Mean Value Property

In this section we shall prove the following property of a harmonic function $u$ on an open domain $\Omega \subset \mathbb{R}^n$: the value $u(x)$ of $u$ at the center of any ball $B(x,r)$ with compact closure in $\Omega$ is equal to the mean of $u$ on the boundary of the ball. Conversely, if this holds for all balls with compact closure in $\Omega$, then $u$ is harmonic. The mean of $u$ on the ball $B(x,r)$ is the mean over $r' \in [0,r]$ of the means of $u$ on the boundary of $B(x,r')$. Therefore the same statement holds for the means of $u$ on the balls $B(x,r)$. This relation is called mean value property and has many important consequences.

**Mean Value Property 3.3.** *Let $u \in C^2(\Omega)$ be harmonic on an open domain $\Omega \subset \mathbb{R}^n$ containing the closure of the ball $B(x,r)$. The mean of $u$ on the ball $B(x,r)$ and on its boundary is equal to the value $u(x)$ of $u$ at the center. Conversely, if the means of a function $u \in C^2(\Omega)$ on all balls $B(x,r)$ with compact closure in $\Omega$ or on all boundaries of such balls is equal to the value $u(x)$ at the center of the ball, then $u$ is harmonic.*

*Proof.* We define $\Phi(r)$ for $x \in \Omega$ as the mean of $u$ on $\partial B(x,r) \subseteq \Omega$:

$$\Phi(r) := \frac{1}{r^{n-1}n\omega_n}\int_{\partial B(x,r)} u(y)\,\mathrm{d}\sigma(y) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} u(x+rz)\,\mathrm{d}\sigma(z).$$

Here $\omega_n$ denotes the volume of the unit ball in Euclidean space $\mathbb{R}^n$. We apply the Divergence Theorem and calculate the derivative $\Phi'(r) =$

$$= \frac{1}{n\omega_n}\int_{\partial B(0,1)} \nabla u(x+rz)\cdot z\,\mathrm{d}\sigma(z) = \frac{1}{r^{n-1}n\omega_n}\int_{\partial B(x,r)} \nabla u(y)\cdot N\,\mathrm{d}\sigma(y) = \frac{1}{r^{n-1}n\omega_n}\int_{B(x,r)} \triangle u\,\mathrm{d}\mu.$$

Hence for harmonic $u$ this function is constant as long as $B(x,r)$ has compact closure in $\Omega$. By continuity of $u$ this function $\Phi(r)$ converges in the limit $\lim r \to 0$ to $u(x)$. This shows that the means of $u$ on all spheres $\partial B(x,r)$ are equal to the values $u(x)$ of $u$ at the center $x$. Now we claim that the integral of the function $u$ over $B(x,r)$ is

$$\int_{B(x,r)} u(y)d\mu(y) = \int_{B(0,r)} u(x+y)d\mu(y) = \int_0^r\int_{\partial B(0,s)} u(x+y)d\sigma(y)ds = \int_0^r\int_{\partial B(x,s)} u(y)d\sigma(y)ds.$$

In fact for any linear rotation $\mathsf{O}$ the square of the determinant of the Jacobean of the map $(z,r) \mapsto \mathsf{O}(z,\sqrt{r^2-z^2})$ is equal to $\frac{r^2}{r^2-z^2}$ which is equal to $\det\big((\Phi'(z))^T\Phi'(z)\big) = 1+(\nabla_z\sqrt{r^2-z^2})^2$ of the pasteurisation $z \mapsto \Phi(z) = \mathsf{O}(z,\sqrt{r^2-z^2})$ of $\partial B(0,r)$. Hence the claim follows from Jacobi's transformation formula and the Definition 2.3 of the integral over $\partial B(0,r)$. So the mean of $u$ on the ball $B(x,r)$ is equal to

$$\frac{1}{r^n\omega_n}\int_{B(x,r)} u(y)\,\mathrm{d}^\mathrm{n}y = \frac{n}{r^n}\int_0^r \frac{1}{s^{n-1}n\omega_n}\int_{\partial B(x,s)} s^{n-1}u(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s = \frac{n}{r^n}\int_0^r s^{n-1}\Phi(s)\,\mathrm{d}s.$$

For constant $\Phi$ this is again equal to the value $u(x)$ of $u$ at the center $x$.

Conversely, if the means of $u$ on all balls $B(x,r)$ with compact closure in $\Omega$ is equal to the values $u(x)$ of $u$ at the center $x$, then we have

$$u(x) = \frac{1}{\omega_n r^n}\int_0^r n\omega_n s^{n-1}\Phi(s)\,\mathrm{d}s = \frac{n}{r^n}\int_0^r s^{n-1}\Phi(s)\,\mathrm{d}s.$$

The vanishing of the derivative with respect to $r$ of the right hand side yields

$$0 = -\frac{n^2}{r^{n+1}} \int_0^r s^{n-1}\Phi(s)\,\mathrm{d}s + \frac{n}{r^n} r^{n-1}\Phi(r) = -\frac{n}{r}u(x) + \frac{n}{r}\Phi(r).$$

Therefore also the means $\Phi(r)$ of $u$ on the boundaries $\partial B(x,r)$ are equal to the value $u(x)$ of $u$ at the center $x$. Since $u$ is twice continuously differentiable, the function $\Phi$ is twice continuously differentiable. We have seen that $\Phi'(r)$ is the mean of $\triangle u$ on $B(x,r)$. In particular, if the mean of $u$ on all $\partial B(x,r) \subset \Omega$ is equal to the value $u(x)$ of $u$ at the center $x$, then the integral of $\triangle u$ over the balls with compact closure in $\Omega$ vanish. If there exists $x \in \Omega$ with $\triangle u(x) \neq 0$, then there is a ball $B(x,r)$ with compact closure in $\Omega$, such that either $\triangle u < -|\triangle u(x)|/2$ or $\triangle u > |\triangle u(x)|/2$ on $B(x,r)$ and $\int_{B(x,r)} \triangle u(y) d^n y \neq 0$. Therefore $\triangle u$ vanishes and $u$ is harmonic on $\Omega$. **q.e.d.**

**Corollary 3.4.** *Let $u$ be a smooth*[1] *harmonic function on an open domain $\Omega \subset \mathbb{R}^n$ and $B(x,r)$ a ball with compact closure in $\Omega$. For all multi-indices $\alpha$ we have the estimate*

$$|\partial^\alpha u(x)| \leq C(n,|\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(\overline{B(x,r)})} \quad \textit{with} \quad C(n,|\alpha|) = 2^{\frac{|\alpha|(1+|\alpha|)}{2}} n^{|\alpha|}.$$

*Proof.* All partial derivatives of a harmonic function are harmonic. The Mean Value Property and the Divergence Theorem yield for $i = 1, \ldots, n$

$$|\partial_i \partial^\alpha u(x)| = \left| \frac{2^n}{\omega_n r^n} \int_{B(x,r/2)} \partial_i \partial^\alpha u d\mu \right| = \left| \frac{2^n}{\omega_n r^n} \int_{\partial B(x,r/2)} \partial^\alpha u N_i d\sigma \right| \leq \frac{2n}{r} \|\partial^\alpha u\|_{L^\infty(\partial B(x,r/2))}.$$

The inductive application gives first $C(n,1) = 2n$, and using the induction hypothesis

$$\|\partial^\alpha u(y)\| \leq 2^{|\alpha|} C(n,|\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(B(x,r))} \quad \text{for all} \quad y \in \partial B(x,r/2)$$

the relation $C(n, 1+|\alpha|) = 2^{1+|\alpha|} n C(n,|\alpha|)$. The given $C(n,|\alpha|)$ is the solution. **q.e.d.**

**Liouville's Theorem 3.5.** *On $\mathbb{R}^n$ a bounded harmonic function is constant.*

*Proof.* The foregoing corollary shows that $|\partial_i u(x)|$ is bounded by $2n\|u\|_{L^\infty(\mathbb{R}^n)} r^{-1}$ for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$. In the limit $r \to \infty$ the first partial derivatives vanish identically. Therefore $u$ is constant. **q.e.d.**

Let us now transfer the Mean Value Property to a property of distributions. If $u \in C^2(\Omega)$ is harmonic, then for $B(x,r) \subset \Omega$ and $\psi \in C_0^\infty((0,r))$ we have

$$\int_{B(x,r)} u(y) \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} \mathrm{d}^{\mathrm{n}} y = \int_0^r \frac{\psi(s)}{n s^{n-1}\omega_n} \int_{\partial B(x,s)} u(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s = \left(\int_0^r \psi(s)\,\mathrm{d}s\right) u(x).$$

So the distribution $F_u$ has the following property:

[1] We shall see in Weyl's Lemma that this assumption can be replaced by the assumption $u \in L^\infty_{\mathrm{loc}}(\Omega)$.

**Weak Mean Value Property 3.6.** *Let $U \in \mathcal{D}'(\Omega)$ be a harmonic distribution on an open domain $\Omega \subset \mathbb{R}^n$. For each ball $B(x,r)$ with $B(x,r) \subset \Omega$ and each $\psi \in C_0^\infty((0,r))$ with $\int \psi \, \mathrm{d}\mu = 0$ the distribution $U$ vanishes on the following test function:*

$$f \in C_0^\infty(\Omega), \quad y \mapsto f(y) = \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} \quad \textit{with} \quad \operatorname{supp} f \subset B(x,r) \subset \Omega.$$

*Proof.* It suffices to show that there exists a test function $g \in C_0^\infty(\Omega)$ with $\triangle g = f$. By the assumptions on $\psi$ there exists a test function $\Psi \in C_0^\infty((0,r))$ with $\Psi' = \psi$. We define

$$g(y) = v(|y-x|) \quad \text{with} \quad v(t) = \int_r^t \frac{\Psi(s)}{ns^{n-1}\omega_n} \, \mathrm{d}s.$$

This function $g$ has compact support in $B(x,r) \subset \Omega$, depends only on $|y-x|$ and is constant on $B(x,\epsilon)$ for some $\epsilon > 0$. We calculate for $y \neq x$:

$$\nabla_y g(y) \qquad\qquad = v''(|y-x|) + \frac{n-1}{|y-x|} v'(|y-x|)$$

This implies

$$\triangle_y g(y) = \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} - \frac{(n-1)\Psi(|y-x|)}{n|y-x|^n\omega_n} + \frac{n-1}{|y-x|}\frac{\Psi(|y-x|)}{n|y-x|^{n-1}\omega_n} = f(y).$$ **q.e.d.**

**Weyl's Lemma 3.7.** *On an open domain $\Omega \subset \mathbb{R}^n$ for each harmonic distribution $U \in \mathcal{D}'(\Omega)$ there exists a harmonic function $u \in C^\infty(\Omega)$ with $U = F_u$.*

*Proof.* Let us first define $u$. For all $x \in \Omega$ choose a ball $B(x,r) \subset \Omega$ and a test function $\psi \in C_0^\infty((0,r))$ with $\int_0^r \psi(s) \, \mathrm{d}s = 1$. Then we define

$$u(x) := U(g_x) \quad \text{with} \qquad\qquad g_x(y) := \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n}.$$

If $U$ is harmonic, then the Weak Mean Value Property implies that $u(x)$ does not depend on the choice of $r$ and $\psi$. Hence we can use in the formula for $u(x)$ the same $r$ and $\psi$ for all $x$ in a small neighbourhood of each $x_0$. Then $u$ is the convolution of the test function $g_0 = \mathsf{P}g_0$ with the distribution $U$. Due to Lemma 2.7, $u$ is smooth.

Next we prove that the distribution $\tilde{U} = F_u$ has the Weak Mean Value Property. By the discussion which motivates Lemma 2.7 this distribution $\tilde{U}$ is the convolution of $g_0 = \mathsf{P}g_0$ with $U$, conceived not as the function $u$ but as a distribution $F_u$. The functions $f$ in the Weak Mean Value Property are characterised by three properties:

1. they depend only on the distance $|y - x|$ of the variable $y$ to some center $x \in \Omega$,
2. they vanish on a neighbourhood of the center $x$ and
3. their integrals vanish.

The first property is equivalent to the invariance of $f$ with respect to all rotations around the center. We show that the convolution $g * f$ of a test function $g$ which is invariant with respect to all rotations around the center $x$ with a test function f which is invariant with respect to all rotations around a center $z$ is invariant with respect to all rotations around the center $x + z$. In fact, for all $\mathsf{O} \in O(n, \mathbb{R})$ we use the invariance of the Lebesgue measure with respect to translations and $\mathsf{O}$ and obtain

$$(g*f)(x+z+\mathsf{O}y) = \int_{\mathbb{R}^n} g(x+z+\mathsf{O}y-y')f(y')\,\mathrm{d}^{\mathrm{n}}y' = \int_{\mathbb{R}^n} g(x+\mathsf{O}(y-z'))f(z+\mathsf{O}z')\,\mathrm{d}^{\mathrm{n}}z'.$$

Furthermore, the integral of $g * f$ is the product of the integrals of $g$ and $f$:

$$\begin{aligned}\int_{\mathbb{R}^n} (g * f)(x)\,\mathrm{d}^{\mathrm{n}}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^{\mathrm{n}}y\,\mathrm{d}^{\mathrm{n}}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^{\mathrm{n}}x\,\mathrm{d}^{\mathrm{n}}y \\ &= \left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}^{\mathrm{n}}x\right)\left(\int_{\mathbb{R}^n} f(y)\,\mathrm{d}^{\mathrm{n}}y\right).\end{aligned}$$

In particular, the convolution $g_0 * f$ of the test functions $g_0$ with the test function $f$ in the Weak Mean Value Property is again a test function which is invariant with respect to the rotations around the same center as $f$ with vanishing total integral. Since $\psi$ has compact support in $(0, r)$ the function $f$ vanishes on $B(x, \epsilon)$ for sufficiently small $\epsilon > 0$. If the support of $g_0$ is contained in $B(0, \epsilon/2)$, then $g_0 * f$ vanishes on $B(x, \epsilon/2)$. This implies that $g_0 * f$ is again a function of the form considered in the Weak Mean Value Property and $\tilde{U}(f) = U(g_0 * f) = 0$. So $\tilde{U}$ has the Weak Mean Value Property.

Next we prove that $u$ is harmonic. Let $\phi_{B(0,\epsilon)} \in C_0^\infty(\mathbb{R})$ be the mollifier defined in Section 2.4. For any $B(x, r)$ with compact closure in $\Omega$, there exists $R > r$ with $B(x, R) \subset \Omega$. For all $0 < r_1 < r_2 < R$ and sufficiently small $\epsilon$ the function $\psi(t) = \phi_{B(0,\epsilon)}(t-r_1)-\phi_{B(0,\epsilon)}(t-r_2)$ has compact support in $(0, R)$ and vanishing total integral. Let $f_\epsilon$ denote the corresponding functions in the kernel of $\tilde{U}$. Since $u$ is continuous, the limit $\epsilon \downarrow 0$ of $\tilde{U}(f_\epsilon)$ exists and converges to the difference of the means of $u$ on $\partial B(x, r_2)$ and $\partial B(x, r_1)$. Since $\tilde{U}$ has the Weak Mean Value Property these differences vanish for all $0 < r_1 < r_2 < R$ and the means of $u$ on all $\partial B(x, r) \subset \Omega$ coincide. In the limit $r \downarrow 0$ these means converge to $u(x)$, since $u$ is continuous. Therefore $u$ has the Mean Value Property and is a smooth harmonic function.

Finally we prove $\tilde{U} = U$. The functions $\psi(t) = \phi_{B(0,\epsilon/3)}(t - 2/3\epsilon)$ have support $[\epsilon/3, \epsilon]$ and total integral 1. The corresponding functions $g_0$ are smooth mollifiers $\lambda_\epsilon$. By definition of $\tilde{U}$ we have $\tilde{U} = \lambda_\epsilon * U$. Now Lemma 2.8 implies $\tilde{U} = U$. **q.e.d.**

Actually we have proven that any distribution $U$ that has the Weak Mean Value Property corresponds to a smooth harmonic function. Therefore the weak solutions of the Laplace equations coincide with the strong solutions, and all solutions are smooth.

Let us finish this section with a proof of Harnack's inequality. This inequality estimates the values of a positive harmonic on a path–connected domain in terms of the value at any point in the domain.

**Harnack's Inequality 3.8.** *Let $\Omega'$ be an open path–connected domain with compact closure in the open domain $\Omega \subset \mathbb{R}^n$. Then there exists a constant $C > 0$ depending only on $\Omega'$ and $\Omega$, such that any non-negative harmonic function $u$ on $\Omega$ satisfies the Harnack inequality*

$$\sup_{x\in\Omega'} u(x) \leq C \inf_{x\in\Omega'} u(x).$$

*In particular, for all $x, y \in \Omega'$ we have* $$\frac{1}{C}u(y) \leq u(x) \leq Cu(y).$$

*Proof.* Let $r$ be the minimal value of the continuous function on the compact set $\bar{\Omega}'$:

$$\bar{\Omega}' \to \mathbb{R}^+, \quad x \mapsto \sup\{R > 0 \mid B(x, 2R) \subset \Omega\}.$$

For $x \in \Omega'$ and $y \in B(x, r)$ we have $B(y, r) \subset B(x, 2r) \subset \Omega$. The Mean Value Property implies

$$u(x) = \frac{1}{2^n r^n \omega_n} \int_{B(x,2r)} u \, \mathrm{d}\mu \geq \frac{2^{-n}}{r^n \omega_n} \int_{B(y,r)} u \, \mathrm{d}\mu = 2^{-n} u(y).$$

Since $\bar{\Omega}'$ is compact and path–connected, it can be covered by finitely many balls $B_1, \ldots, B_N$ of radius $\frac{r}{2}$ such that $B_{n+1} \cap B_n \neq \emptyset$ for $n = 1, \ldots, N-1$. An $N$-fold application of the special case implies for general $x, y \in \Omega'$

$$u(x) \geq 2^{-nN} u(y).$$

Taking the infimum over $x \in \Omega'$ and the supremum over $y \in \Omega'$ gives

$$\sup_{x\in\Omega'} u(x) \leq 2^{nN} \inf_{x\in\Omega'} u(x).$$ **q.e.d.**

**Harnack's Principle 3.9.** *On an open and path–connected domain $\Omega \subset \mathbb{R}^n$ a monotone sequence of harmonic functions $(u_n)_{n\in\mathbb{N}}$ converges uniformly on all compact subsets if and only if there exists $x \in \Omega$ such that $(|u_n(x)|)_{n\in\mathbb{N}}$ is bounded.*

*Proof.* If $(u_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets, then $(u_n(x))_{n\in\mathbb{N}}$ converges for all $x \in \Omega$. Conversely, let $(|u_n(x)|)_{n\in\mathbb{N}}$ be bounded for some $x \in \Omega$. By monotonicity the sequence $(u_n(x))_{n\in\mathbb{N}}$ converges. Furthermore, we may assume that $(u_n - u_m)_{n\geq m}$ is non-negative. Harnack's Inequality implies that $(u_n - u_m)_{n\geq m}$ is uniformly bounded on compact subsets of $\Omega$ and monotonic. Hence it converges uniformly there. The limit has together with all $u_n$ the Mean Value Property and is harmonic. **q.e.d.**

## 3.3 Maximum Principle

Let the harmonic function $u$ take in a point $x$ of an open path–connected domain $\Omega \subset \mathbb{R}^n$ a maximum. The Mean Value Property implies on all balls $B(x,r) \subset \Omega$

$$\frac{1}{r^n\omega_n}\int_{B(x,r)} |u(x)-u(y)| d^n y = \frac{1}{r^n\omega_n}\int_{B(x,r)} (u(x)-u(y)) d^n y = 0.$$

Hence $u$ takes the maximum on all these balls $B(x,r) \subset \Omega$. Since $\Omega$ is path–connected every other point $y \in \Omega$ is connected with $x$ by a continuous path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. The compact image $\gamma[0,1]$ is covered by finitely many balls $B(\gamma(t_1), r_1), \ldots, B(\gamma(t_N), r_N) \subset \Omega$ with $0 \le t_1 < \ldots t_N \le 1$ and $r_1, \ldots, r_N > 0$. Hence $u$ is constant along $\gamma$, and on $\Omega$ since this is true for all $y \in \Omega$. This proves

**Strong Maximum Principle 3.10.** *If a harmonic function $u$ has on a path–connected open domain $\Omega \subset \mathbb{R}^n$ a maximum, then $u$ is constant.* **q.e.d.**

**Weak Maximum Principle 3.11.** *Let the harmonic function $u$ on a bounded open domain $\Omega \subset \mathbb{R}^n$ extend continuously to the boundary $\partial\Omega$. The maximum of $u$ is taken on the boundary $\partial\Omega$.*

*Proof.* By Heine Borel the closure $\bar{\Omega}$ is compact and the continuous function $u$ takes on $\bar{\Omega}$ a maximum. If it does not belong to $\partial\Omega$, then $u$ is constant on the corresponding path–connected component and the maximum is also taken on $\partial\Omega$. **q.e.d.**

Since the negative of a harmonic function is harmonic the same conclusion holds for minima. Now we generalise the Maximum Principle, but not the Mean Value Property.

**Definition 3.12.** *On an open domain $\Omega \subset \mathbb{R}^n$ an differential operator $L$*

$$Lu = \sum\nolimits_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum\nolimits_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i}$$

*with symmetric coefficients $a_{ij} = a_{ji}$ is called elliptic, if*

$$\sum\nolimits_{i,j=1}^{n} a_{ij}(x)k_i k_j > 0 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n \setminus \{0\}.$$

If we replace $a_{ij}$ by $\frac{1}{2}(a_{ij} + a_{ji})$, then the assumption $a_{ij} = a_{ji}$ is fulfilled. Due to the commutativity of the second derivatives this replacement does not change $L$.

**Theorem 3.13.** *Let $L$ be an elliptic operator on a bounded open domain $\Omega \subset \mathbb{R}^n$ whose coefficients $a_{ij}$ and $b_i$ extend continuously and elliptic to $\partial\Omega$. Every twice differentiable solution $u$ of $Lu \ge 0$ which extends continuously to $\partial\Omega$ takes its maximum on $\partial\Omega$.*

*Proof.* Let us first show that $L$ is uniform elliptic, i.e. there exists $\lambda > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(x)k_i k_j \geq \lambda \sum_{i=1}^{n} k_i^2 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n.$$

The continuous function $(x,k) \mapsto \sum_{i,j=1}^{n} a_{ij}(x)k_i k_j$ attains on the compact set $(x,k) \in \bar{\Omega} \times S^{n-1} \subset \bar{\Omega} \times \mathbb{R}^n$ a minimum $\lambda > 0$. Hence $L$ is uniform elliptic.

For $v(x) = \exp(\alpha x_1)$ with $\alpha > 0$ we conclude

$$Lv = \alpha(\alpha a_{11}(x) + b_1(x))v \geq \alpha(\alpha\lambda + b_1(x))v.$$

The continuous coefficients $b_i$ are bounded on the compact set $\bar{\Omega}$. Therefore there exists $\alpha > 0$ with $Lv > 0$. By linearity of $L$ we obtain $L(u + \epsilon v) > 0$ on $\Omega$ for all $\epsilon > 0$. The continuous functions $u+\epsilon v$ attains on $\bar{\Omega}$ a maximum. The first derivative of the function $u+\epsilon v$ which is twice differentiable on $\Omega$ vanishes at a maximum $x_0 \in \Omega$ and the Hessian is negative semi-definite. In particular there exists an orthogonal matrix $B \in O(n)$ and non-positive $\lambda_1, \ldots, \lambda_n$ with $\frac{\partial^2(u+\epsilon v)(x_0)}{\partial x_i \partial x_j} = \sum_k B_{ki}\lambda_k B_{kj}$. Now the ellipticity implies $-L(u+\epsilon v)(x_0) \geq -\lambda \sum_{ki} \lambda_k B_{ki}^2 \geq 0$ and contradicts to $L(u+\epsilon v) > 0$. Therefore for all $\epsilon > 0$ the maximum belongs to the boundary $x_0 \in \partial\Omega$:

$$\sup_{x\in\Omega} u(x) + \epsilon \inf_{x\in\Omega} v(x) \leq \sup_{x\in\Omega}(u(x)+\epsilon v(x)) = \max_{x\in\partial\Omega}(u(x)+\epsilon v(x)) \leq \max_{x\in\partial\Omega} u(x) + \epsilon \max_{x\in\partial\Omega} v(x).$$

Because this holds for all $\epsilon > 0$ the boundedness of $v$ on $\bar{\Omega}$ implies the theorem. **q.e.d.**

The negative of the functions $u$ in the theorem obey $Lu \leq 0$ and take a minimum on the boundary. In particular, the solutions $u$ of $Lu = 0$ take the maximum and the minimum on the boundary.

## 3.4 Green's Function

In this section we try to find some conditions which ensure the existence and uniqueness of a harmonic function on a path–connected, open and bounded domain $\Omega \subset \mathbb{R}^n$. A natural candidate for further conditions are boundary value problems. This means that we assume that either the harmonic function or some of its derivatives extends continuously to the boundary and coincides there with a given function. We call a function $u$ on the closure $\bar{\Omega}$ of an domain $m$ times continuously differentiable, if it is $m$ times continuously differentiable on $\Omega$ and all partial derivatives of order at most $m$ extend continuously to $\partial\Omega$.

In the following formula we apply the Divergence Theorem to $x \mapsto v(x)\nabla u(x)$:

**Green's First Formula 3.14.** *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_\Omega v(y)\triangle u(y)d^ny + \int_\Omega \nabla v(y)\cdot\nabla u(y)d^ny = \int_{\partial\Omega} v(z)\nabla u(z)\cdot N d\sigma(z). \qquad \textbf{q.e.d.}$$

If we subtract the formula for interchanged $u$ and $v$, then we obtain:

**Green's Second Formula 3.15.** *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_\Omega \left(v(y)\triangle u(y) - u(y)\triangle v(y)\right) d^ny = \int_{\partial\Omega} \left(v(z)\nabla u(z) - u(z)\nabla v(z)\right)\cdot N d\sigma(z). \qquad \textbf{q.e.d.}$$

Let us apply this formula to the fundamental solution $v(y) = \Phi(x-y)$. This solution is harmonic only for $y \neq x$. Like in the proof of Theorem 3.2 we restrict the integral to the compliment of $B(x,\epsilon)$. For $u \in C^2(\mathbb{R}^n)$ we showed in the proof of Theorem 3.2

$$\lim_{\epsilon\to 0} \int\limits_{\partial(\mathbb{R}^n\setminus B(x,\epsilon))} u(z)\nabla_z\Phi(x-z)\cdot N d\sigma(z) = \lim_{\epsilon\to 0} \int\limits_{\partial(\mathbb{R}^n\setminus B(0,\epsilon))} u(x-z)\nabla_z\Phi(z)\cdot N d\sigma(z) = u(x).$$

We also showed that the other integral over $\partial(\mathbb{R}^2 \setminus B(x,\epsilon))$ converges to zero in the limit $\epsilon \to 0$ and $\triangle_y\Phi(x-y)$ vanishes on $\mathbb{R}^n \setminus B(x,\epsilon)$. This proves

**Green's Representation Theorem 3.16.** *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for $x \in \Omega$ and a function $u \in C^2(\bar{\Omega})$ we have*

$$u(x) = -\int_\Omega \Phi(x-y)\triangle u(y)d^ny + \int_{\partial\Omega} \left(\Phi(x-z)\nabla_z u(z) - u(z)\nabla_z\Phi(x-z)\right)\cdot N d\sigma(z).$$

This implies that on $\Omega$ each solution of the Poisson equation is uniquely determined by the values of $u$ and the normal derivative $\nabla u\cdot N$ on $\partial\Omega$. Conversely, we look for such functions, such that there exists a solution of Poisson's equation with the additional condition that $u$ and $\nabla u\cdot N$ take on $\partial\Omega$ the given values. The Weak Maximum Principle implies the harmonic function is already uniquely determined by the values of $u$ on $\partial\Omega$. So we formulate the following boundary value problem:

**Dirichlet Problem 3.17.** *For a given function $f$ on an open domain $\Omega \subset \mathbb{R}^n$ and $g$ on $\partial\Omega$ we look for a solution $u$ of $-\triangle u = f$ on $\Omega$ which extends continuously to $\partial\Omega$ and coincides there with $g$.*

**Green's Function 3.18.** *A function $G_\Omega : \{(x,y) \in \Omega\times\Omega \mid x \neq y\} \to \mathbb{R}$ is called Green's function for the open domain $\Omega \subset \mathbb{R}^n$, if it has the following two properties:*

**(i)** *For $x \in \Omega$ the function $y \mapsto G_\Omega(x,y) - \Phi(x-y)$ is harmonic on $y \in \Omega$.*

**(ii)** *For $x \in \Omega$ the function $y \mapsto G_\Omega(x,y)$ extends continuously to $\partial\Omega$ and vanishes on $y \in \partial\Omega$.*

Green's Second Formula yields for the function $v(y) = G_\Omega(x,y) - \Phi(x-y)$:

$$-\int_\Omega \Phi(x-y)\triangle u(y) d^n y + \int_{\partial\Omega} (\Phi(x-z)\nabla_z u(z) - u(z)\nabla_z \Phi(x-z)) \cdot N d\sigma(z)$$
$$= -\int_\Omega G_\Omega(x,y)\triangle u(y) d^n y - \int_{\partial\Omega} u(z)\nabla_z G_\Omega(x,z) \cdot N d\sigma(z).$$

Now Green's Representation Theorem implies

$$u(x) = -\int_\Omega G_\Omega(x,y)\triangle_y u(y) d^n y - \int_{\partial\Omega} u(z)\nabla_z G_\Omega(x,z) \cdot N d\sigma(z).$$

If, conversely, the functions $f : \bar{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ have sufficient regularity, then

$$u(x) = \int_\Omega G_\Omega(x,y) f(y) d^n y - \int_{\partial\Omega} g(z)\nabla_z G_\Omega(x,z) \cdot N d\sigma(z)$$

solves the Dirichlet Problem. In fact by Theorem 3.2 the first term solves the Dirichlet Problem for $g = 0$. If $g : \partial\Omega \to \mathbb{R}$ is the boundary value of a function on $\Omega$ with sufficient regularity, then the difference of $g$ minus the corresponding first term is harmonic and coincides with the second term. Therefore the Dirichlet Problem reduces to the search of the Green's Function.

For $x \in \Omega$ the difference $y \mapsto G_\Omega(x,y) - \Phi(x-y)$ is harmonic on $y \in \Omega$ and on the boundary $\partial\Omega$ equal to $-\Phi(x-y)$. Hence the difference is the solution of the Dirichlet Problem for $f = 0$ and $g(x) = -\Phi(x-y)$.

**Theorem 3.19** (Symmetry of the Green's Function)**.** *If there is a Green's Function $G_\Omega$ for the domain $\Omega$, then $G_\Omega(x,y) = G_\Omega(y,x)$ holds for all $x \neq y \in \Omega$.*

*Proof.* For $x \neq y \in \Omega$ let $\epsilon > 0$ be sufficiently small, such that both balls $B(x,\epsilon)$ and $B(y,\epsilon)$ are disjoint subsets of $\Omega$. Green's Second Formula implies for the domain $\Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))$ and the functions $u(z) = G(x,z)$ and $v(z) = G(y,z)$

$$\int_{\partial B(x,\epsilon)} (G(y,z)\nabla_z G(x,z) - G(x,z)\nabla_z G(y,z)) \cdot N d\sigma(z)$$
$$= \int_{\partial B(y,\epsilon)} (G(x,z)\nabla_z G(y,z) - G(y,z)\nabla_z G(x,z)) \cdot N d\sigma(z).$$

For $\epsilon \to 0$ the estimate for $L_\epsilon$ in the proof of Theorem 3.2 shows that both second terms converge to zero. The calculation of $K_\epsilon$ in the proof of Theorem 3.2 carries over and shows that the first terms converge to $G(y,x)$ and $G(x,y)$, respectively. **q.e.d.**

We shall calculate Green's function for all balls in $\mathbb{R}^n$. Let us first restrict to the unit ball $\Omega = B(0,1)$. We may use the inversion $x \mapsto \tilde{x} = \frac{x}{|x|^2}$ in the unit sphere $\partial B(0,1)$ in order to solve the corresponding Dirichlet Problem. The inversion maps the inside of the unit ball to the outside and vice versa. It helps to solve the Dirichlet Problem for $f = 0$ and $g(x) = \Phi(x - y)$:

**Green's Function of the unit ball 3.20.** *The Green's Function of $B(0,1)$ is*

$$G_{B(0,1)}(x,y) = \Phi(x-y) - \Phi(|x|(\tilde{x}-y)) = \begin{cases} \Phi(x-y) - |x|^{2-n}\Phi(\tilde{x}-y) & \text{for } n > 2 \\ \Phi(x-y) - \Phi(\tilde{x}-y) - \Phi(x) & \text{for } n = 2. \end{cases}$$

*Proof.* For $|y| = 1$ we have $|x|^2|\tilde{x} - y|^2 = 1 - 2y \cdot x + |x|^2 = |x - y|^2$. Hence $\Phi(|x|(\tilde{x} - y))$ and $\Phi(x - y)$ coincide on the boundary $y \in \partial B(0,1)$. **q.e.d.**

**Poisson's Representation Formula 3.21.** *For $f \in C^2(\overline{B(z,r)})$ and $g \in C(\partial B(z,r))$ the unique solution of the Dirichlet Problem on $\Omega = B(z,r)$ is given by*

$$u(x) = \frac{1}{r^{n-2}} \int_{B(z,r)} G_{B(0,1)}(\tfrac{x-z}{r}, \tfrac{y-z}{r}) f(y) d^n y + \frac{1 - \frac{|x-z|^2}{r^2}}{n\omega_n} \int_{\partial B(0,1)} \frac{g(z+ry)}{|\frac{x-z}{r} - y|^n} d\sigma(y).$$

*Proof.* The affine map $x \mapsto \frac{x-z}{r}$ is a homeomorphism from $B(z,r)$ onto $B(0,1)$ and from $\partial B(z,r)$ onto $\partial B(0,1)$. The difference $r^{2-n}\Phi(\frac{x-z}{r} - \frac{y-z}{r}) - \Phi(x-y)$ vanishes for $n > 2$ and is constant for $n = 2$. Therefore the Green's function of $B(z,r)$ is equal to

$$G_{B(z,r)}(x,y) = r^{2-n} G_{B(0,1)}(\tfrac{x-z}{r}, \tfrac{y-z}{r}).$$

It suffices to consider the two cases $g = 0$ and $f = 0$ separately. The properties of the Green's function together with Theorem 3.2 show, that for $g = 0$ the function $u$ differs by a harmonic function from a solution of Poisson's equation. By the symmetry of the Green's Function the map $x \mapsto G_{B(z,r)}(y,x)$ extends continuously to $\overline{B(z,r)}$ and vanishes on the boundary $x \in \partial B(z,r)$. This finishes the proof for $g = 0$.

For $|y| = 1$ and $n > 2$ we observe (the reader should check this formula for $n = 2$):

$$\begin{aligned} K(x,y) &= -\nabla_y G_{B(0,1)}(x,y) \cdot \frac{y}{|y|} \\ &= \frac{-1}{n(n-2)\omega_n} \frac{y}{|y|} \cdot \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}\,|\tilde{x}-y|^{n-2}} \right) \\ &= \frac{1}{n\omega_n} \frac{y}{|y|} \cdot \left( \frac{y-x}{|x-y|^n} - \frac{|x|^2(y-\tilde{x})}{|x|^n\,|\tilde{x}-y|^n} \right) \\ \text{(for } |y| = 1\text{)} \quad &= \frac{1 - x\cdot y - |x|^2 + x \cdot y}{n\omega_n |x-y|^n} = \frac{1 - |x|^2}{n\omega_n |x-y|^n}. \end{aligned}$$

By the Symmetry of the Green's Function the function $x \mapsto K(x,y)$ is harmonic. Hence for $f = 0$ the given function $u$ is harmonic. For finishing the proof we show that

$$u(z+rx) = \int_{\partial B(0,1)} g(z+ry)K(x,y)d\sigma(y)$$

extends continuously to $x \in \partial B(0,1)$ and coincides there with $g(z+rx)$. We observe

(i) the integral kernel $K(x,y)$ is positive for $(x,y) \in B(0,1) \times \partial B(0,1)$.

(ii) For all $x \in \partial B(0,1)$ and $\epsilon > 0$ the family of functions $y \mapsto K(\lambda x, y)$ converge uniformly to zero for $\lambda \uparrow 1$ on $y \in \partial B(0,1) \setminus B(x,\epsilon)$, and

(iii) The formula which follows from Green's Second Formula and Green's Representation Formula yields for the function $u = 1$ on the domain $\Omega = B(0,1)$

$$\int_{\partial B(0,1)} K(x,y)d\sigma(y) = 1 \quad \text{for} \quad x \in B(0,1).$$

For continuous $g$ the properties (i)-(iii) ensure that in the limit $\lambda \uparrow 1$ the family of functions $x \mapsto \int_{\partial B(0,1)} g(y)K(\lambda x,y)d\sigma(y)$ converge on $\partial B(0,1)$ uniformly to $g$. **q.e.d.**

A harmonic function $u$ on $B(z,r)$ which extends continuously to $\partial B(z,r)$ obeys

$$u(x) = \frac{1 - \frac{|x-z|^2}{r^2}}{n\omega_n} \int_{\partial B(0,1)} \frac{u(z+ry)}{|\frac{x-z}{r} - y|^n} d\sigma(y) = \frac{r^2 - |x-z|^2}{nr\omega_n} \int_{\partial B(z,r)} \frac{u(y)}{|x-y|^n} d\sigma(y).$$

In particular, $u$ is completely determined by the values on $\partial B(z,r)$. Partial derivatives with respect to $x$ yield similar formulas for the values of all partial derivatives of $u$. This formula implies the Mean Value property. For all $y \in \partial B(z,r)$ the Taylor series of $x \mapsto |x-y|^{-n} = (y^2 - 2xy + x^2)^{-\frac{n}{2}}$ in $x = z$ converge on all balls $B(z,r')$ with $r' < r$ uniformly to $|x-y|^{-n}$. Consequently all harmonic functions are analytic.

**Corollary 3.22.** *Harmonic functions on an open domain $\Omega \subset \mathbb{R}^n$ are analytic.* **q.e.d.**

**Exercise 3.23.** *1. Show the estimate $|\partial^\alpha |x|^{-n}| \leq C(n,|\alpha|)|x|^{-n-|\alpha|}$ for all $|x| \neq 0$ and all multi-indices $\alpha$ with a constant $C(n,|\alpha|)$ depending only on $n$ and $|\alpha|$.*

*2. Give another proof of Corollary 3.4.*

**Lemma 3.24.** *Let $\Omega \subset \mathbb{R}^n$ be an open neighbourhood of $0$ and $u$ a bounded harmonic function on $\Omega \setminus \{0\}$. Then $u$ extends as a harmonic function to $\Omega$.*

*Proof.* On a ball $B(0,r)$ with compact closure in $\Omega$, Theorem 3.21 gives a harmonic function $\tilde{u}$ which coincides on $\partial B(0,r)$ with $u$. The family of harmonic functions $u_\epsilon(x) = \tilde{u}(x) - u(x) + \epsilon G_{B(0,r)}(x,0)$ on $B(0,r)\setminus\{0\}$ vanish on $\partial B(0,r)$. If for any $\epsilon > 0$ the function $u_\epsilon$ takes on $B(0,r)\setminus\{0\}$ a negative value, then due to the boundedness of $u$ and $\tilde{u}$ and the unboundedness of $G_{B(0,r)}(\cdot,0)$ the harmonic function $u_\epsilon$ has a negative minimum on $B(0,r)\setminus\{0\}$. This contradicts the Strong Maximum Principle. Hence $u_\epsilon$ is non-negative. Analogously $u_\epsilon$ us for negative $\epsilon$ non-positive. Otherwise $u_\epsilon$ would have a positive maximum in $B(0,r)\setminus\{0\}$. In both limits $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$ $u_0 = \tilde{u} - u$ vanishes identically on $B(0,r)\setminus\{0\}$ and $\tilde{u}$ is a harmonic extension of $u$ to $\Omega$. **q.e.d.**

The proof shows a slightly stronger statement. Each harmonic function on $\Omega\setminus\{0\}$ whose absolute value $|u(x)|$ is for all $\epsilon > 0$ bounded by $\epsilon G_{B(0,r)}(x,0)$ on $B(0,\delta)\setminus\{0\}$ with sufficiently small $\delta > 0$ depending on $\epsilon$ has an harmonic extension to $\Omega$.

## 3.5 Dirichlet's Principle

The unique solution of Dirichlet's Problem solves also a variational problem.

**Dirichlet's Prinzip 3.25.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and obey the assumptions of the Divergence Theorem. For continuous real functions $f$ on $\bar{\Omega}$ and $g$ on $\partial\Omega$ the solution $u$ of the Dirichlet Problem 3.4 is the minimizer of the following functional:*

$$I : \{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\} \to \mathbb{R}, \qquad w \mapsto I(w) = \int_\Omega \left(\frac{1}{2}\nabla w \cdot \nabla w - wf\right) d^n x.$$

*Proof.* Let $u$ be a solution of the Dirichlet Problem and $w$ another function in the domain $\{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\}$ of $I$. An integration by parts yields

$$0 = \int_\Omega (-\triangle u - f)(u-w) d^n x = \int_\Omega (\nabla u \cdot \nabla(u-w) - f(u-w))\, d^n x.$$

$$\begin{aligned}\int_\Omega (\nabla u \cdot \nabla u - fu)\, d^n x &= \int_\Omega (\nabla u \cdot \nabla w - fw)\, d^n x \leq \\ &\leq \int_\Omega \frac{1}{2}\nabla u \cdot \nabla u d^n x + \int_\Omega \left(\frac{1}{2}\nabla w \cdot \nabla w - fw\right) d^n x\end{aligned}$$

Here we used the Cauchy-Schwarz inequality:

$$\int\limits_\Omega \nabla u \cdot \nabla w d^n x \leq \int_\Omega \nabla u \cdot \nabla w d^n x + \frac{1}{2}\int_\Omega (\nabla u - \nabla w)\cdot(\nabla u - \nabla w) d^n x =$$

$$\int_\Omega \frac{1}{2}\nabla u \cdot \nabla u d^n x + \int_\Omega \frac{1}{2}\nabla w \cdot \nabla w d^n x.$$

This shows $I(u) \leq I(w)$.

If, conversely, $u$ is a minimum, then all $v \in C^2(\bar{\Omega})$ which vanish on $\partial\Omega$ obey

$$0 = \frac{d}{dt} I(u + tv)\bigg|_{t=0} = \frac{d}{dt}\left(I(u) + t\int_\Omega (\nabla u \cdot \nabla v - fv)\, d^n x + \frac{t^2}{2}\int_\Omega \nabla v \cdot \nabla v d^n x\right)\bigg|_{t=0}$$
$$= \int_\Omega (\nabla u \cdot \nabla v - fv)\, d^n x = \int_\Omega (-\triangle u - f) v d^n x.$$

The final integration by parts shows $-\triangle u = f$ on $\Omega$. **q.e.d.**

Finally we remark that one can also prove the uniqueness of the solution with the help of this functional. The difference of two solutions solves the Dirichlet Problem for $f = 0$ and $g = 0$. In this case the functional is non-negative, and vanishes if and only if $u$ is constant. The boundary conditions forces this constant to be zero. By using this principle one can show in a larger class of functions, that this functional has a unique minimizer, which thereby solves the Dirichlet Problem.

# Chapter 4

# Heat Equation

In this chapter we investigate the heat equation

$$\dot{u} - \triangle u = 0$$

and the corresponding inhomogeneous variant

$$\dot{u} - \triangle u = f.$$

The unknown function $u$ is defined on an open domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and the inhomogeneity $f$ is a given function on $\Omega$. We shall extend some statements about harmonic functions to solutions of the heat equation.

This heat equation describes a diffusion process. This means a time-like evolution of space-like distributed quantities like heat, chemical concentration and others. Here the flow density is proportional to the negative of the gradient. Then the heat equation follows from the scalar conservation law.

## 4.1 Fundamental Solution

Since the heat equation is linear and contains only a first order derivative with respect to time and only second derivatives with respect to space, for any solution $u(x,t)$ and any $\lambda \in \mathbb{R}$ the function $u(\lambda x, \lambda^2 t)$ is also a solution. This scaling behaviour suggests to look for solutions which depend only on $\frac{x^2}{t}$. We invoke the following ansatz:

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad x \in \mathbb{R}^n, t \in \mathbb{R}^+.$$

Here $\alpha$ and $\beta$ are constants and $v : \mathbb{R}^n \to \mathbb{R}$ an unknown function. This ansatz is justified by the scaling behaviour $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$. With $\lambda = \frac{1}{t}$ we obtain

$v(\frac{x}{t^\beta}) = u(\frac{x}{t^\beta}, 1)$. This ansatz transforms the heat equation into the following PDE

$$-\alpha \cdot t^{-(\alpha+1)} v(y) - \beta t^{-(\alpha+1)} y \cdot \nabla v(y) - t^{-(\alpha+2\beta)} \triangle v(y) = 0 \quad \text{mit} \quad y = \frac{x}{t^\beta}.$$

If we set $\beta = \frac{1}{2}$, then this equation does not depend on $t$ and reduces to

$$\alpha v + \tfrac{1}{2} y \cdot \nabla v + \triangle v = 0.$$

Again we assume that $v$ is a function of $|y|$. With $v(y) = w(|y|)$ we obtain:

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0 \quad \text{with} \quad r = \frac{|\,x\,|}{\sqrt{t}}.$$

If we set $\alpha = \frac{n}{2}$, then we may integrate once:

$$\left(r^{n-1} w'\right)' + \tfrac{1}{2} \left(r^n w\right)' = 0 \qquad\qquad r^{n-1} w' + \tfrac{1}{2} r^n w = a.$$

The constant $a$ vanishes, if $w$ and $w'$ vanish at infinity.

$$w' = -\tfrac{1}{2} r w \qquad\qquad w = b \cdot e^{\frac{-r^2}{4}}.$$

For a special choice of the constants $a$ and $b$ we again obtain the fundamental solution.

**Definition 4.1.** *The fundamental solution of the heat equation is defined as*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \textit{for} \quad x \in \mathbb{R}^n, t > 0 \\ 0 & \textit{for} \quad x \in \mathbb{R}^n, t < 0 \end{cases}.$$

**Lemma 4.2.** *For all $t > 0$ the fundamental solution satisfies* $\displaystyle\int_{\mathbb{R}^n} \Phi(x,t) d^n x = 1$.

*Proof.* $$\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4t}} d^n x = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-x^2} d^n x = \frac{1}{\pi^{n/2}} \left( \int_{\mathbb{R}} e^{-x^2} dx \right)^n = 1.$$ **q.e.d.**

The fundamental solution is similar to a mollifier on $\mathbb{R}^n$. So we may expect that the convolution with $\Phi$ converges in the limit $t \downarrow 0$ like the identity.

**Theorem 4.3.** *For $h \in C_b(\mathbb{R}^n, \mathbb{R})$ the following function $u$ has the properties (i)-(iii):*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) h(y) d^n y$$

**(i)** $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$

**(ii)** $\dot{u} - \triangle u = 0$ *on* $\mathbb{R}^n \times \mathbb{R}^+$

**(iii)** $u$ *extends continuously and bounded to* $\mathbb{R}^n \times [0, \infty)$ *with* $\lim_{t \to 0} u(x,t) = h(x)$.

*Proof.* Since $\Phi(x,t)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^+$ the foregoing lemmas and the boundedness of $h$ implies that $u(x,t)$ is well defined, bounded and continuous on $\mathbb{R}^n \times [0, \infty)$. On $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ all partial derivatives of $(x,t) \mapsto \Phi(x-y,t)$ belong to $L^1(\mathbb{R}^n)$ considered as functions on $y \in \mathbb{R}^n$ and depend continuously on $(x,t) \in \mathbb{R}^n$. So they define a smooth map from $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ into $L^1(\mathbb{R}^n)$. The integral is a linear continuous operator from $L^1(\mathbb{R}^n)$ to $\mathbb{R}$. So $u$ is smooth. No (ii) follows, since $\Phi$ solves the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$. The continuity of $h$ implies uniform continuity on compact subsets. For any $\epsilon > 0$ and any $x$ in a compact subset of $\mathbb{R}^n$ there exists $\delta > 0$, such that $|h(x) - h(y)| < \epsilon$ for all $|x - y| < \delta$. Furthermore there exists $T > 0$, such that

$$\int_{\mathbb{R}^n \setminus B(0,\delta)} \Phi(y,t) d^n y = \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} \Phi(y,1) d^n y < \epsilon \qquad \text{for all } t < T.$$

This implies
$$|u(x,t) - h(x)| \leq \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(h(y) - h(x)) d^n y \right|$$
$$\leq \int_{B(x,\delta)} \Phi(x-y,t) \mid h(y) - h(x) \mid d^n y \quad + \int_{\mathbb{R}^n \setminus B(x,\delta)} \Phi(x-y,t) |h(y) - h(x)| d^n y$$
$$\leq \epsilon + 2\epsilon \sup\{|h(y)| \mid y \in \mathbb{R}^n\} \qquad \text{for all } t < T.$$

So $u(x,t)$ converges in the limit $t \downarrow 0$ uniformly on compact subsets of $\mathbb{R}^n$ to $h$. **q.e.d.**

In this limit $t \downarrow 0$ $\Phi$ converges as a distribution (and as a measure) to the $\delta$-distribution. Note that by this formula the speed of propagation is unbounded.

## 4.2 Inhomogeneous Initial value problem

In the forgoing section we constructed a solution of the initial value problem

$$\dot{u} - \triangle u = 0 \quad \text{and} \quad u(x,0) = h(x).$$

Duhamel's principle derives solutions of the inhomogeneous initial value problem from solutions of the homogeneous initial values problem. If we write the heat equation as $\dot{u} = \triangle u$ and recall that the Laplace operator is a linear map from the space of smooth functions on $\mathbb{R}^n$ into itself, then the heat equation becomes a linear ODE in the (infinite-dimensional) space of smooth functions on $\mathbb{R}^n$. For linear ODEs the variation of constants is also a method to obtain the solutions of the inhomogeneous equation in terms of homogeneous solutions. In fact if we take the integral over the interval $[0,t]$ of the corresponding homogeneous solutions which are at $s \in [0,t]$ equal to the inhomogeneity at $s$, then we obtain a solution of the inhomogeneous equation which vanishes at $t = 0$. Now Duhamel's principle is just the application of the variation of constants to the heat equation considered as an ODE in the space of functions on $\mathbb{R}^n$:

Let $\quad u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) d^n y ds.$ Then formally we obtain

$$\dot{u}(x,t) - \triangle u(x,t) = \lim_{s\to 0} \int_{\mathbb{R}^n} \Phi(x-y,s) f(y,t-s) d^n y +$$
$$+ \int_0^t \int_{\mathbb{R}^n} \left( \dot{\Phi}(x-y,t-s) - \triangle_x \Phi(x-y,t-s) \right) f(y,s) d^n y ds = f(x,t).$$

**Theorem 4.4** (Solution of the inhomogeneous initial value problem). *If $f$ is twice continuously and bounded differentiable on $\mathbb{R}^n \times [0,\infty)$, then*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) d^n y ds = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s) d^n y ds$$

*solves the inhomogeneous initial value problem*

$$\dot{u} - \triangle u = f \text{ on } \mathbb{R}^n \times \mathbb{R}^+ \text{ and } \qquad \lim_{t\to 0} u(x,t) = 0.$$

*Proof.* We already proved that $v_s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) d^n y$ solves on $\mathbb{R}^n \times (s,\infty)$ the initial value problem $\dot{v}_s - \triangle v_s = 0$ with $\lim_{t\to s} v_s(x,t) = f(x,t)$. So $v_s$ is on $\mathbb{R}^n \times [s,\infty)$ continuous. This implies for all $\epsilon > 0$ the relation

$$u_\epsilon(x,t) = \int_0^{t-\epsilon} v_s(x,t) ds = \int_0^{t-\epsilon} \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) d^n y ds$$

$$\dot{u}_\epsilon(x,t) - \triangle u_\epsilon(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t-(t-\epsilon)) f(y,t-\epsilon) d^n y = \int_{\mathbb{R}^n} \Phi(x-y,\epsilon) f(y,t-\epsilon) d^n y.$$

Theorem 4.3 (iii) implies $\lim_{\epsilon\to 0} \dot{u}_\epsilon - \triangle u_\epsilon = f$ on $\mathbb{R}^n \times \mathbb{R}^+$. On the other hand we have

$$u_\epsilon(x,t) = \int_0^{t-\epsilon} \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) d^n y ds = \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s) d^n y ds.$$

By the second integral in the Theorem and the assumptions on $f$ we conclude that

$$\lim_{\epsilon\to 0} \left( \dot{u}_\epsilon(x,t) - \triangle u_\epsilon(x,t) \right) = \left( \frac{\partial}{\partial t} - \triangle \right) \lim_{\epsilon\to 0} u_\epsilon(x,t) = \left( \frac{\partial}{\partial t} - \triangle \right) u(x,t)$$

holds. The continuity of $v$ gives $u(x,0) = 0$. **q.e.d.**

**Corollary 4.5.** *The inhomogeneous initial value problem has the following solution:*

$$\dot{u} - \triangle u = f \qquad\qquad u(x,0) = h(x)$$
$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)h(y)d^n y \;+ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds. \qquad \textbf{q.e.d.}$$

## 4.3 Mean Value Property

We use the fundamental solution $\Phi(x,t)$ in order to determine the value $u(x,t)$ as a mean value on some ball like domain which has to be chosen properly.

**Definition 4.6.** *For all* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ *and all* $r > 0$ *we define*

$$E(x,t,r) = \left\{(y,s) \in \mathbb{R}^{n+1} \mid s \le t, \Phi(x-y,t-s) \ge \tfrac{1}{r^n}\right\}$$

$$\begin{aligned}
e^{-\frac{|x-y|^2}{4(t-s)}} \ge \frac{(4\pi)^{n/2}(t-s)^{n/2}}{r^n} \quad &\iff \quad e^{\frac{|x-y|^2}{4(t-s)}} \le \frac{1}{\pi^{n/2}}\left(\frac{r^2}{4(t-s)}\right)^{n/2} \\
&\iff \quad \frac{\mid x-y\mid^2}{4(t-s)} \le \frac{n}{2}(2\ln(r) - \ln(4(t-s)) - \ln(\pi)) \\
&\iff \quad \mid x-y\mid^2 \le 2(t-s)n(2\ln(r) - \ln(t-s) - \ln(4\pi)).
\end{aligned}$$

**Theorem 4.7** (mean value property of the heat equation)**.** *Let* $u$ *be a solution of the heat equation on an open domain* $\Omega \subset \mathbb{R}^n \times \mathbb{R}$*. For any* $(x,t) \in \Omega$ *and any* $r > 0$ *with* $E(x,t,r) \subset \Omega$ *we have*

$$u(x,t) = \frac{1}{C_n r^n}\int_{E(x,t,r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}d^n y ds \qquad \text{with} \qquad C_n = \int_{E(0,0,1)} \frac{|y|^2}{s^2}d^n y ds.$$

*Proof.* Due to the translation invariance we may assume $(x,t) = (0,0)$. We define

$$\phi(r) = \frac{1}{r^n}\int_{E(0,0,r)} u(y,s)\frac{|y|^2}{s^2}d^n y ds = \frac{1}{r^n}\int_{E(0,0,r)} u(ry,r^2 s)\frac{|ry|^2}{(r^2 s)^2}d^n(ry)d(r^2 s) = \int_{E(0,0,1)} u(ry,r^2 s)\frac{|y|^2}{s^2}d^n y ds.$$

Here we used the fact that the bijective map $(y,t) \mapsto (ry, r^2 t)$ maps $E(x,t,1)$ onto $E(rx, r^2 t, r)$ since $\Phi(r(x-y), r^2 t) = r^{-n}\Phi(x-y,t)$. We calculate

$$\begin{aligned}
\phi'(r) &= \int_{E(0,0,1)} \frac{|y|^2}{s^2}\left(y\cdot\nabla u(ry,r^2 s) + 2rs\dot{u}(ry,r^2 s)\right)d^n y ds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \frac{|y|^2}{s^2} y\cdot\nabla u(y,s)d^n y ds + \frac{1}{r^{n+1}}\int_{E(0,0,r)} 2\dot{u}(y,s)\frac{\mid y\mid^2}{s}d^n y ds
\end{aligned}$$

For $\psi = -\frac{n}{2}\ln(-4\pi s) + \frac{|y|^2}{4s} + n\ln r$ we obtain $E(0,0,r) = \{(y,s) \mid \psi(y,s) \geq 0\}$. Furthermore $\psi$ vanishes on the boundary of $E(0,0,r)$.

$$\begin{aligned}
\frac{1}{r^{n+1}}\int_{E(0,0,r)} 2\dot{u}\frac{|\, y\,|^2}{s}d^nyds &= \frac{1}{r^{n+1}}\int_{E(0,0,r)} 4\dot{u}y\cdot\nabla\psi d^nyds \\
&= -\frac{1}{r^{n+1}}\int_{E(0,0,r)} (4n\dot{u}\psi + 4\psi y\cdot\nabla\dot{u})d^nyds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} (-4n\dot{u}\psi + 4\dot{\psi}y\cdot\nabla u)d^nyds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \left(-4n\dot{u}\psi + 4\left(-\frac{n}{2s} - \frac{|\, y\,|^2}{4s^2}\right) y\cdot\nabla u\right)d^nyds.
\end{aligned}$$

Hence we have
$$\begin{aligned}
\phi'(r) &= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \left(-4n\triangle u\psi - \frac{2n}{s}y\cdot\nabla u\right)d^nyds \\
&= \frac{1}{r^{n+1}}\int_{E(0,0,r)} \left(4n\nabla u\cdot\nabla\psi - \frac{2n}{s}y\cdot\nabla u\right)d^nyds = 0.
\end{aligned}$$

This shows that $\phi$ is constant. By the continuity of $u$ and by the equation

$$\frac{1}{r^n}\int_{E(0,0,r)} \frac{|\, y\,|^2}{s^2}d^nyds = \frac{1}{r^n}\int_{E(0,0,r)} \frac{|\, ry\,|^2}{(r^2s)^2}d^nrydr^2s = \int_{E(0,0,1)} \frac{|\, y\,|^2}{s^2}d^nyds = C_n$$

we obtain $\lim\limits_{r\to 0}\phi(r) = C_n u(0,0)$. **q.e.d.**

It is possible to calculate the constant explicitly. The heat ball $E(0,0,1)$ contains all $(y,s) \in \mathbb{R}^n \times (-\infty, 0]$ with $s \leq 0$ and $|y|^2 \leq -2sn(2\ln(1) - \ln(-s) - \ln(4\pi)) = 2ns\ln(-4\pi s)$. By the positivity of $|y|^2$ we have $-4\pi s < 1$ and $-\frac{1}{4\pi} < s < 0$. This gives

$$\begin{aligned}
C_n &= \int_{-\frac{1}{4\pi}}^0 \frac{1}{s^2}\int_{B(0,\sqrt{2ns\ln(-4\pi s)})\subset\mathbb{R}^n} |y|^2 d^nyds \\
&= \int_{-\frac{1}{4\pi}}^0 \frac{1}{s^2}\int_0^{\sqrt{2ns\ln(-4\pi s)}} n\omega_n r^{n+1}drds = \int_0^{\frac{1}{4\pi}} \frac{1}{s^2}\int_0^{\sqrt{-2ns\ln(4\pi s)}} n\omega_n r^{n+1}drds \\
&= n\omega_n\int_0^{\frac{1}{4\pi}} \frac{1}{s^2}\left[\frac{r^{n+2}}{n+2}\right]_0^{\sqrt{2ns\ln\left(\frac{1}{4\pi s}\right)}} ds = \frac{n\omega_n(2n)^{\frac{n+2}{2}}}{n+2}\int_0^{\frac{1}{4\pi}} \left(s\ln\left(\frac{1}{4\pi s}\right)\right)^{\frac{n+2}{2}}\frac{ds}{s^2}.
\end{aligned}$$

Now we substitute $4\pi s = e^{-\frac{2}{n}t^2}$ with $\frac{2}{n}t^2 = \ln(\frac{1}{4\pi s})$ and $\frac{4}{n}tdt = -\frac{ds}{s}$.

$$\begin{aligned}
C_n &= \frac{n\omega_n(2n)^{\frac{n+2}{2}}}{n+2}\int_0^\infty \left(\frac{e^{-\frac{2}{n}t^2}}{4\pi}\right)^{\frac{n}{2}} \left(\tfrac{2}{n}t^2\right)^{\frac{n+2}{2}} \tfrac{4}{n}tdt = \frac{n\omega_n 2^{n+2-n+1}}{(n+2)n\pi^{\frac{n}{2}}}\int_0^\infty 2te^{-t^2}t^{n+2}dt \\
&= -\frac{8\omega_n}{(n+2)\pi^{\frac{n}{2}}}\left[e^{-t^2}t^{n+2}\right]_0^\infty + \frac{8\omega_n}{\pi^{\frac{n}{2}}}\int_0^\infty e^{-t^2}t^{n+1}dt = \frac{4\omega_n}{\pi^{\frac{n}{2}}}\int_0^\infty 2te^{-t^2}t^n dt \\
&= -\frac{4\omega_n}{\pi^{\frac{n}{2}}}\left[e^{-t^2}t^n\right]_0^\infty + \frac{4}{\pi^{\frac{n}{2}}}\int_0^\infty n\omega_n e^{-t^2}t^{n-1}dt = \frac{4}{\pi^{\frac{n}{2}}}\left(\int_{-\infty}^\infty e^{-x^2}dx\right)^n = 4.
\end{aligned}$$

## 4.4 Maximum Principle

For any open domain $\Omega \subset \mathbb{R}^n$ we define the parabolic cylinder as $\Omega_T = \Omega \times (0,T]$. The parabolic boundary $\partial\Omega_T$ of $\Omega_T$ is defined as $\bar{\Omega}_T \setminus \Omega_T$. It is the union of $(\partial\Omega \times (0,T]) \cup \big(\bar{\Omega} \times 0\big)$ and does not contain at time $t = T$ points inside of $\Omega$.

**Theorem 4.8** (strong maximum principle of the heat equation)**.** *Let $\Omega$ be path connected (i.e. any $x, x' \in \Omega$ are connected by a continuous path from $x$ to $x'$) and let $u$ be twice continuously differentiable solution of the heat equation on $\Omega_T$ with continuous extension to $\bar{\Omega}_T$. If $u$ takes the maximal value in $\Omega_T$, then $u$ is constant on $\bar{\Omega}_T$.*

*Proof.* Let $(x_0, t_0)$ be an element of $\Omega_T$ at which $u$ takes the maximal value. Then there exists $r_0 > 0$ such that $E(x_0, t_0, r_0)$ is contained in $\Omega_T$. By the mean value property $u$ is constant on $E(x_0, t_0, r_0)$. Since $\Omega$ is path connected there exists for any $(x,t) \in \Omega \times (0, t_0)$ finitely many $E(x_0, t_0, r_0), E(x_1, t_1, r_1), \ldots, E(x_n, t_n, r_n)$ in $\Omega \times (0, t_0)$ containing the points $(x_1, t_1), \ldots, (x_n, t_n), (x, t)$. So $u$ is constant on $\bar{\Omega}_T$. **q.e.d.**

**Theorem 4.9** (weak maximum prinziple for the heat equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u$ a twice differentiable solution of the heat equation on $\Omega_T$ which extends continuously to $\bar{\Omega}_T$. Then the maximum of $u$ is taken on $\partial\Omega_T$.* **q.e.d.**

Again this Maximum principle implies the uniqueness of a boundary value problem:

**Theorem 4.10** (uniqueness of the boundary value problem)**.** *On an open and bounded domain $\Omega \subset \mathbb{R}^n$ there exists at most one solution $u$ of the inhomogeneous heat equation which extends continuously to $\bar{\Omega}_T$ and coincides on $\partial\Omega_T$ with a given function.*

*Proof.* Apply the weak maximum principle to the difference of two solutions. **q.e.d.**

In order to prove on $\mathbb{R}^n \times \mathbb{R}^+$ the uniqueness of the initial value problem we need as in the case of the Poisson problem a bound on the growth at infinity.

**Theorem 4.11** (maximum prinziple for the Cauchy problem)**.** *For a bounded and continuous initial value $h$ on $\mathbb{R}^n$ let $u$ be a solution on $\mathbb{R}^n \times (0,T]$ of the problem:*

$$\dot{u} - \triangle u = 0 \textit{ on } \mathbb{R}^n \times (0,T) \qquad\qquad u(x,0) = h(x) \textit{on } \mathbb{R}^n \times \{0\},$$

*which is bounded by* $\quad u(x,t) \leq Ae^{a|x|^2} \quad$ *on* $\quad \mathbb{R}^n \times [0,T]$
*for some positive constants $A, a > 0$. Then $u$ is bounded by* $\quad \sup\limits_{\mathbb{R}^n \times [0,T]} u = \sup\limits_{\mathbb{R}^n} h.$

*Proof.* We first consider the case where $a$ and $T$ obey $4aT < 1$. Then there exists an $\epsilon > 0$ with $4a(T+\epsilon) < 1$. For all $y \in \mathbb{R}^n$ and $\mu > 0$ the following function $v$ solves together with the fundamental solution on $\mathbb{R}^n \times (0, T+\epsilon)$ the heat equation:

$$v(x,t) = u(x,t) - \mu(T+\epsilon-t)^{-\frac{n}{2}} \exp\left(\tfrac{|x-y|^2}{4(T+\epsilon-t)}\right)$$

On any domain of the form $\Omega_T = B(y,r) \times (0,T]$ the weak maximum principle applies. Due to the assumptions both function $u$ and $h$ are bounded by $Ae^{a|x|^2}$. Since the inequality $\frac{1}{4(T+\epsilon-t)} > a$ holds for $t > 0$ there exists for any $\mu > 0$ a $R > 0$ such that $v(x,t) \leq \sup\{h(x) \mid x \in \mathbb{R}\}$ holds for all $r > R$ on $\partial B(y,r)_T = B(y,r)\times\{0\} \cup \partial B(y,r) \times (0,T]$. Hence the weak maximum principle implies $v(x,t) \leq \sup\{h(x) \mid x \in \mathbb{R}^n\}$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. This holds for all $\mu > 0$ and by continuity also for $\mu = 0$.

For $4aT \geq 1$ we decompose the time interval into $[0,T] = [0,T_1] \cup \ldots \cup [T_M, T]$ with the property $4a(T_{m+1} - T_m) < 1$. By induction the general case follows. **q.e.d.**

**Theorem 4.12** (existence and uniqueness of the initial value problem)**.** *For $h \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$ there exists at most one solution of the initial value problem*

$$\dot{u} - \triangle u = f \textit{ on } \mathbb{R}^n \times (0,T) \qquad\qquad u = h \textit{ on } \mathbb{R}^n \times \{0\}$$

*which is bounded by $|u(x,t)| \leq Ae^{a|x|^2}$ on $\mathbb{R}^n \times [0,T_0]$ for some $A>0$, $a>0$ and $T_0>0$.*

*If $h$ and $f$ are bounded by $|h(x)| \leq Ae^{a|x|^2}$ and $f(x,t) \leq Ae^{a|x|^2}$ on $(x,t) \in \mathbb{R}^n \times [0,T]$ for some $A > 0$, $a > 0$, and $T > 0$ then this unique solution is given by*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)h(y)d^n y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)d^n y ds.$$

*This solution might explode at some finite $t \uparrow T_0 \geq \frac{1}{16a}$.*

*Proof.* By the maximum principle for for the Cauchy problem Theorem 4.11 the difference of any two solutions vanishes. This shows uniqueness.

In order to prove existence we apply Corollary 4.5 and show that the given $u(x,t)$ has a bound as stated. For $0 \le t-s \le \frac{1}{16a}$ we have $-\frac{|x-y|^2}{4(t-s)} \le -2a|x-y|^2 - \frac{\epsilon|x-y|^2}{8(t-s)}$ and

$$e^{-2a|x|^2}e^{a|y|^2}\Phi(x-y,t-s) \le \frac{e^{-2a|x|^2+a|y|^2-2a|x-y|^2}e^{-\frac{|x-y|^2}{8(t-s)}}}{(4\pi(t-s))^{n/2}} = \frac{2^{n/2}e^{-a|2x-y|^2}e^{-\frac{|x-y|^2}{8(t-s)}}}{(8\pi(t-s))^{n/2}},$$
$$\Phi(x-y,t-s) \le 2^{n/2}\Phi(x,y,2(t-s))e^{2a|x|^2}e^{-a|y|^2}.$$

The inequalities $|h(x)| \le Ae^{a|x|^2}$ and $f(x,t) \le Ae^{a|x|^2}$ which hold for $(x,t) \in \mathbb{R}^n \times [0,T]$ first imply $u(x,t) \le A'e^{2a|x|^2}$ for $t \in [0,T_0]$ with $T_0 = \min\{T, \frac{1}{16a}\}$ and some $A' > 0$. For $f = 0$ the maximum principle for the Cauchy problem Theorem 4.11 implies

$$\begin{aligned}\sup_{(x,t)\in\mathbb{R}^n\times[0,T_0]} e^{-2a|x|^2}|u(x,t)| &\le 2^{\frac{n}{2}} \sup_{(x,t)\in\mathbb{R}^n\times[0,T_0]} \int_{\mathbb{R}^n} \Phi(x-y,2t)e^{-a|y|^2}|h(y)|d^ny \\ &\le 2^{\frac{n}{2}} \sup_{y\in\mathbb{R}^n} e^{-a|y|^2}|h(y)| \le 2^{\frac{n}{2}}A.\end{aligned}$$

For non vanishing $f$ we get $\sup\limits_{(x,t)\in\mathbb{R}^n\times[0,T_0]} e^{-2a|x|^2}|u(x,t)| \le 2^{\frac{n}{2}}A\left(1+\int_0^t ds\right) \le 2^{\frac{n}{2}}A(1+T)$. So the given $u$ obeys locally in $t \in [0,T]$ a bound as stated and is the unique solution, as long as it obeys such a bound. The solution $u(x,t) = (T_0-t)^{-\frac{n}{2}}\exp\left(\frac{|x|^2}{4(T_0-t)}\right)$ of the homogeneous heat equation shows that this might not be true for all $t \in [0,T]$. **q.e.d.**

Improved arguments yields the sharp bound on the extinction time $T_0 \ge \frac{1}{4a}$.

**Example 4.13.** *We show by a counterexample the non uniqueness of solutions without any bound of the initial value problem. For $n = 1$ we make the ansatz*

$$u(x,t) = \sum_{l=0}^{\infty} g_l(t)x^l, \qquad \dot{u}(x,t) - \triangle u(x,t) = \sum_{l=0}^{\infty}(\dot{g}_l(t) - (l+2)(l+1)g_{l+2}(t))x^l.$$

*For a given function $g_0(t) = g(t)$ we thus obtain the following formal solution of the homogeneous heat equation:*

$$u(x,t) = \sum_{l=0}^{\infty} \frac{g^{(l)}(t)}{(2l)!}x^{2l}.$$

*We now show that for $g(t) = \exp(-t^{-2})$ this power series indeed converges to a solution such that on every compact subset of $\mathbb{R}^n$ the uniform limit $t \downarrow 0$ vanishes. We first calculate $g^{(l)}(t)$ for any $l \in \mathbb{N}_0$ by a real polynomial $p_l$ of degree $l$ solving the relation*

$$g^{(l)}(t) = t^{-l}p_l(t^{-2})\exp(-t^{-2}) \quad \textit{with} \quad p_{l+1}(z) = 2zp_l(z) - lp_l(z) - 2zp_l'(z).$$

*This recursion relation for $p_l$ follows by differentiating by $t$ The first two polynomials are $p_0(z) = 1$ and $p_1(z) = 2z$. We claim that the coefficient of $p_l(z)$ in front of $z^k$ is bounded by $\frac{l!7^l}{2^k k!}$. For $l = 0$, $k = 0$ this is clear. By induction we obtain with $k \le l+1$*

$$2\frac{l!7^l}{2^{k-1}(k-1)!} + l\frac{l!7^l}{2^k k!} + 2k\frac{l!7^l}{2^k k!} = \frac{l!7^l(4k+l+2k)}{2^k k!} \le \frac{l!7^l 7(l+1)}{2^k k!} \le \frac{(l+1)!7^{l+1}}{2^k k!}.$$

*This proves the claim. Using the inequalities $\frac{l!}{(2l)!} = \frac{1}{2^l 1\cdot 3\cdots(2l-1)} \le \frac{1}{2^l l!}$ we conclude*

$$|u(x,t)| \le \sum_{l=0}^{\infty} \frac{l!7^l x^{2l}}{(2l)!t^l} \sum_{k=0}^{l} \frac{g(t)}{2^k k! t^{2k}} \le \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{7x^2}{2t}\right)^l \sum_{k=0}^{\infty} \frac{g(t)}{k!}\left(\frac{1}{2t^2}\right)^k = \exp\left(\frac{7x^2}{2t} - \frac{1}{2t^2}\right).$$

*Therefore the series converges absolutely and for $t \downarrow 0$ uniformly on compact sets to $0$.*

In analogy to the Laplace equation one can show the uniqueness of the boundary value problem Theorem 4.10 and of the initial value problem Theorem 4.12 also with the monotonicity of an energy functional. We define

$$e(t) = \int_\Omega u^2(x,t)d^n x.$$

If $u$ solves the homogeneous heat equation and vanishes at the boundary of $\Omega$, then this functional is monotonically decreasing with respect to time:

$$\dot{e}(t) = 2\int_\Omega u(x,t)\dot{u}(x,t)d^n x = 2\int_\Omega u(x,t)\triangle u(x,t)d^n x = -2\int_\Omega (\nabla u(x,t))^2 d^n x \le 0.$$

If $u(x,t)$ vanishes at $t = 0$, and if $u(\cdot,t)$ and $\nabla u(\cdot,t)$ are square integrable for $t > 0$, then $u$ vanishes identically since $\nabla u(\cdot,t)$ vanishes and $u(\cdot,t)$ is constant for $t > 0$.

## 4.5 Heat Kernel

In analogy to the Green's function of the Laplace equation we define for open subsets $\Omega \subset \mathbb{R}^n$ the heat kernel $H_\Omega$.

**Definition 4.14.** *For an open domain $\Omega \subset \mathbb{R}^n$ the heat kernel $H_\Omega : \Omega \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$ of $\Omega$ is characterised by the following two properties:*

**(i)** *For $(x,t) \in \Omega \times \mathbb{R}^+$ $y \mapsto H_\Omega(x,y,t)$ extends continuously to $\bar{\Omega}$ with value $0$ on $\partial\Omega$.*

**(ii)** *For $x \in \Omega$ the function $(y,t) \mapsto H_\Omega(x,y,t) - \Phi(x-y,t)$ solves the homogeneous heat equation and extends continuously to $\bar{\Omega} \times \mathbb{R}_0^+$ with value $0$ on $(y,t) \in \bar{\Omega} \times \{0\}$.*

**Lemma 4.15.** *If $u$ and $v$ are two functions on $\Omega \times \mathbb{R}^+$ with an open domain $\Omega \subset \mathbb{R}^n$ which all three have appropriate regularity, then we have*

$$\begin{aligned}
& \int_0^T \int_\Omega u(x,t)(\partial_t v(x,T-t) + \triangle v(x,T-t))d^n x dt \\
+ \quad & \int_0^T \int_\Omega (\partial_t u(x,t) - \triangle u(x,t)) v(x,T-t) d^n x dt = \\
= \quad & \int_0^T \int_{\partial\Omega} (u(y,t)\nabla_y v(y,T-t) - \nabla_y u(y,t) v(y,T-t)) \cdot N(y) d\sigma(y) dt \\
+ \quad & \int_\Omega (u(x,T)v(x,0) - u(x,0)v(x,T)) d^n x.
\end{aligned}$$

*Proof.* The fundamental theorem of calculus gives for the terms with $t$-derivatives the final integral over $\Omega$ and the boundary terms of a partial integration with respect to $y$ yields the two gradients with respect to $x$ in the integral over $\partial\Omega$. **q.e.d.**

The function $v(y,t) = H_\Omega(x,y,t)$ has at $v(x,0)$ a singularity and is not defined there. Hence we integrate with respect to $dt$ over the interval $t \in [0, T-\epsilon]$ instead of $t \in [0,T]$ and take afterwards the limit $\epsilon \downarrow 0$. Then Theorem 4.3 gives

$$\begin{aligned}
& \int_0^T \int_\Omega (\dot{u}(y,t) - \triangle u(y,t)) H_\Omega(x,y,T-t) d^n y dt = \\
= & \int_0^T \int_{\partial\Omega} u(z,t) \nabla_z H_\Omega(x,z,T-t) \cdot N(z) d\sigma(z) dt + u(x,T) - \int_\Omega u(y,0) H_\Omega(x,y,T) d^n y.
\end{aligned}$$

This shows also $$u(x,T) = \int_0^T \int_\Omega (\dot{u}(y,t) - \triangle u(y,t)) H_\Omega(x,y,T-t) d^n y dt$$
$$- \int_0^T \int_{\partial\Omega} u(z,t) \nabla_z H_\Omega(x,z,T-t) N(z) d\sigma(z) dt + \int_\Omega u(y,0) H_\Omega(x,y,T) d^n y.$$

**Theorem 4.16** (solution of the initial and boundary value problem)**.** *Let $f$ be a function on $\Omega \times (0,T)$, $g$ a function on $\partial\Omega \times [0,T]$ and $h$ a function on $\Omega$ which together with the open domain $\Omega \subset \mathbb{R}^n$ have appropriate regularity such that all appearing integrals converge absolutely. Then*

$$\begin{aligned}
u(x,T) = & \int_0^T \int_\Omega f(y,t) H_\Omega(x,y,T-t) d^n y dt \\
& - \int_0^T \int_{\partial\Omega} g(z,t) \nabla_z H_\Omega(x,z,T-t) N(z) d\sigma(z) dt + \int_\Omega h(y) H_\Omega(x,y,T) d^n y
\end{aligned}$$

*is the unique solution of the initial and boundary value problem*

$$\dot{u} - \triangle u = f \text{ on } \Omega \times (0,T) \qquad u = g \text{ on } \partial\Omega \times [0,T] \qquad u(x,0) = h(x) \text{ on } \Omega.$$

We prepare the proof by showing that the heat kernel is symmetric:

**Lemma 4.17.** *For all $T > 0$ and $x, y \in \bar{\Omega}$ we have $H_\Omega(x,y,T) = H_\Omega(y,x,T)$.*

*Proof.* We insert $u(z,t) = H_\Omega(x,z,t)$ and $v(z,t) = H_\Omega(y,z,t)$ in Lemma 4.15. By Theorem 4.3 (iii) and the property (ii) of the heat kernel the following integral vanishes:

$$\int_\Omega (H_\Omega(x,z,T)H_\Omega(y,z,0) - H_\Omega(x,z,0)H_\Omega(y,z,T))d^nz = H_\Omega(x,y,T) - H_\Omega(y,x,T). \textbf{q.e.d.}$$

*Sketch of the proof of Theorem 4.16.* The case $f = 0 = g$ follows from the defining properties of the heat kernel. This implies that in the second case $g = 0 = h$

$$v(x,T) = \int_\Omega H_\Omega(x,y,T-t)f(y,t)d^ny \text{ solves the initial value problem}$$

$$\dot{v} - \triangle v = 0 \text{ on } \Omega \times (t,\infty) \quad v(x,t) = f(x,t) \text{ on } \Omega \times \{t\} \quad v(x,t) = 0 \text{ on } \partial\Omega \times [0,\infty].$$

If we assume that $f$ has appropriate regularity and extends twice continuously differentiable to $\bar{\Omega} \times [0,T]$ as in the homogeneous initial value problem Theorem 4.4, then

$$u(x,T) = \int_0^T \int_\Omega H_\Omega(x,y,T-t)f(y,t)d^nydt \qquad \text{solves the initial value problem}$$

$$\dot{u}(x,t) - \triangle u(x,t) = f \text{ on } \Omega \times (0,T) \quad u(x,0) = 0 \text{ on } \Omega \quad u(x,t) = 0 \text{ on } \partial\Omega \times [0,T].$$

Finally we consider the inhomogeneous boundary value problem: In this case $u$ solves

$$\dot{u}(x,t) - \triangle u(x,t) = 0 \text{ on } \Omega \times (0,T) \quad u(x,0) = 0 \text{ on } \Omega \quad u(x,t) = g \text{ on } \partial\Omega \times [0,T].$$

We first extend any function $g$ on $\partial\Omega \times [0,T]$ with appropriate regularity to $\Omega \times [0,T]$ such that it vanishes outside a tubular neighbourhood of $\partial\Omega \times [0,T]$. If we subtract from this extension $\tilde{u}$ the solution of $f = \dot{\tilde{u}} - \triangle\tilde{u}$ and $h(x) = \tilde{u}(x,0)$ then we obtain a solution of the desired boundary value problem. **q.e.d.**

The appropriate regularity conditions depend on the heat kernel and therefore also on the domain. All the time we assumed that the divergence theorem holds for the open domain $\Omega \subset \mathbb{R}^n$. Before we construct the heat kernel for some special domains, we prove the following general property of the heat kernel:

**Lemma 4.18.** *For any bounded connected open domain $\Omega \subset \mathbb{R}^n$ the corresponding heat kernel is positive on the corresponding parabolic cylinder, if it exists.*

*Proof.* The fundamental solution $\Phi(x,t)$ is positive on $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. For bounded open domains $\Omega \subset \mathbb{R}^n$ and given $x \in \Omega$ the difference $\Phi(x-y,t) - H_\Omega(x,y,t)$ of the fundamental solution minus the heat kernel is the unique solution of the heat equation on $\Omega \times [0,T]$ which vanishes on $\Omega \times \{t=0\}$ and coincides on $\partial\Omega \times [0,T]$ with $\Phi(x-y,t)$. This solution is for all $\epsilon > 0$ on $\Omega \times \{t=\epsilon\}$ and on $\partial\Omega \times [0,T]$ not larger than $\Phi(x-y,t)$. By the Maximum Principle it is smaller than $\Phi(x-y,t)$ and $H_\Omega(x,y,t)$ is positive. **q.e.d.**

## 4.6 Spectral Theory and the Heat Equation

In this section we solve the initial value problem

$$\dot{u} - \triangle u = 0 \quad \text{on} \quad \Omega \times [0,T] \qquad u = 0 \quad \text{on } \partial\Omega \times [0,T] \qquad u = h \quad \text{on } \Omega \times \{0\}$$

with the help of the Laplace operator on $\Omega$. If $h$ is an eigenfunction of the Laplace operator:

$$-\triangle h = \lambda h \quad \text{on} \quad \Omega \quad \text{and} \quad h|_{\partial\Omega} = 0,$$

then the initial value problem can be solved by the following ansatz:

$$u(x,t) = \varphi(t)h(x) \quad \dot{\varphi}(t)h(x) + \lambda\varphi(t)h(x) = 0.$$

The general solution is $\dot{\varphi} = -\lambda\varphi$, $\varphi(t) = e^{-\lambda(t-t_o)}$. With $\varphi(0) = 1$ we obtain the unique solution of the corresponding initial value problem $u(x,t) = e^{-\lambda t}h(x)$. By linearity the corresponding solution for initial value $h = h_1 + \ldots + h_M$ with $-\triangle h_i = \lambda_i h_i$ on $\Omega$ and $h_i|_{\partial\Omega} = 0$ is given by $u(x,t) = e^{-\lambda_1 t}h_1(x) + \ldots + e^{-\lambda_M t}h_M(x)$. Hence it suffices to decompose $h$ into a sum of eigenfunctions of the Laplace operator on $\Omega$ with Dirichlet boundary conditions.

To explain this strategy we first interpret the fundamental solution as such a decomposition. On $\mathbb{R}^n$ the Laplace operator has the following eigenfunctions:

$$-\triangle e^{2\pi ik\cdot x} = 4\pi^2 k^2 e^{2\pi ikx}.$$

The equality $\displaystyle\int_{\mathbb{R}^n} e^{\left(2\pi ik\sqrt{t}+\frac{x}{2\sqrt{t}}\right)^2} d^n k = \frac{1}{(2\pi\sqrt{t})^n}\int_{\mathbb{R}^n} e^{(ik+\frac{x}{2\sqrt{t}})^2} d^n k = \frac{1}{(4\pi t)^{n/2}},$
holds for all imaginary $x$ and by analytic continuation for all $x \in \mathbb{R}^n$. This implies

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} = \int_{\mathbb{R}^n} e^{\left(2\pi ik\sqrt{t}+\frac{x-y}{2\sqrt{t}}\right)^2} e^{-\frac{(x-y)^2}{4t}} d^n k = \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i(x-y)k} d^n k.$$

So by our considerations above the solution of the initial value problem

$$\dot{u} - \triangle u = 0 \quad \text{on} \quad \mathbb{R}^n \times [0,T] \qquad\qquad u(x,0) = h \quad \text{on} \quad \mathbb{R}^n$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-4\pi^2k^2t}e^{2\pi i(x-y)k}h(y)d^nkd^ny.$$

For an integrable function $h$ we can apply Fubini's Theorem. So for continuous and integrable $h$ we conclude from $\lim\limits_{t\downarrow 0} u(x,t) = h(x)$ also

$$h(x) = \lim_{t\downarrow 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-4\pi^2k^2t}e^{2\pi i(x-y)k}h(y)d^nyd^nk.$$

We define the Fourier transform of $h$ as $\quad \hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi iky}h(y)d^ny.\quad$ This gives

$$u(x,t) = \int_{\mathbb{R}^n} e^{-4\pi^2k^2t}e^{2\pi ikx}\hat{h}(k)d^nk \quad \text{and} \quad h(x) = \lim_{t\downarrow 0}\int_{\mathbb{R}^n} e^{-4\pi^2k^2t}e^{2\pi ikx}\hat{h}(k)d^nk.$$

**Definition 4.19.** *Let $\mathcal{S}$ be the so called Schwartz space which contains all smooth complex valued functions on $\mathbb{R}^n$ all whose partial derivatives decay faster than every negative power of the coordinate.*

**Lemma 4.20.** *The Fourier transformation maps the Schwartz space into itself. The inverse map is given by*

$$\mathsf{P}\circ\mathcal{F}:\mathcal{S}\to\mathcal{S}, \quad \hat{h}\mapsto h, \quad \textit{with} \quad h(x) = \int_{\mathbb{R}^n} e^{2\pi ikx}\hat{h}(k)d^nk.$$

*Proof.* By two partial integrations we calculate

$$-\widehat{\triangle h}(k) = -\int_{\mathbb{R}^n} e^{-2\pi iky}\triangle h(y)d^ny = \int_{\mathbb{R}^n} 4\pi^2k^2e^{-2\pi iky}h(y)d^ny = 4\pi^2k^2\hat{h}(k).$$

So by $|\hat{h}(k)| \le \int_{\mathbb{R}^n}|h(y)|d^ny$ the Fourier transform of any Schwartz function decays faster then every inverse power of the coordinate. For any $h \in C_0^\infty(\mathbb{R}^n,\mathbb{C})$ we obtain

$$\|\hat{h}\|_\infty \le \|h\|_{L^1(\mathbb{R}^n)}.$$

Since $C_0^\infty(\mathbb{R}^n,\mathbb{C})$ is dense in $L^1(\mathbb{R}^n)$, the Fourier transform extends to a continuous linear map from $L^1(\mathbb{R}^n)$ into the Banach space $C_b(\mathbb{R}^n,\mathbb{C})$. Furthermore, we have

$$|\partial_i\hat{h}(k)| = \left|\int_{\mathbb{R}^n} -2\pi iy_ie^{-2\pi iyk}h(y)d^ny\right| \le 2\pi\||y|h(y)\|_{L^1(\mathbb{R}^n)}.$$

So $\hat{h}$ is smooth, if $h$ decays faster than every inverse power of the coordinate. So the Fourier transform of an integrable function is continuous and the Fourier transform of a Schwartz function is smooth.

Theorem 4.3 implies for any $h \in \mathcal{S}$

$$h(x) = \lim_{t \downarrow 0} \int_{\mathbb{R}^n} e^{-4\pi^2 k^2 t} e^{2\pi i k x} \hat{h}(k) d^n k \quad \text{with} \quad \hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k y} h(y) d^n y.$$

Since $e^{-4\pi^2 k^2 t}$ converges in the limit $t \downarrow 0$ on any compact subset $K \subset \mathbb{R}^n$ uniformly to 1 and since $\hat{h} \in \mathcal{S}$ belongs to $L^1(\mathbb{R}^n)$, we also have $P \circ \mathcal{F} \circ \mathcal{F} = \mathbb{1}_{\mathcal{S}}$ and $\mathcal{F} \circ \mathcal{F} = \mathsf{P}$, respectively. Now the equation

$$\int_{\mathbb{R}^n} e^{2\pi i k x} \hat{h}(k) d^n k = \int_{\mathbb{R}^n} e^{-2\pi i k x} \hat{h}(-k) d^n k$$

implies $\mathsf{P} \circ \mathcal{F} = \mathcal{F} \circ \mathsf{P}$ and henceforth also $\mathcal{F} \circ \mathsf{P} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F} \circ \mathsf{P} = \mathbb{1}_{\mathcal{S}}$. **q.e.d.**

For any Schwartz function $h$ we apply Fubini's Theorem and obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \hat{h}(k) \bar{\hat{h}}(k) d^n k &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{h}(k) \bar{h}(y) e^{2\pi i k y} d^n y d^n k \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{h}(k) e^{2\pi i k y} \bar{h}(y) d^n k d^n y = \int_{\mathbb{R}^n} h(y) \bar{h}(y) d^n y.
\end{aligned}$$

This shows that the Fourier transform preserves the $L^2(\mathbb{R}^n)$-norm. Since the Schwartz space is dense in $L^2(\mathbb{R}^n)$ this implies that the Fourier transform extends to an unitary operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

**Definition 4.21.** *For any open connected domain $\Omega \subset \mathbb{R}^n$ let $W^{2,2}_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in the Hilbert space with the scalar product*

$$\langle f, g \rangle_{W^{2,2}_0(\Omega)} = \int_\Omega (\triangle f) \triangle \bar{g} d^n x + \int_\Omega f \bar{g} d^n x.$$

All functions $h \in C^\infty_0(\Omega)$ obey

$$\langle \triangle h, \triangle h \rangle_{L^2(\Omega)} = \int_\Omega (\triangle h) \triangle \bar{h} d^n x \leq \langle h, h \rangle_{W^{2,2}_0(\Omega)}.$$

Therefore for any $h \in W^{2,2}_0(\Omega)$ the function $\triangle h$ belongs to $L^2(\Omega)$. For $f \in L^2(\Omega)$ the Cauchy Schwarz inequality implies

$$\left| \langle f, \triangle h \rangle_{L^2(\Omega)} \right| \leq \|f\|_{L^2(\Omega)} \cdot \|h\|_{W^{2,2}_0(\Omega)}.$$

A sequence $(h_n)_{n \in \mathbb{N}}$ in $C^\infty_0(\Omega)$ converges together with $(\triangle h_n)_{n \in \mathbb{N}}$ in $L^2(\Omega)$, if and only if it converges in $W^{2,2}(\Omega)$. So the operator $H = -\triangle$ is a closed self adjoint operator on $L^2(\Omega)$ with domain $W^{2,2}_0(\Omega) \subset L^2(\Omega)$. By the inequality

$$\int_\Omega (-\triangle h) \bar{h} d^n x = \int_\Omega |\nabla h|^2 \geq 0 \quad \text{for all} \quad h \in C^\infty_0(\Omega),$$

$H$ is non negative. Hence the operator $H$ has a spectral decomposition and $e^{-tH}$ is a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$ such that the following equation holds:

$$\|e^{-tH}h\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)}.$$

This shows that $u(x,t) = (e^{-tH}h)(x)$ solves $\dot{u}(x,t) = -(He^{-tH})(x) = \triangle u(x,t)$ with Dirichlet boundary condition

$$u(x,0) = h(x) \qquad\qquad u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega.$$

We shall calculate with the help of this relation between the spectral theory of the Laplace operator with Dirichlet boundary condition and the heat equation the heat kernel of the circle $S^1$ and the interval $[-1,1]$.

## 4.7 Heat Kernel of $S^1$

We identify the circle $S^1$ with the quotient $\mathbb{R}/\mathbb{Z}$. The eigenfunctions of $-\frac{d^2}{dx^2}$ on $\mathbb{R}/\mathbb{Z}$ are equal to $e^{2\pi ikx}$ with $k \in \mathbb{Z}$ with eigenvalues $4\pi^2k^2$. This eigenfunctions build an orthogonal system of the Hilbert space $L^2(\mathbb{R}/\mathbb{Z})$. By the Theorem of Stone and Weierstraß the algebra of polynomials with respect to $\sin(2\pi x)$ and $\cos(2\pi x)$ are dense in the real Banach space $C(\mathbb{R}/\mathbb{Z},\mathbb{R})$. This in turn implies that the same holds for polynomials with respect to $e^{2\pi\imath x}$ and $e^{-2\pi\imath x}$ in the complex Banach space $C(\mathbb{R}/\mathbb{Z},\mathbb{C})$. Therefore the orthogonal complement in $L^2(\mathbb{R}/\mathbb{Z})$ of the former orthogonal system is trivial, and this system is an orthogonal basis. So any $h \in L^2(\mathbb{R}/\mathbb{Z})$ may be decomposed into a series of the eigenfunctions $e^{2\pi ikx}$ of $-\frac{d^2}{dx^2}$ on $\mathbb{R}/\mathbb{Z}$ with eigenvalues $4\pi^2k^2$:

$$h(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx} \qquad \text{with} \qquad a_k = \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi iky}h(y)dy.$$

Therefore the heat kernel of $\mathbb{R}/\mathbb{Z}$ is given by

$$H_{\mathbb{R}/\mathbb{Z}}(x,y,t) = \sum_{k\in\mathbb{Z}} e^{-4\pi^2k^2t+2\pi ik(x-y)} = \Theta(x-y,4\pi it) \quad \text{with } \Theta(x,\tau) = \sum_{k\in\mathbb{Z}} e^{2\pi ikx+\pi i\tau k^2}.$$

Here $\Theta(x,\tau)$ is Jacobi's Theta function. This sum converges on the domain $(x,\tau) \in \mathbb{C} \times \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ to a holomorphic functions since $e^{\pi i\tau k^2}$ decays exponentially with respect to $k^2$. This Theta function is characterised by the following properties:

$$\Theta(x+1,\tau) = \Theta(x,\tau), \qquad \Theta(x+\tau,\tau) = \Theta(x,\tau)e^{-\pi i\tau-2\pi ix}, \qquad \Theta(\tfrac{1}{2}+\tfrac{1}{2}\tau,\tau) = 0.$$

The first property follows from the periodicity of $e^{2\pi ikx}$ with period 1. The other two properties we show by direct calculation:

$$\begin{aligned}
\Theta(x+\tau,\tau) &= \sum_{k\in\mathbb{Z}} e^{2\pi ik(x+\tau)+\pi ik^2\tau} && = \sum_{k\in\mathbb{Z}} e^{2\pi ikx+\pi i(k+1)^2\tau-\pi i\tau}\\
&= \sum_{k\in\mathbb{Z}} e^{2\pi i(k+1)x+\pi i(k+1)^2\tau-2\pi ix-\pi i\tau} && = \Theta(x,\tau)e^{-2\pi ix-\pi i\tau}\\
\Theta(\tfrac{1}{2}+\tfrac{\tau}{2},t) &= \sum_{k\in\mathbb{Z}}(-1)^k e^{\pi i\tau((k+\frac{1}{2})^2-\frac{1}{4})} && = e^{-\frac{4\pi i\tau}{4}}\sum_{l\in\mathbb{N}_0} e^{\pi i\tau(l+\frac{1}{2})^2}(1-1)=0.
\end{aligned}$$

**Exercise 4.22.** **(i)** *Show that for all $t>0$ the fundamental solution $\Phi(x,t)$ belongs to the Schwartz space considered as a function on $x\in\mathbb{R}^n$.*

**(ii)** *Calculate for all $t>0$ the Fourier transform of the fundamental solution $\Phi(x,t)$ considered as a function on $x\in\mathbb{R}^n$.*

**(iii)** *Show that for any Schwartz function $f$ on $\mathbb{R}$ the following series converges to a smooth function $\tilde{f}$ on $\mathbb{R}$ which is periodic with period 1:*

$$\tilde{f}(x)=\sum_{n\in\mathbb{Z}} f(x+n).$$

**(iv)** *Let $h$ be a periodic continuous functions on $\mathbb{R}$ with period 1. Show that the solution of the heat equation with initial values $h$ preserves periodicity with period 1 for all $t>0$. Conclude that the following series is the heat kernel of $S^1$:*

$$\sum_{n\in\mathbb{Z}}\Phi(x-y+n,t).$$

**(v)** *Due to Poisson's summation formula every Schwartz function on $\mathbb{R}$ satisfies*

$$\sum_{n\in\mathbb{Z}} f(x+n)=\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi inx}.$$

*Show with the help of this formula the relation*

$$H_{S^1}(x,y,t)=\sum_{n\in\mathbb{Z}}\Phi(x-y+n,t).$$

**(vi)** *Show that $f(x)=e^{-x^2}(e^{-x^2}+\sin^2 x)$ is a positive Schwartz function on $\mathbb{R}$, whose square root does not belong to the Schwartz space.*

## 4.8 Heat Kernel of $[0,1]$

The eigenfunctions of $-\frac{d^2}{dx^2}$ on $[0,1]$ with Dirichlet boundary conditions, this means roots at $\partial[0,1]=\{0,1\}$, are given by

$$\sqrt{2}\sin(k\pi x)\quad\text{with}\quad k\in\mathbb{N}$$

These functions again build an orthogonal system:

$$\int_0^1\sqrt{2}\sin(k\pi x)\sqrt{2}\sin(k'\pi x)dx=\int_0^1(\cos((k-k')\pi x)-\cos((k+k')\pi x))dx=\delta_{k,k'}.$$

For any continuous functions $f$ on $[0,1]$ with roots at $\partial[0,1]$ the function

$$\tilde{f}(x)=\begin{cases}f(x-2n) & \text{for } x\in[2n,2n+1] \text{ with } n\in\mathbb{Z}\\ -f(2n-x) & \text{for } x\in[2n-1,2n] \text{ with } n\in\mathbb{Z}\end{cases}$$

is continuous on $\mathbb{R}$ with roots at $\mathbb{Z}$ and is periodic with period 2. By the Theorem of Stone and Weierstraß the finite linear combinations of $(x\mapsto\exp(k\pi\imath x))_{k\in\mathbb{N}}$ build a dense subalgebra of $C(\mathbb{R}/2\mathbb{Z})$ and therefore are also dense in $L^2(\mathbb{R}/2\mathbb{Z})$. The map $f\mapsto\tilde{f}$ maps $L^2[0,1]$ onto the following closed subspace $\mathcal{A}\subset L^2(\mathbb{R}/2\mathbb{Z})$:

$$\left\{f\in L^2(\mathbb{R}/2\mathbb{Z})\,\middle|\, f(n+x)=\begin{cases}f(x) & \text{for even } n\in 2\mathbb{Z} \text{ and } x\in\mathbb{R}\\ -f(1-x) & \text{for odd } n\in 2\mathbb{Z}+1 \text{ and } x\in\mathbb{R}\end{cases}\right\}.$$

This space $\mathcal{A}$ consists of all periodic odd functions on $\mathbb{R}$ with period 2, since $n=-1$ gives $f(x-1)=-f(1-x)$. A linear combination $\sum_k a_k\exp(k\pi\imath x)$ belongs to $\mathcal{A}$, if and only if $a_{-k}=-a_k$ holds for all $k\in\mathbb{Z}$. Hence the linear combinations of $(\sqrt{2}\sin(k\pi x))_{k\in\mathbb{N}}$ are dense in $\mathcal{A}$ and build an orthogonal basis of $L^1[0,1]$. This implies

$$h=\sum_{k\in\mathbb{N}}a_k\sqrt{2}\sin(k\pi x)\qquad\text{with}\qquad a_k=\int_0^1\sqrt{2}\sin(k\pi y)h(y)dy\qquad\text{for } h\in L^2[0,1].$$

We conclude that the unique solution of the initial value problem

$$\dot{u}(x,t)-\triangle u(x,t)=0\quad u(x,0)=h(x)\quad u(0,t)=u(1,t)=0\quad\text{for } (x,t)\in(0,1)\times\mathbb{R}^+$$

is given by $$u(x,t)=\int_0^1 H_{[0,1]}(x,y,t)h(y)dy$$ with

$$\begin{aligned}H_{[0,1]}(x,y,t)&=\sum_{k=1}^{\infty}e^{-\pi^2k^2t}2\sin(k\pi x)\sin(k\pi y)\\&=\sum_{k=1}^{\infty}e^{-\pi^2k^2t}(\cos(k\pi(x-y))-\cos(k\pi(x+y)))=\frac{1}{2}\Theta(\tfrac{x-y}{2},\pi it)-\frac{1}{2}\Theta(\tfrac{x+y}{2},\pi it).\end{aligned}$$

**Exercise 4.23.** **(i)** *Show that the heat kernel $H_{[0,1]}$ is given by*

$$H_{[0,1]}(x,y,t) = \frac{1}{2}\Theta(\tfrac{x-y}{2},\pi\imath t) - \frac{1}{2}\Theta(\tfrac{x+y}{2},\pi\imath t).$$

**(ii)** *Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:*

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z}+1 \text{ and } x \in \mathbb{R}. \end{cases}$$

*Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2.*

**(iii)** *Show that for any Schwartz function $f$ on $\mathbb{R}$ the following series converges to a smooth functions $\tilde{f}$ in $\mathcal{A}$:*

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(2n+x) - \sum_{n\in\mathbb{Z}} f(2n-x).$$

**(iv)** *Show for any $h \in \mathcal{A}$, that the solutions of the heat equation with initial value $h$ is for all $t > 0$ a smooth function in $\mathcal{A}$. Conclude from this that the following sum has the properties of the Heat kernel of $[0,1]$:*

$$\sum_{n\in\mathbb{Z}} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t).$$

**(v)** *Show the relation*

$$H_{[0,1]}(x,y,t) = \sum_{n\in\mathbb{Z}} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t).$$

The heat kernel of the Cartesian product of two domains can be easily calculated in terms of the heat kernels of both domains:

**Lemma 4.24.** *If $\Omega \subset \mathbb{R}^m$ and $\Omega' \subset \mathbb{R}^n$ are two open, bounded and connected domains with given heat kernels $H_\Omega$ and $H_{\Omega'}$, then the heat kernel of $\Omega \times \Omega'$ is given by*

$$H_{\Omega\times\Omega'}((x,x'),(y,y'),t) = H_\Omega(x,y,t)H_{\Omega'}(x',y',t) \quad (x,x'),(y,y') \in \bar{\Omega}\times\bar{\Omega}' \quad t \in \mathbb{R}^+.$$

*Proof.* For any $(x,x',t) \in \Omega \times \Omega' \times \mathbb{R}^+$ the function $(y,y') \mapsto H_\Omega(x,y,t)H_{\Omega'}(x',y',t)$ extends by the value zero continuously to $\partial(\Omega \times \Omega') = (\partial\Omega \times \Omega') \cup (\omega \times \partial\Omega')$. The Laplace operator of the Cartesian product is the sum of the corresponding Laplace operators. Hence for all $(x, x') \in \Omega \times \Omega'$ the function $(y, y', t) \mapsto H_\Omega(x, y, t) H_{\Omega'}(x'y', t)$ solves the homogeneous heat equation. The product of both fundamental solutions is the fundamental solution on $\mathbb{R}^{m+n}$. Hence for all $(x, x') \in \Omega \times \Omega'$ the function $(y, y', t) \mapsto H_\Omega(x, y, t) H_{\Omega'}(x', y't) - \Phi(x - y, t)\Phi(x' - y', t)$ extends continuously to $\bar{\Omega} \times \bar{\Omega}' \times \mathbb{R}^+_0$ by setting it zero on $(y, y', t) \in \bar{\Omega} \times \bar{\Omega}' \times \{0\}$. **q.e.d.**

So we might have a formula for the heat kernels all tori $(\mathbb{R}/\mathbb{Z})^n$ and all Cartesian products $[0, 1]^n$. However the boundaries of the Cartesian products $[0, 1]^n \subset \mathbb{R}^n$ are no continuously differentiable submanifolds of $\mathbb{R}^n$ and our proof of the divergence theorem does not apply to these Cartesian products. However, the divergence theorem holds for these Cartesian products and we prove this in the lecture Partial Differential Equations. So we have determined the heat kernel of all tori $(\mathbb{R}/\mathbb{Z})^n$ and all Cartesian products $[0, 1]^n$. Hence the unique solution of the initial value problem

$$\dot{u} - \triangle u = 0 \text{ on } (0, 1)^n \times (0, T], \quad u(x, 0) = h(x) \text{ on } [0, 1]^n, \quad u = 0 \text{ on } \partial[0, 1]^n \times [0, T]$$

is given by $\qquad u(x, t) = \int_{[0,1]^n} \prod_{i=1}^n H_{[0,1]}(x_i, y_i, t) h(y) d^n y.$

From $\Phi(x-y, t) = \frac{1}{r^n}\Phi(\frac{x}{r} - \frac{y}{r}, \frac{t}{r^2})$ we obtain $H_{[0,r]^n}(x, y, t) = \dfrac{1}{r^n} \displaystyle\prod_{i=1}^n H_{[0,1]}\left(\frac{x_i}{r}, \frac{y_i}{r}, \frac{t}{r^2}\right)$.

**Corollary 4.25.** *Any solution $u(x, t)$ of the homogeneous heat equation on a neighbourhood of $[0, r]^n \times [0, T] \subset \mathbb{R}^n \times \mathbb{R}$ satisfies*

$$u(x, T) = -\int_0^T \int_{\partial[0,r]^n} u(z, t) \nabla_z H_{[0,r]^n}(x, z, T - t) N(z) d\sigma(z) dt + \int_{[0,r]^n} u(y, 0) H_{[0,r]^n}(x, y, T) d^n y.$$

**q.e.d.**

In the proof of Theorem 4.3 we show that in the limit $t \downarrow 0$ $\Phi(x - y, t)$ converges on the complement of $y \in B(x, \delta)$ uniformly to zero. The same is true for all partial derivatives and due to condition (ii) in Definition 4.14 also for $H_{[0,1]^n}(x, y, t)$. By Lemma 4.17 the integral for $u(x, T)$ is smooth at all $x \in (0, r)^n$. For $(z, t) \in \partial[0, r]^n \times [0, T]$ the Taylor series of $x \mapsto H_{[0,r]^n}(x, z, T - t)$ converges uniformly on compact subsets of $x \in (0, r)^n$ to $H_{[0,r]^n}(x, z, T - t)$. This implies the following Corollary:

**Corollary 4.26.** *Any solution $u$ of the homogeneous heat equation on an open domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is smooth and for fixed $t$ analytic with respect to $x$.* **q.e.d.**

# Chapter 5

# Wave Equation

In this final chapter we consider the homogeneous and inhomogeneous wave equation $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0$ and $\frac{\partial^2 u}{\partial t^2} - \triangle u = f$ on open subsets of $\mathbb{R}^n \times \mathbb{R}$. The wave equation is a linear second order PDE. The coefficient matrix for the second derivatives has one positive and $n$ negative eigenvalues and is neither definite nor semi definite. In the second chapter we introduced this differential equation as the simplest hyperbolic differential equation. The method which solves this PDE is completely different from the methods which solve the Laplace equation or the heat equation. The wave equation describes phenomena which propagate with finite speed through space time. The example of electrodynamic waves motivated the investigation of this equation. Later these methods were generalised to non linear hyperbolic equations in order to describe gravitational waves.

## 5.1 D'Alembert's Formula for $n = 1$

First we solve the following initial value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 && && \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\
&u(x,0) = g(x) && \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x \in \mathbb{R},
\end{aligned}$$

for given functions $g$ and $h$ on $\mathbb{R}$. If we consider this PDE as an ODE on the vector space of smooth functions on $\mathbb{R}$, then the theory of ODEs suggests that we should fix the initial value $u(\cdot, t_0)$ and the first derivative $\frac{\partial u}{\partial t}(\cdot, t_0)$ with respect to $t$ for a given initial time $t_0$. Here the initial values $g$ and $h$ are exactly of this form for $t_0 = 0$. So we expect that this initial value problem should have exactly one solution.

For $n = 1$ we may factorise the wave operator (also called D'Alembert's operator)

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right).$$

If $u$ solves the homogeneous wave equation, then $v(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)$ solves $\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0$. This is the transport equation with constant coefficient with the unique solution

$$v(x,t) = a(x-t) \quad \text{with} \quad a(x) = v(x,0).$$

So the solution $u(x,t)$ of the wave equation solves the first order linear PDE

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = a(x-t).$$

This is an inhomogeneous transport equation with constant coefficients with the solution

$$u(x,t) = \int_0^t a(x + (t-s) - s)ds + b(x+t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t)$$

with $b(x) = u(x,0)$. The initial values $u(x,0) = g(x)$ and $\frac{\partial u}{\partial t}(x,0) = h(x)$ yields

$$b(x) = g(x) \quad \text{and} \qquad a(x) = v(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{\partial u}{\partial x}(x,0) = h(x) - g'(x).$$

If we insert this in our solutions, then we obtain

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \left(h(y) - g'(y)\right) dy + g(x+t)$$

Hence the solution of the initial value problem of the wave equation is given by

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

In this way we derived with the help of the homogeneous and inhomogeneous transport equation the D'Alembert's Formula:

**Theorem 5.1** (D'Alembert's Formula)**.** *If $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $h : \mathbb{R} \to \mathbb{R}$ continuously differentiable, then*

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

*is a twice continuously differentiable function on* $\mathbb{R} \times \mathbb{R}_0^+$*, which is the unique solution of the initial value problem*

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \qquad \textit{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$
$$u(x,0) = g(x) \ \textit{and} \qquad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \textit{for} \quad x \in \mathbb{R}.$$

D'Alembert's Formula shows that for $n = 1$ the general solution of the wave equation takes the form

$$u(x,t) = F(x+t) + G(x-t).$$

Conversely, every function of this form is a solution of the wave equation if $F$ and $G$ are twice differentiable. This fact is a consequence of the factorisation of the wave operator into the two first order linear PDEs'

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} = 0$$

whose solutions are differentiable functions of the form $F(x+t)$ and $G(x-t)$.

We interpret the fact, that the value of the solution at $(x,t)$ depends only on the values of $g$ at $x \pm t$ and the values of $h$ at points in the interval $[x-t, x+t]$ as the bound 1 on the length of the speed of propagation, since the straight lines from all these points to $(x,t)$ propagate with speed of length not larger than 1. If we insert instead of $h$ an anti-derivative $H$, then the value $u(x,t)$ of the solution at $(x,t)$ depends only on the values of $g$ and $H$ at $x \pm t$ and the propagation speed has length 1. Furthermore, the solution is $k$-times differentiable, if $g$ and $H$ are $k$ times differentiable, or equivalently if $g$ is $k$ times differentiable and $h$ is $(k-1)$ times differentiable. So the regularity of the solution does not improve with time, as it does for solutions of the heat equation.

Let us use a reflection in order to derive the solution of the following initial value problem, which will show up later in this chapter:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \qquad u(0,t) = 0 \qquad \text{for} \quad t \in \mathbb{R}_0^+,$$
$$u(x,0) = g(x) \quad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h(x) \quad \text{for} \quad x \in \mathbb{R}^+.$$

The functions $u$, $g$ and $h$ extend to odd functions on the whole space $\mathbb{R} \times \mathbb{R}_0^+$:

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{for } x \geq 0, \\ -u(-x,t) & \text{for } x \leq 0, \end{cases}$$
$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } x \geq 0, \\ -g(-x) & \text{for } x \leq 0, \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x) & \text{for } x \geq 0, \\ -h(-x) & \text{for } x \leq 0. \end{cases}$$

For any solution $\tilde{u}$ of the initial value problem

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

$$\tilde{u}(x,0) = \tilde{g}(x) \quad \text{and} \qquad \frac{\partial \tilde{u}}{\partial t}(x,0) = \tilde{h}(x) \qquad \text{for} \quad x \in \mathbb{R},$$

the function $(x,t) \mapsto -\tilde{u}(-x,t)$ is another solution. Due to the uniqueness of the solution both solutions coincide: $\tilde{u}(-x,t) = -\tilde{u}(x,t)$. In particular for the unique solution of the first initial value problem the solution $\tilde{u}$ of the former initial value problem extends to a solution of the latter initial value problem on $\mathbb{R} \times \mathbb{R}^+_0$. Because the functions $\tilde{u}$, $\tilde{g}$ and $\tilde{h}$ are odd with respect to $x$, for $x \leq t$ the first integral on the right hand side vanishes:

$$\int_{x-t}^{x+t} \tilde{h}(y)dy = \int_{x-t}^{t-x} \tilde{h}(y)dy + \int_{t-x}^{t+x} \tilde{h}(y)dy.$$

Hence the solution of the former initial value problem is given by

$$u(x,t) = \begin{cases} \frac{1}{2}\left(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy\right) & \text{for } 0 \leq t \leq x \\ \frac{1}{2}\left(g(t+x) - g(t-x) + \int_{t-x}^{t+x} h(y)dy\right) & \text{for } 0 \leq x \leq t. \end{cases}$$

Note that the waves propagating towards the boundary at $x = 0$ are reflected at the boundary and propagate back.

## 5.2 Spherical Means of the Wave Equation

We shall derive now the PDE which is obeyed by the spherical means of the wave equation. This PDE is similar to the one-dimensional wave equation, which we shall solve later. This will lead to the general solution of the initial value problem of the wave equation in all dimensions. This initial value problem is the search for the solution $u$ of the wave equation $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0$ on $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ which obeys $u(x,0) = g(x)$ and $\frac{\partial u}{\partial t}(x,0) = h(x)$. We define for all $x \in \mathbb{R}^n, t \geq 0, r > 0$

$$U(x,r,t) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y,t)d\sigma(y) = \fint_{\partial B(x,r)} u(y,t)d\sigma(y).$$

Here the symbol $\fint$ denotes the mean on the domain $\Omega$, i.e. the integral over the domain $\Omega$ divided by the integral of the function 1 over the domain $\Omega$. Analogously we define

$$G(x,r) = \fint_{\partial B(x,r)} g(y)d\sigma(y) \quad \text{and} \qquad H(x,r) = \fint_{\partial B(x,r)} h(y)d\sigma(y).$$

**Lemma 5.2.** *If $u \in C^m(\mathbb{R}^n \times \mathbb{R}^+)$ is a $m$-times continuously differentiable solution of the initial value problem (with continuous partial derivatives of order $\leq m$ on $\mathbb{R}^n \times \mathbb{R}^+_0$)*

$$\frac{\partial^2 u}{\partial t} - \triangle u = 0 \qquad \text{on} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \qquad \text{with}$$

$$u(x,0) = g(x) \qquad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x),$$

*then the spherical mean $U(x,r,t)$ is for fixed $x \in \mathbb{R}^n$ a $m$-times differentiable function on $(r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, which solves the following initial value problem of the Euler-Poisson-Darboux Equation (with continuous partial derivatives of order $\leq m$):*

$$\frac{\partial^2 U}{\partial t^2}(x,r,t) - \frac{\partial^2 U}{\partial r^2}(x,r,t) - \frac{n-1}{r}\frac{\partial U}{\partial r}(x,r,t) = 0 \quad \text{on} \quad (r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$$

$$U(x,r,0) = G(x,r) \quad \text{and} \quad \frac{\partial U}{\partial t}(x,r,0) = H(x,r)$$

*Proof.* By a substitution the domain of the integral becomes independent of $t$ and $r$:

$$U(x,r,t) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(ry + x, t) d\sigma(y).$$

Hence we may calculate the derivative

$$\begin{aligned}
\frac{\partial U}{\partial r}(x,r,t) &= \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla u(ry + x, t) \cdot y d\sigma(y) \\
&= \frac{r}{n\omega_n} \int_{B(0,1)} \triangle u(ry + x, t) d^n y \\
&= \frac{r}{n} \int_{B(x,r)} \triangle u(y,t) d^n y.
\end{aligned}$$

In the second line we used that the divergence of $y \mapsto \nabla u(ry + x, t)$ is $r \triangle u(ry + x, t)$. This shows that the partial derivative of the spherical mean with respect to the radius is $\frac{r}{n}$ times the mean of the Laplace operator applied to the original function on the corresponding ball. In particular we have $\lim_{r \to 0} \frac{\partial U}{\partial r}(x,r,t) = 0$. Analogously we obtain

$$\begin{aligned}
\frac{\partial^2 U}{\partial r^2}(x,r,t) &= \frac{\partial}{\partial r} \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \triangle u(y,t) d^n y \\
&= \frac{1-n}{n\omega_n r^n} \int_{B(x,r)} \triangle u(y,t) d^n y + \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} \triangle u(y,t) d\sigma(y). \\
&= \left(\frac{1}{n} - 1\right) \int_{B(x,r)} \triangle u(y,t) d^n y + \int_{\partial B(x,r)} \triangle u(y,t) d\sigma(y).
\end{aligned}$$

In particular we have $\lim_{r\to 0}\frac{\partial^2 U}{\partial r^2}(x,r,t)=\frac{1}{n}\triangle u(x,t)$. Furthermore, the partial derivative of the mean over a ball of radius $r$ with respect to $r$ is equal to $\frac{n}{r}$ times the corresponding spherical mean minus $\frac{n}{r}$ times this mean:

$$\begin{aligned}\frac{\partial}{\partial r}\frac{1}{\omega_n r^n}\int_{B(x,r)}u(y,t)d^ny &= -\frac{n}{\omega_n r^{n+1}}\int_{B(x,r)}u(y,t)d^ny+\frac{1}{\omega_n r^n}\int_{\partial B(x,r)}u(y,t)d\sigma(y)\\ &= -\frac{n}{r}\int_{B(x,r)}u(y,t)d^ny+\frac{n}{r}\int_{\partial B(x,r)}u(y,t)d\sigma(y).\end{aligned}$$

These formulas allow to calculate all partial derivatives of the spherical means in terms of the mean of powers of the Laplace operator applied to the original functions over spheres and balls. On the other hand we also have

$$\frac{\partial}{\partial r}r^{n-1}\frac{\partial U}{\partial r}(x,r,t)=\frac{\partial}{\partial r}\frac{1}{n\omega_n}\int_{B(x,r)}\triangle u(y,t)d^ny=\frac{1}{n\omega_n}\int_{\partial B(x,r)}\frac{\partial^2 u}{\partial t^2}(y,t)d\sigma(y)=r^{n-1}\frac{\partial^2 U}{\partial t^2}(x,r,t).$$

This implies
$$r^{n-1}\frac{\partial^2 U}{\partial t^2}=(n-1)r^{n-2}\frac{\partial U}{\partial r}+r^{n-1}\frac{\partial^2 U}{\partial r^2}.$$
**q.e.d.**

## 5.3 Solution in Dimension 3

We shall see that for odd dimensions the spherical means of solutions of the wave equation can be transformed into solutions of the one-dimensional wave equation, but not for even dimensions. For this reason we shall next solve the initial value problem of the wave equation in three dimensions. In this section we consider for any $x\in\mathbb{R}^3$ the following initial value problem for the spherical means of a solution of the wave equation:

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial r^2}-\frac{2}{r}\frac{\partial U}{\partial r}=0 \quad\text{on}\quad (x,r,t)\in\{x\}\times\mathbb{R}^+\times\mathbb{R}^+\\ U=G\quad\text{and}\quad\frac{\partial U}{\partial t}=H\quad\text{on}\quad (x,r,t)\in\{x\}\times\mathbb{R}^+\times\{0\}.\end{aligned}$$

The substitution $\tilde U=rU$ transforms the former initial value problem into the following:

$$\begin{aligned}\frac{\partial^2\tilde U}{\partial t^2}-\frac{\partial^2\tilde U}{\partial r^2}=0\text{ on }(x,r,t)\in\{x\}\times\mathbb{R}^+\times\mathbb{R}^+,\qquad \tilde U(x,0,t)=0\text{ for }t\in\mathbb{R}^+_0,\\ \tilde U(x,r,0)=\tilde G(x,r)=rG(x,r)\quad\text{and}\quad\frac{\partial\tilde U}{\partial t}(x,r,0)=\tilde H(x,r)=rH(x,r)\text{ for }r\in\mathbb{R}^+.\end{aligned}$$

We solved this initial value problem in the first section. The solution is

$$\tilde U(x,r,t)=\frac{1}{2}\left(\tilde G(x,r+t)-\tilde G(x,t-r)\right)+\frac{1}{2}\int_{-r+t}^{r+t}\tilde H(x,s)ds\quad\text{for }0\le r\le t.$$

The continuity of $u(x,t)$ implies

$$u(x,t) = \lim_{r\downarrow 0} U(x,r,t) = \lim_{r\downarrow 0} \frac{\tilde{U}(x,r,t)}{r}.$$

Therefore we obtain for all $x \in \mathbb{R}^3, t > 0$.

$$\begin{aligned}
u(x,t) &= \lim_{r\downarrow 0} \tfrac{1}{2}\left(\tfrac{\tilde{G}(x,t+r)-\tilde{G}(x,t)}{r} + \tfrac{\tilde{G}(x,t-r)-\tilde{G}(x,t)}{-r}\right) && + \lim_{r\downarrow 0} \tfrac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(x,s)ds \\
&= \frac{\partial \tilde{G}(x,t)}{\partial t} && + \tilde{H}(x,t) \\
&= \frac{\partial}{\partial t} t \fint_{\partial B(x,t)} g(y)d\sigma(y) && + t\fint_{\partial B(x,t)} h(y)d\sigma(y) \\
&= \frac{\partial}{\partial t} t \fint_{\partial B(0,1)} g(x+tz)d\sigma(z) && + t\fint_{\partial B(x,t)} h(y)d\sigma(y) \\
&= \fint_{\partial B(0,1)} \nabla_y g(x+tz)\cdot tz d\sigma(z) && + \fint_{\partial B(x,t)} (th(y)+g(y))\, d\sigma(y) \\
&= \fint_{\partial B(x,t)} (th(y)+g(y))\, d\sigma(y) && + \fint_{\partial B(x,t)} \nabla_y g(y)\cdot(y-x)d\sigma(y)
\end{aligned}$$

The last line is Krichoff's Formula for the solution of the initial value problem of the three dimensional wave equation.

## 5.4 Solution in Dimension 2

In two dimensions the Euler-Poisson-Darboux equations cannot be transformed into the one-dimensional wave equation. We present another method and transform the initial value problem of the two-dimensional wave equation into an initial value problem of the three-dimensional wave equation, by choosing initial vales which depend only on the coordinates $x_1$ and $x_2$ and not on the coordinate $x_3$: Let $\bar{u}(x,t)$ be on $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$ the solution of the initial value problem

$$\frac{\partial^2 \bar{u}(x,t)}{\partial t^2} - \triangle \bar{u}(x,t) = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$$

$$\bar{u}(x,0) = g(\bar{x}) \quad \text{and} \quad \frac{\partial \bar{u}}{\partial t}(x,0) = h(\bar{x}) \qquad \text{for} \quad x \in \mathbb{R}^3.$$

Here we set $\bar{x} = (x_1, x_2)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We observe that the mean over $\partial B(x,r)$ of a function $f$ depends only on $\bar{x}$, if $f$ depends only on $\bar{x}$:

$$\frac{\partial}{\partial x_3}\fint_{\partial B(x,r)} f(y)d\sigma(y) = \frac{\partial}{\partial x_3}\fint_{\partial B(0,r)} f(x+y)d\sigma(y) = \fint_{\partial B(0,r)} \frac{\partial f(x+y)}{\partial x_3} d\sigma(y) = 0.$$

This shows that $\bar{u}(x,t)$ is of the form $u(\bar{x},t)$ and the latter function $u(\bar{x},t)$ yields the desired solution of the two-dimensional initial value problem. Let us now calculate this function. The function $\gamma(y)=\sqrt{t^2-(y-\bar{x})^2}$ on the two-dimensional ball $B(\bar{x},t)$ yields by the formula $\Psi(y)=(y,\pm\gamma(y))$ a parametrisations of both hemispheres of the boundary of the three-dimensional ball $B\left((\bar{x},0),t\right)$ by the two-dimensional ball $B(\bar{x},t)$ as in Definition 2.3. The $3\times 2$-matrix $\Psi'(y)$ has the same form $\Psi'(y)=\left(\begin{smallmatrix}\mathbb{1}_{\mathbb{R}^2}\\ \pm\nabla^T\gamma(y)\end{smallmatrix}\right)$ as $\phi'$ after Lemma 2.4 with $\mathsf{O}=\mathbb{1}$ and $g=\pm\gamma$. Hence the determinant of $(\Psi'(y))^T\Psi(y)$ is $1+(\nabla\gamma(y))^2$. So the intgerals over both hemispheres is equal to the integral over $B(\bar{x},t)$ with the measure $d\sigma(y,\pm\gamma(y))=\sqrt{1+(\nabla_y\gamma(y))^2}dy^2$:

$$\fint_{\partial B(x,t)} g(\bar{y})d\sigma(y)=\frac{1}{4\pi t^2}\int_{\partial B(x,t)} g(\bar{y})d\sigma(y)=\frac{2}{4\pi t^2}\int_{B(\bar{x},t)} g(y)\sqrt{1+(\nabla_y\gamma(y))^2}d^2y$$

$$\text{with}\quad \sqrt{1+\left(\nabla_y\gamma(y)\right)^2}=\sqrt{\frac{t^2-(y-\bar{x})^2+(y-\bar{x})^2}{t^2-(y-\bar{x})^2}}=\frac{t}{\sqrt{t^2-(y-\bar{x})^2}}.$$

Both hemispheres do not cover the boundary $\partial B((\bar{x},0),t)$ completely, but the missing equator is one-dimensional and has measure zero with respect to $\sigma(y)$. This implies

$$\fint_{\partial B(x,t)} g(\bar{y})d\sigma(y)=\frac{1}{2\pi t}\int_{B(\bar{x},t)}\frac{g(y)}{\sqrt{t^2-(y-\bar{x})^2}}d^2y=\frac{t}{2}\fint_{B(\bar{x},t)}\frac{g(y)}{\sqrt{t^2-(y-\bar{x})^2}}d^2y.$$

This gives finally the following formula for $u(\bar{x},t)$ on $(\bar{x},t)\in\mathbb{R}^2\times\mathbb{R}^+$:

$$\begin{aligned}
u(\bar{x},t) &= \frac{\partial}{\partial t}t\fint_{\partial B(x,t)} g(\bar{y})d\sigma(y) && + t\fint_{\partial B(x,t)} h(\bar{y})d\sigma(y)\\
&= \frac{\partial}{\partial t}\frac{t^2}{2}\fint_{B(\bar{x},t)}\frac{g(y)}{\sqrt{t^2-(y-\bar{x})^2}}d^2y && + \frac{t^2}{2}\fint_{B(\bar{x},t)}\frac{h(y)}{\sqrt{t^2-(y-\bar{x})^2}}d^2y\\
&= \frac{\partial}{\partial t}\frac{t}{2}\fint_{B(0,1)}\frac{g(\bar{x}+tz)}{\sqrt{1-z^2}}d^2z && + \frac{t^2}{2}\fint_{B(\bar{x},t)}\frac{h(y)}{\sqrt{t^2-(y-\bar{x})^2}}d^2y\\
&= \frac{t}{2}\fint_{B(\bar{x},t)}\frac{g(y)+th(y)+\nabla_y g(y)(y-\bar{x})}{\sqrt{t^2-(y-\bar{x})^2}}d^2y.
\end{aligned}$$

The last line is Poisson's formula for the solution of the inital value problem of the two-dimensional wave equation. It shows that in two dimensions the propagation speed of solutions of the wave equation has all lengths bounded by 1. In fact the value $u(x,t)$ depends only on the values of $g$ and $h$ on the ball $B(\bar{x},t)$ and the straight lines connecting these points with $(x,t)$ have all speeds whose lengths are bounded by 1.

This method of deriving the solution of the initial value problem in a lower dimension by transfoming the initial value problem into an initial value problem in the higher dimensional space, is called the method of descent. Here the initial values do not depend on some of the coordinates of the higher dimensional space. We have to show that the corresponding solutions do not depend on these coordinates as well. A natural question is, whether we may obtain the solution of the one-dimensional wave equation by this method of descent from Poisson's formula?

## 5.5 Solution in odd Dimensions

In any odd dimension we can transform the Euler-Poisson-Darboux equation into the one-dimensional wave equation. As a preparation we first show the following Lemma:

**Lemma 5.3.** *Let $\phi$ ba a $(k+1)$ times continuously differentiable function on $\mathbb{R}$. We obtain for all $k \in \mathbb{N}$*

**(i)** $\left(\frac{d}{dr}\right)^2 \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} r^{2k-1}\phi(r) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k} r^{2k}\frac{d\phi}{dr}(r)$

**(ii)** $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} r^{2k-1}\phi(r) = \sum_{j=0}^{k-1}\beta_{k,j}r^{j+1}\frac{d^j\phi}{dr^j}(r)$ *with numbers* $\beta_{k,j}$ $(j = 0,\ldots,k-1)$*, which do not depend on* $\phi$.

**(iii)** $\beta_{k,0} = 1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1) = \frac{(2k-1)!}{2\cdot 4\cdot\ldots\cdot(2k-2)} = \frac{(2k-1)!}{2^{(k-1)}(k-1)!}$

*Proof.* First we proof (i) by induction. For $k = 1$ we have

$$\frac{d^2}{dr^2}r\phi(r) = 2\frac{d\phi}{dr}(r) + r\frac{d^2\phi}{dr^2}(r) = \frac{1}{r}\frac{d}{dr}r^2\frac{d\phi}{dr}(r).$$

Now we assume that the statement is true for $k \in \mathbb{N}$. Then we obtain for $k+1$:

$$\begin{aligned}(\tfrac{d}{dr})^2(\tfrac{1}{r}\tfrac{d}{dr})^k r^{2k+1}\phi(r) &= (\tfrac{d}{dr})^2(\tfrac{1}{r}\tfrac{d}{dr})^{k-1}r^{2k-1}((2k+1)\phi(r) + r\tfrac{d\phi}{dr}(r))\\ &= (\tfrac{1}{r}\tfrac{d}{dr})^k r^{2k}\tfrac{d}{dr}((2k+1)\phi(r) + r\tfrac{d\phi}{dr}(r))\\ &= (\tfrac{1}{r}\tfrac{d}{dr})^k((2k+2)r^{2k}\tfrac{d\phi}{dr}(r) + r^{2k+1}\tfrac{d^2\phi}{dr^2}(r)) = (\tfrac{1}{r}\tfrac{d}{dr})^{k+1}r^{2k+2}\tfrac{d\phi}{dr}(r).\end{aligned}$$

By the Leibniz rule every derivative in (ii) results in two contributions: The first is a deminishing of the power of $r$ by one and does not change $\phi$. The other does not change the power of $r$ and acts as a derivative on $\phi$. The total power of $r$ is $r^{2k-1-(k-1)} = r^k$ and the total order of derivatives is $(\frac{d}{dr})^{k-1}$. This implies (ii).

In the term with the coefficient $\beta_{k,0}$ all derivatives act only on the powers of $r$. The first derivative acts on $r^{2k-1}$, the second derivatives acts on $r^{2k-3}$ and the last derivative acts on $r^3$. This shows (iii). **q.e.d.**

Let the dimension $n = 2k + 1 \geq 3$ be odd and let $u \in C^{k+1}(\mathbb{R}^{2k+1} \times \mathbb{R}_0^+)$ obey

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$

$$u(x,0) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \text{for} \quad x \in \mathbb{R}^n.$$

We define
$$\tilde{U}(x,r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} U(x,r,t)$$

$$\tilde{G}(x,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} G(x,r) \qquad \tilde{H}(x,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} H(x,r).$$

**Lemma 5.4.** *If $u \in C^{k+1}(\mathbb{R}^{2k+1} \times \mathbb{R}_0^+)$ solves the initial value problem of the wave equation, then $\tilde{U}(x,r,t)$ solves for any $x \in \mathbb{R}^{2k+1}$ the following initial value problem:*

$$\frac{\partial^2 \tilde{U}}{\partial t^2} - \frac{\partial^2 \tilde{U}}{\partial r^2} = 0 \text{ on } (x,r,t) \in \{x\} \times \mathbb{R}^+ \times \mathbb{R}^+ \qquad \tilde{U}(x,0,t) = 0 \text{ for } t \in \mathbb{R}_0^+$$

$$\tilde{U}(x,r,0) = \tilde{G}(x,r) \text{ and } \qquad \frac{\partial \tilde{U}}{\partial t}(x,r,0) = \tilde{H}(x,r) \text{ for } r \in \mathbb{R}^+.$$

*Proof.* Let $U(x,r,t)$ solve in the dimension $2k = n-1$ the corresponding initial value problem of the Euler-Poisson-Darboux equation:

$$\begin{aligned}
\frac{\partial^2 \tilde{U}}{\partial r^2}(x,r,t) &= \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} U(x,r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k} r^{2k} \frac{\partial U}{\partial r}(x,r,t) \\
&= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\frac{\partial^2 U}{\partial r^2}(x,r,t) + 2kr^{2k-2}\frac{\partial U}{\partial r}(x,r,t)\right) \\
&= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}\frac{\partial^2 U}{\partial t^2}(x,r,t) = \frac{\partial^2 \tilde{U}}{\partial t^2}(x,r,t)
\end{aligned}$$

Due to the Lemmas 5.2 and 5.3 (iii) the values of $\tilde{U}(x,r,t)$ vanish for $r = 0$. **q.e.d.**

For any $(x,r,t) \in \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with $r \leq t$ the solution of the initial value problem is

$$\tilde{U}(x,r,t) = \frac{1}{2}\left(\tilde{G}(x,t+r) - \tilde{G}(x,t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(x,s)ds.$$

We recall Lemma 5.3 (ii): $\tilde{U}(x,r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} U(x,r,t) = \sum_{j=0}^{k-1} \beta_{k,j} r^{j+1} \frac{\partial^j U}{\partial r^j}(x,r,t).$

Lemma 5.2 implies $\lim_{r\to 0} r^{j+1}\frac{\partial^j U}{\partial r^j}(x,r,t) = 0$ for all $j \in \mathbb{N}_0$. We conclude

$$u(x,t) = \lim_{r\to 0} U(x,r,t) = \lim_{r\to 0} \frac{\tilde{U}(x,r,t)}{\beta_{k,0} r}$$

Alltogether the solution of the initial value problem in odd dimensions is given by

$$u(x,t) = \frac{2^{k-1}(k-1)!}{(2k-1)!}\lim_{r\to 0}\left(\frac{\tilde{G}(x,t+r)-\tilde{G}(x,t-r)}{2r} + \fint_{t-r}^{t+r}\tilde{H}(x,s)ds\right) = \frac{2^{k-1}(k-1)!}{(2k-1)!}\left(\frac{\partial\tilde{G}}{\partial t}(x,t)+\tilde{H}(x,t)\right).$$

**Theorem 5.5.** *For odd $n \geq 3$ the solution of the initial value problem*

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 && \text{for} \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+ \\ &u(x,0) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x\in\mathbb{R}^n \end{aligned}$$

*with $g \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$ and $h \in C^{\frac{n+1}{2}}(\mathbb{R}^n)$ has at any $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+_0$ the value*

$$u(x,t) = \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} g(y)d\sigma(y) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} h(y)d\sigma(y)\right).$$

*Proof.* First we consider the case $g = 0$. In this case Lemma 5.3 (i) implies

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\frac{\partial^2}{\partial t^2}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} h(y)d\sigma(y) \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} t^{n-1}\frac{\partial}{\partial t}\fint_{\partial B(x,t)} h(y)d\sigma(y) \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} t^{n-1}\frac{\partial}{\partial t}\frac{1}{n\omega_n}\int_{\partial B(0,1)} h(x+tz)d\sigma(z) \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \frac{t^{n-1}}{n\omega_n}\int_{\partial B(0,1)} \nabla_x h(x+tz)\cdot z d\sigma(z) \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \frac{1}{n\omega_n}\int_{\partial B(x,t)} \nabla_y h(y)\cdot N(y) d\sigma(y) \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \frac{1}{n\omega_n}\int_{B(x,t)} \triangle h(y) d^n y \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \frac{1}{n\omega_n t}\int_{\partial B(x,t)} \triangle h(y) d\sigma(y) \\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(0,t)} \triangle_x h(x+y) d\sigma(y) \\
&= \triangle u(x,t).
\end{aligned}$$

Here we used the divergence theorem and polar coordinates for the calculation of the derivative of integrals over $B(x,t)$ with respect to $t$. If we replace $h$ by $g$ and $u(x,t)$ by $v(x,t)$ with $u(x,t) = \frac{\partial v}{\partial t}(x,t)$, then we obtain the solution in the case with $h = 0$. This shows that $v(x,t)$ and therefore also $u(x,t)$ solves the wave equation. The Lemmas 5.2 and 5.3 (iii) finally imply

$$\begin{aligned} u(x,0) &= \lim_{t\to 0}\left(\frac{\partial}{\partial t}t\fint_{\partial B(x,t)} g(y)d\sigma(y) + t\fint_{\partial B(x,t)} h(y)d\sigma(y)\right) + \mathbf{O}(t) &&= g(x) \\ \frac{\partial u}{\partial t}(x,0) &= \lim_{t\to 0}\left(t\fint_{\partial B(x,t)} \triangle g(y)d\sigma(y) + \frac{\partial}{\partial t}t\fint_{\partial B(x,t)} h(y)d\sigma(y)\right) + \mathbf{O}(t) &&= h(x). \quad \textbf{q.e.d.} \end{aligned}$$

To the limit $t \downarrow 0$ only the lowest powers of $t$ with $j = 0$ in Lemma 5.3 (ii) contribute and in the second formula we used that the integral solves the homogeneous wave equation. The solution $u(x,t)$ depends only on the values of $g$ and $h$ at elements of $\partial B(x,t)$. Therefore the speed of propagation has length 1. This is the content of Huygen's principle. However the formula for the solution of the initial value problem of the heat equation shows that the speed of propagation of solutions of the heat equation can be arbitrary large. Besides this difference there exists another difference to the heat equation: The solution $u$ of the homogeneous wave equation is at $(x,t)$ $\frac{n-1}{2}$ times less differentiable than $g$ and $\frac{n-3}{2}$ times less differentiable than $h$. This is a general property of solutions of hyperbolic PDEs in contrast to parabolic PDEs: for a large class of initial values the solutions of the homogeneous heat equation are smooth independent of the regularity of the initial values.

## 5.6 Solution in even Dimensions

We again use the method of descent, and derive the solutions in the dimension $n$ as special solutions in the dimension $n+1$ with initial values $g(x)$ and $h(x)$ on $\mathbb{R}^{n+1}$ not depending on $x_{n+1}$ for $x = (x_1, \ldots, x_{n+1})$. By the general formula of the last section we see that in this case also the solution does not depend on $x_{n+1}$. Indeed if the partial derivative $\partial_{n+1}g(x)$ and $\partial_{n+1}h(x)$ vanish, then the same is true for

$$\fint_{\partial B(x,t)} g(y)d\sigma(y) \quad \text{and} \quad \fint_{\partial B(x,t)} h(y)d\sigma(y).$$

If the function $f$ obeys $\frac{\partial f}{\partial x_{n+1}} = 0$, then we obtain

$$\frac{\partial}{\partial x_{n+1}}\fint_{\partial B(x,t)} f(y)d\sigma(y) = \fint_{\partial B(0,t)} \frac{\partial}{\partial x_{n+1}} f(x+y)d\sigma(y) = 0.$$

This holds for all $t \in \mathbb{R}^+$ and therefore also for all partial derivatives with respect to $t$. So the solution in dimension $n = 2k$ is obtained as the composition of the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}, x \mapsto (x, 0)$ with the solution of the corresponding initial value problem in dimension $n+1$. As in the application of the method of descent to the descent from dimension three to dimension two we parameterise both hemispheres of $\partial B(x,t)$ by the maps $\Psi : B(\bar{x}, t) \to \partial B(x,t)$ with $y \mapsto (y, \pm\gamma(y))$ with $\gamma(y) = \sqrt{t^2 - (y - \bar{x})^2}$. The Jacobi matrix of this map is the $(n+1) \times n$ Matrix $\left(\begin{smallmatrix} \mathbb{1}_{\mathbb{R}^n} \\ \pm(\nabla_y \gamma(y))^t \end{smallmatrix}\right)$ and the determinan of $(\Psi'(y))^T \Psi'(y)$ is again $1 + (\nabla_y \gamma(y))^2$. Hence we obtain

$$\fint_{\partial B(x,t)} \bar{f}(y) d\sigma(y) = \frac{1}{(n+1)\omega_{n+1} t^n} \int_{\partial B(x,t)} \bar{f}(y) d\sigma(y) = \frac{2}{(n+1)\omega_{n+1} t^n} \int_{B(\bar{x},t)} f(y) \sqrt{1+(\nabla\gamma(y))^2} d^n y$$
$$= \frac{2t}{(n+1)\omega_{n+1} t^n} \int_{B(\bar{x},t)} \frac{f(y) d^n y}{\sqrt{t^2 - (y-\bar{x})^2}} = \frac{2t\omega_n}{(n+1)\omega_{n+1}} \fint_{B(\bar{x},t)} \frac{f(y) d^n y}{\sqrt{t^2 - (y-\bar{x})^2}}.$$

**Theorem 5.6** (Solution in even dimension)**.** *Let $n$ be a positive even integer and $g \in C^{\frac{n+4}{2}}(\mathbb{R}^n)$ and $h \in C^{\frac{n+2}{2}}(\mathbb{R}^n)$. Then the solution of the initial value problem*

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 && && \textit{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ &u(x,0) = g(x) \quad \textit{and} && \frac{\partial u}{\partial t}(x,0) = h(x) && \textit{for} \quad x \in \mathbb{R}^n \end{aligned}$$

*takes at any $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+_0$ the following value:*

$$u(x,t) = \frac{1}{2^{\frac{n}{2}} \frac{n}{2}!} \left( \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} t^n \fint_{B(x,t)} \frac{g(y) d^n y}{\sqrt{t^2 - (y-x)^2}} + \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} t^n \fint_{B(x,t)} \frac{h(y) d^n y}{\sqrt{t^2 - (y-x)^2}} \right).$$

*Proof.* The foregoing formula yields with the function $f = 1$ and for $x = 0$ and $t = r$:

$$(n+1)\omega_{n+1} r^n = \int_{\partial B(0,r)} d\sigma(y) = 2r \int_{B(\bar{0},r)} \frac{d^n y}{\sqrt{r^2 - y^2}} = 2r \int_0^r \int_{\partial B(\bar{0},s)} \frac{d\sigma(y)}{\sqrt{r^2 - s^2}} ds$$
$$= 2r \int_0^r \frac{n\omega_n s^{n-1} ds}{\sqrt{r^2 - s^2}} = 2n\omega_n r^n \int_0^1 (1-x^2)^{\frac{n-2}{2}} dx$$

with the substitution $x = \frac{\sqrt{r^2 - s^2}}{r} = \sqrt{1 - (\frac{s}{r})^2}$ , $rdx = -\frac{sds}{rx}$ and $s = r\sqrt{1-x^2}$. We calculate the remaining integral by inductive partial integration and insert the result:

$$\int_0^1 (1-x^2)^m dx = x(1-x^2)^m\Big|_{x=0}^1 + 2m\int_0^1 (1-x^2)^{m-1}x^2 dx$$
$$= -2m\int_0^1 (1-x^2)^m dx + 2m\int_0^1 (1-x^2)^{m-1}dx = \frac{2m}{2m+1}\int_0^1 (1-x^2)^{m-1} = \frac{(m!2^m)^2}{(2m+1)!}.$$
$$\frac{2^{\frac{n+1-3}{2}}\frac{n+1-3}{2}!}{(n+1-2)!}\frac{2\omega_n}{(n+1)\omega_{n+1}} = \frac{2^{\frac{n-2}{2}}\frac{n-2}{2}!}{(n-1)!}\frac{(2\frac{n-2}{2}+1)!}{2\frac{n}{2}(2^{\frac{n-2}{2}}\frac{n-2}{2}!)^2} = \frac{1}{2^{\frac{n}{2}}\frac{n}{2}!}.$$

**q.e.d.**

By this formula the value of the solution at $(x,t)$ depends on the values of $g$ and $h$ on $B(x,t)$ rather than the values on $\partial B(x,t)$ like in odd dimensions. Hence the length of the speed of propagation is bounded by 1, but not equal to 1. Furthermore the solution is at $(x,t)$ $\frac{n}{2}$ times less differentiable as $g$ and $\frac{n-2}{2}$ times less differentiable then $h$.

We close this section by showing that for any $k \in \mathbb{N}$ the solution in dimenson $n = 2k-1$ is obtained by the method of descent from the solution in dimension $n+1 = 2k$. The initial values $g$ and $h$ are functions on $\mathbb{R}^n$ and again they define functions on $\mathbb{R}^{n+1}$ $\bar{g}(x) = g(\bar{x})$ and $\bar{h}(x) = h(\bar{x})$ with $\overline{(x_1,\ldots,x_{n+1})} = (x_1,\ldots,x_n)$ which do not depend on $x_{n+1}$. For a differentiable function $f$ on $\mathbb{R}^n$ we have

$$\frac{\partial}{\partial x_{n+1}}\int_{B(x,t)} \frac{f(\bar{y})}{\sqrt{t^2-(y-x)^2}}d^{n+1}y = \int_{B(0,t)} \frac{\frac{\partial f}{\partial x_{n+1}}(\bar{x}+\bar{z})}{\sqrt{t^2-z^2}}d^{n+1}z = 0.$$

Hence the solution of the corresponding initial value problem in dimension $n+1$ does not depend on $x_{n+1}$ and again defines a solutions of the initial value problem in dimension $n$. The map $y \mapsto \bar{y}$ maps the ball $B(0,t) \subset \mathbb{R}^{n+1}$ onto the corresponding ball $\bar{B}(0,t) \subset \mathbb{R}^n$. The preimage of $\bar{y} \in \bar{B}(0,t)$ with repsect to this map is $\{(\bar{y}, y_{n+1}) \mid y_{n+1} \in (-\sqrt{t^2-\bar{y}^2}, \sqrt{t^2-\bar{y}^2})\}$. The substitution $z = \frac{y_{n+1}}{\sqrt{t^2-\bar{y}^2}}$ yields $dy_{n+1} = \sqrt{t^2-\bar{y}^2}dz$ and $\sqrt{t^2-\bar{y}^2-y_{n+1}^2} = \sqrt{t^2-\bar{y}^2}\sqrt{1-z^2}$:

$$\int_{B(0,t)} \frac{\bar{f}(x+y)}{\sqrt{t^2-y^2}}d^{n+1}y = \int_{\bar{B}(0,t)}\int_{-\sqrt{t^2-\bar{y}^2}}^{\sqrt{t^2-\bar{y}^2}} \frac{dy_{n+1}}{\sqrt{t^2-\bar{y}^2-y_{n+1}^2}}f(\bar{x}+\bar{y})d^n\bar{y} =$$
$$= \int_{\bar{B}(0,t)}\int_{-1}^1 \frac{dz}{\sqrt{1-z^2}}f(\bar{x}+\bar{y})d^n\bar{y} = \pi\int_{B(\bar{x},t)} f(\bar{y})d^n\bar{y}.$$

So for odd $n \geq 3$ we indeed recover the formula for the solution in even dimension:

$$\begin{aligned}
u(x,t) &= \frac{\pi}{2^{\frac{n+1}{2}}\frac{n+1}{2}!\omega_{n+1}}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}}\int_{B(x,t)} g(y)d^n y + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}}\int_{B(x,t)} h(y)d^n y\right)\\
&= \frac{\pi n\omega_n}{2^{\frac{n+1}{2}}\frac{n+1}{2}!\omega_{n+1}}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} g(y)d^n y + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} h(y)d^n y\right),
\end{aligned}$$

$$\frac{\pi n\omega_n}{2^{\frac{n+1}{2}}\frac{n+1}{2}!\omega_{n+1}} = \frac{\pi}{2^{\frac{n+1}{2}}\frac{n-1}{2}!}\frac{2n\omega_n}{(n+1)\omega_{n+1}} = \frac{\pi}{2^{\frac{n+1}{2}}\frac{n-1}{2}!}\frac{(2^{\frac{n-3}{2}}\frac{n-3}{2}!)^2(n-1)}{(n-2)!\frac{\pi}{2}} = \frac{2^{\frac{n-3}{2}}\frac{n-3}{2}!}{(n-2)!}.$$

Here we use the same formulas as for even $n$. However, for odd $n \geq 3$ the final integral is

$$\begin{aligned}
\frac{(n+1)\omega_{n+1}}{2n\omega_n} &= \int_0^1 (1-x^2)^{\frac{n}{2}-1}dx = \frac{n-2}{n-1}\int_0^1 (1-x^2)^{\frac{n}{2}-2}dx\\
&= \frac{(n-2)!}{(2^{\frac{n-3}{2}}\frac{n-3}{2}!)^2(n-1)}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{(n-2)!}{(2^{\frac{n-3}{2}}\frac{n-3}{2}!)^2(n-1)}\frac{\pi}{2}.
\end{aligned}$$

For $n = 1$ the first formula gives with $\omega_2 = \pi$ D'Alembert's Formula. In particular, the formula for the solution in dimension $n \in \mathbb{N}$ gives by the method of descent the formulas for the solutions in all dimensions less than $n$. The iterated application of the method of descent shows that the solutions in dimensions $m < n$ are obtained by considering solutions $u$ in dimension $n$ with initial values which depend only on the first $m$ variables $x_1, \ldots, x_m$. These solutions also depend only on $x_1, \ldots, x_m$ and define the corresponding solutions as functions depending on $(x_1, \ldots, x_m, t) \in \mathbb{R}^m \times \mathbb{R}^+$.

## 5.7 Inhomogeneous Wave Equation

Duhamel's principle also applies to the initial value problem of the wave equation. We conceive the wave equation as a first order linear ODE on pairs of functions on $x \in \mathbb{R}^n$:

$$\frac{d}{dt}\begin{pmatrix} u(\cdot,t)\\ \frac{\partial u}{\partial t}(\cdot,t)\end{pmatrix} = \begin{pmatrix} 0 & 1\\ \triangle & 0\end{pmatrix}\begin{pmatrix} u(\cdot,t)\\ \frac{\partial u}{\partial t}(\cdot,t)\end{pmatrix} + \begin{pmatrix} 0\\ f(\cdot,t)\end{pmatrix}.$$

So we may calculate the special solution of the inhomogeneous wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \triangle u = f && \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+\\
&u(x,0) = 0 \quad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0 && \text{for} \quad x \in \mathbb{R}^n
\end{aligned}$$

as an integeral of the family of solutions of the homogeneous wave equation whose initial values is given by the inhomogeneity: If $u(x,t,s)$ solves for any $s \in \mathbb{R}^+$

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 && \text{for} \quad (x,t) \in \mathbb{R}^n \times (s,\infty)\\
&u(x,s,s) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,s,s) = f(x,s) && \text{for} \quad x \in \mathbb{R}^n,
\end{aligned}$$

then $u(x,t) = \int_0^t u(x,t,s)ds$ solves the former inhomogeneous wave equation since

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}\left(u(x,t,t) + \int_0^t \frac{\partial u}{\partial t}(x,t,s)ds\right) = \frac{\partial}{\partial t}\int_0^t \frac{\partial u}{\partial t}(x,t,s)ds =\\
&= \frac{\partial u}{\partial t}(x,t,t) + \int_0^t \frac{\partial^2 u}{\partial t^2}(x,t,s)ds = f(x,t) + \int_0^t \triangle u(x,t,s)ds = f(x,t) + \triangle u(x,t).
\end{aligned}$$

Consequently the initial value problem of the inhomogeneous wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \triangle u = f && \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+\\
&u(x,0) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x \in \mathbb{R}^n
\end{aligned}$$

is the sum of the former special solution with trivial initial value and the solution of the corresponding homogeneous initial value problem.

Finally we investigate how the presence determines the past. The wave equations is invariant with respect to time reversal $t \mapsto -t$. However, this transformation replaces $\frac{\partial u}{\partial t}$ by $-\frac{\partial u}{\partial t}$. Therefore the values $u(x,t)$ of the solution of the end value problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \triangle u = f && \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^-\\
&u(x,0) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x \in \mathbb{R}^n
\end{aligned}$$

are given by the values $u(x,-t)$ of the solution of the initial value problem with initial values $g$ and $-h$ and inhomogeneity $(x,t) \mapsto f(x,-t)$. This means that we can derive both the future and the past from the presence. Both solutions fit together and form a solution $u(x,t)$ of the wave equation on $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ which is completely determined by its values $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ on $x \in \mathbb{R}^n$.

## 5.8 Energy Methods

Hyperbolic PDEs do not satisfy a maximum principle. A maximum in the interior of a domain can be only excluded by a second order PDE which ensures that the Hessian cannot be indefinite. This are exactely the elliptic PDEs and theire limiting cases as the parabolic PDEs. Indeed the methods of Theorem 3.13 applies to degenerate elliptic PDEs as well. However, energy methods apply to hyperbolic PDEs as well as to elliptic PDEs and we may prove the uniqeness of solutions with such methods:

**Theorem 5.7** (uniqueness of the solutions of the wave equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following initial values porblem of the wave equation*

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \triangle u &= f && \text{on} \quad \Omega \times (0,T) \\
u(x,t) &= g(x,t) && \text{on} \quad \Omega \times \{t=0\} \qquad \text{and on} \quad \partial\Omega \times (0,T) \\
\frac{\partial u}{\partial t}(x,0) &= h(x) && \text{on} \quad \Omega \times \{t=0\}
\end{aligned}$$

*has a unique solution on $(x,t) \in \Omega \times (0,T)$.*

*Proof.* The difference of two solutions solves the analogous homogeneous initial value problem with $f = g = h = 0$. For such a solution we define the energy as

$$e(t) = \frac{1}{2}\int_\Omega \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d^n x.$$

Then we calculate

$$\begin{aligned}
\frac{de}{dt}(t) &= \int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(x,t)\frac{\partial u}{\partial t}(x,t) + \frac{\partial \nabla u}{\partial t} u(x,t)\nabla u(x,t)\right) d^n x \\
&= \int_\Omega \frac{\partial u}{\partial t}(x,t)\left(\frac{\partial^2 u}{\partial t^2}(x,t) - \triangle u(x,t)\right) d^n x = 0.
\end{aligned}$$

Here we applied once the divergence theorem to the vector field $\frac{\partial u}{\partial t}\nabla u$ which vanishes at $\partial\Omega \times [0,T]$ together with $u$ and $\frac{\partial u}{\partial t}$. Initially the energy is zero $e(0) = 0$. Since the energy is non negative it stays zero for all positive times $t > 0$. This shows that $u$ is constant and vanishes on $\Omega \times [0,T)$ since it vanishes initially. **q.e.d.**

The proof gives the same conclusion if we assume that instead of $u(x,t)$ the normal derivative $\nabla u(x,t) \cdot N(x,t)$ is given on $\partial\Omega \times [0,T]$. Finally we give a simple proof that the length of the speed of propagation is bounded by 1.

**Theorem 5.8.** *If $u$ is any solution of the homogeneous wave equation obeying $u = \frac{\partial u}{\partial t} = 0$ on $B(x_0,t_0)$ for $t = 0$, then $u$ vanishes on the cone $\{(x,t) \mid |x - x_0| \leq t_0 - t, t > 0\}$.*

*Proof.* Again we calculate the time derivative of the energy

$$\begin{aligned}
e(t) &= \frac{1}{2}\int_{B(x_0,t_0-t)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d^n x \quad \text{as}\\
\frac{de}{dt}(t) &= \frac{1}{2}\frac{d}{dt}\int_0^{t_0-t}\int_{\partial B(x_0,s)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d\sigma(x)ds\\
&= \int_{B(x_0,t_0-t)} \left(\frac{\partial^2 u}{\partial t^2}(x,t)\frac{\partial u}{\partial t}(x,t) + \frac{\partial \nabla u}{\partial t}(x,t)\nabla u(x,t)\right) d^n x\\
&\quad - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d\sigma(x)\\
&= \int_{B(x_0,t_0-t)} \frac{\partial u}{\partial t}(x,t)\left(\frac{\partial^2 u}{\partial t^2}(x,t) - \triangle u(x,t)\right) d^n x\\
&\quad + \int_{\partial B(x_0,t_0-t)}\left(\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - \frac{1}{2}\left(\nabla u(x,t)\right)^2\right) d\sigma(x)\\
&= \int_{\partial B(x_0,t_0-t)}\left(\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - \frac{1}{2}(\nabla u(x,t))^2\right) d\sigma(x).
\end{aligned}$$

Since the outer normal has length one we derive

$$\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) \le \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}(\nabla u(x,t))^2$$

with $a = \nabla u(x,t)$ and $b = \frac{\partial u}{\partial t}(x,t)N(x,t)$ from the following inequality:

$$a\cdot b \le a\cdot b + \frac{1}{2}(a-b)\cdot(a-b) = \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

So by $\dot{e}(t) \le 0$ the energy is monotonically decreasing. Because the energy is non-negative and vanishes initially it stays zero for all positive times in $t \in [0,t_0]$. This implies $u = 0$ on $\{(x,t) \mid |x-x_0| \le t_0 - t, t > 0\}$. **q.e.d.**

By the invariance with respect to time reversal we can also deduce the vanishing of $u$ on the cone $\{(x,t) \mid |x-x_0| < t_0 + t, t < 0\}$ from the vanishing of $u$ and $\frac{\partial u}{\partial t} = 0$ on $(x,t) \in B(x_0,t_0)\times\{0\}$.