School of Business Mathematics and Informatics

SOLUTIONS OF THE SINH-GORDON EQUATION OF SPECTRAL GENUS-TWO AND THEIR CONFORMAL CLASS ON THE BOUNDARY $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$

Written by

Nils Wagner

aspired academic degree:

Master of Science (M.Sc.)

Supervisor and Advisor:

Prof. Dr. Martin Schmidt

Course:

Business Mathematics (M.Sc.) and Mathematics (M.Sc.)

Submitted:

March 30th, 2026

Contents

Introduction

The sinh–Gordon equation is a classical integrable partial differential equation whose finite–type solutions admit an algebro–geometric description in terms of spectral data. In the present thesis we study this picture in spectral genus–two. Our main interest is the boundary of the corresponding moduli space, where the spectral data degenerate and the period problem acquires a reduced genus–one form. The central goal of the thesis is to show that this reduced conformal information already determines the boundary data completely. More precisely, we prove that the map

$$S\colon M_2^2 \cup M_2^3 \to F_3, \qquad a \mapsto q_2(a) + iq_1(a),$$

is bijective.

After the preliminaries, Chapter 2 introduces the basic dynamical framework. We define the space $\mathcal{P}_2$ of genus–two potentials and the associated Lax equations, which give rise to two commuting vector fields. Their integral curves are the Polynomial Killing Fields. A first important point is that these flows preserve the determinant polynomial, so that every solution stays inside a fixed isospectral set

$$I(a) = \{\zeta \in \mathcal{P}_2 \mid \det \zeta = \lambda a(\lambda)\}.$$

This chapter therefore provides the basic objects of the thesis: the potentials, the polynomial Killing fields, the determinant polynomial, and the isospectral sets on which the later analysis is carried out.

Chapter 3 is devoted to the structure of these isospectral sets. The space M_2 of determinant polynomials is first described explicitly in terms of its symmetry and positivity properties, and then decomposed into the strata $M_2^1, \ldots, M_2^5$ according to the multiplicities and positions of the roots. The main result of the chapter is a complete description of the geometry of $I(a)$ in each of these cases. In particular, the isospectral sets are shown to be compact, the local flows from Chapter 2 extend to a global $\mathbb{R}^2$-action, and the orbit structure is determined explicitly. This gives the global geometric picture of the genus–two moduli space and shows in which way the boundary strata differ from the generic case.

In Chapter 4 we turn to the corresponding lattices of periods. For the generic stratum these lattices encode the conformal type of the associated solutions and are represented by points in a fundamental domain. To analyze what happens at the boundary, we introduce frames, monodromy, and the spectral curve, and reformulate the period problem in terms of meromorphic differentials and their periods. This leads to an extended lattice map

$$T\colon M_2^1 \cup M_2^2 \cup M_2^3 \to \mathcal{F},$$

whose essential property is continuity. In this way, Chapter 4 explains how conformal data extend to the boundary and how the behavior of the lattices reflects the degeneration of the spectral data.

Chapter 5 develops the Whitham deformation theory needed for the boundary analysis. The point here is that the stratum M_2^2 can be reduced to a genus–one problem by factoring out the double unitary root and passing to suitable reduced data. In this reduced setting one obtains a distinguished pair of periods and hence a conformal class

$$\tau = q_2 + iq_1 \in \mathcal{F}_3.$$

The Whitham equations are then used to describe this reduced deformation problem in a systematic way, and the construction is extended continuously to the limiting stratum M_2^3. Thus, by the end of

Chapter 5, the boundary problem has been translated into the study of a conformal invariant attached to reduced genus–one data.

Finally, Chapter 6 proves the main theorem of the thesis. Because the reduced problem comes with a distinguished ordered basis of periods, the natural target is the enlarged domain F_3 rather than the usual modular domain. The chapter studies the map S in detail, analyzes the curves in the τ–plane obtained by fixing the radial parameter, and shows that these curves fill the whole region F_3 without intersecting. This yields surjectivity, while the monotonicity properties of the construction imply injectivity. As a consequence, S is bijective. The conclusion is that on the boundary of the spectral genus–two moduli space, the reduced genus–one conformal class contains exactly the information needed to recover the spectral data.

Chapter 1

Preliminaries

In this chapter we collect the facts and results that will be needed later on. We state them only briefly, since the proofs that play an essential role for the development will be given in the subsequent chapters. At this stage, the emphasis is on the results themselves rather than on their proofs.

1.1 A little bit of Lie theory

In this section we collect the Lie theoretic facts which will be used later on. We restrict ourselves to the matrix setting, since this is sufficient for the applications in the thesis. In particular, we discuss matrix Lie groups and their Lie algebras, left-invariant vector fields and their flows, matrix-valued differential equations in commutator form, the Maurer–Cartan equation, and the differentiation of the determinant.

The results of this section can be found, for example, in Hall [2000], chapters 1 to 3 and Trofimov [2013], chapter 3. For the determinant identities used later on, see Magnus and Neudecker [2019], Part I.

1.1.1 Matrix groups and Lie algebras

We begin with the basic examples of matrix groups.

Definition 1.1. *For $n \in \mathbb{N}$ we denote by $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ the vector spaces of real and complex $n \times n$ matrices, respectively. Moreover, we set*

$$GL(n,\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}, \qquad GL(n,\mathbb{C}) := \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\}.$$

Further classical matrix groups are

$$SL(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}, \qquad SL(n,\mathbb{C}) := \{A \in GL(n,\mathbb{C}) \mid \det A = 1\},$$

$$O(n) := \{A \in GL(n,\mathbb{R}) \mid A^t A = I\}, \qquad U(n) := \{A \in GL(n,\mathbb{C}) \mid A^* A = I\},$$

and

$$SU(n) := \{A \in U(n) \mid \det A = 1\}.$$

The set $GL(n,\mathbb{K})$, where $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, is an open subset of $M_n(\mathbb{K})$, hence a smooth manifold. The groups listed above are smooth submanifolds of $GL(n,\mathbb{K})$ and simultaneously groups under matrix multiplication. This motivates the following notion.

Definition 1.2. *A* matrix Lie group *is a subgroup $G \subset GL(n,\mathbb{K})$ which is an embedded smooth submanifold. In this case the group operations*

$$G \times G \to G, \quad (g,h) \mapsto gh, \qquad \textit{and} \qquad G \to G, \quad g \mapsto g^{-1},$$

are smooth maps.

The infinitesimal object attached to a Lie group is its Lie algebra.

Definition 1.3. *Let $G \subset GL(n, \mathbb{K})$ be a matrix Lie group and let e denote its identity element. The* Lie algebra *of G is the tangent space*

$$\mathfrak{g} := T_e G.$$

Since G is a submanifold of $M_n(\mathbb{K})$, we may regard $\mathfrak{g}$ as a linear subspace of $M_n(\mathbb{K})$.

In the matrix setting the Lie bracket is given by the commutator.

Definition 1.4. *For $A, B \in M_n(\mathbb{K})$ we define*

$$[A, B] := AB - BA.$$

This bracket restricts to the Lie algebra $\mathfrak{g}$ of every matrix Lie group G.

The Lie algebras of the classical matrix groups are given by

$$\begin{aligned} \mathfrak{gl}(n, \mathbb{R}) &= M_n(\mathbb{R}), & \mathfrak{gl}(n, \mathbb{C}) &= M_n(\mathbb{C}), \\ \mathfrak{sl}(n, \mathbb{R}) &= \{A \in M_n(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}, & \mathfrak{sl}(n, \mathbb{C}) &= \{A \in M_n(\mathbb{C}) \mid \operatorname{tr}(A) = 0\}, \\ \mathfrak{o}(n) &= \{A \in M_n(\mathbb{R}) \mid A^t = -A\}, & \mathfrak{u}(n) &= \{A \in M_n(\mathbb{C}) \mid A^* = -A\}, \end{aligned}$$

and

$$\mathfrak{su}(n) = \{A \in M_n(\mathbb{C}) \mid A^* = -A,\ \operatorname{tr}(A) = 0\}.$$

Lemma 1.5. *The commutator bracket on $M_n(\mathbb{K})$ is bilinear, antisymmetric, and satisfies the Jacobi identity*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

for all $A, B, C \in M_n(\mathbb{K})$.

Proof. Bilinearity and antisymmetry are immediate from the definition. For the Jacobi identity we expand:

$$\begin{aligned} [A, [B, C]] &= A(BC - CB) - (BC - CB)A \\ &= ABC - ACB - BCA + CBA, \end{aligned}$$

and similarly

$$\begin{aligned} [B, [C, A]] &= BCA - BAC - CAB + ACB, \\ [C, [A, B]] &= CAB - CBA - ABC + BAC. \end{aligned}$$

Adding the three expressions yields zero. □

Remark 1.6. *Hence every matrix Lie algebra is a Lie algebra in the usual sense, that is, a vector space endowed with a bilinear, antisymmetric bracket satisfying the Jacobi identity.*

Definition 1.7 (SU(2)-reality condition)**.** *A holomorphic map*

$$g : \mathbb{C}^\times \to \mathrm{SL}(2, \mathbb{C})$$

is said to satisfy the SU(2)-reality condition *if*

$$g(\lambda) = \left(g(\bar{\lambda}^{-1})^*\right)^{-1} \qquad \textit{for all } \lambda \in \mathbb{C}^\times.$$

Remark 1.8. *If $\lambda \in \mathbb{S}^1$, then $\bar{\lambda}^{-1} = \lambda$, and the reality condition becomes*

$$g(\lambda) = \big(g(\lambda)^*\big)^{-1}.$$

Equivalently,

$$g(\lambda)^* g(\lambda) = 1.$$

Hence $g(\lambda) \in \mathrm{U}(2)$ for all $\lambda \in \mathbb{S}^1$. Since moreover

$$g(\lambda) \in \mathrm{SL}(2, \mathbb{C}),$$

it follows that

$$g(\lambda) \in \mathrm{SU}(2) \qquad \text{for all } \lambda \in \mathbb{S}^1.$$

Remark 1.9. *This is the standard reality condition corresponding to the compact real form* SU(2) *in the loop-group setting; compare, for example, Pressley [1985].*

1.1.2 Left-invariant vector fields and flows

We now pass from the algebraic structure of $\mathfrak{g}$ to vector fields on the group.

Definition 1.10. *Let G be a Lie group and $g \in G$. The map*

$$L_g : G \to G, \qquad h \mapsto gh,$$

is called left translation by *g. A smooth vector field X on G is called* left-invariant *if*

$$(dL_g)_h(X(h)) = X(gh)$$

for all $g, h \in G$.

The left-invariant vector fields are completely determined by their value at the identity.

Proposition 1.11. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g} = T_eG$. For every $A \in \mathfrak{g}$ there exists a unique left-invariant vector field X_A on G such that $X_A(e) = A$. It is given by*

$$X_A(g) = (dL_g)_e(A) = gA.$$

Thus the assignment

$$\mathfrak{g} \to \{\textit{left-invariant vector fields on } G\}, \qquad A \mapsto X_A,$$

is a linear isomorphism.

Proof. For $A \in T_eG$ define $X_A(g) := (dL_g)_e(A)$. Since left translations are smooth, X_A is a smooth vector field. Moreover, for all $g, h \in G$ we have

$$(dL_g)_h(X_A(h)) = (dL_g)_h(dL_h)_e(A) = (dL_{gh})_e(A) = X_A(gh),$$

hence X_A is left-invariant. Uniqueness follows by evaluating the left-invariance identity at $h = e$. $\square$

The relevance of global vector fields comes from the fact that they generate flows.

Theorem 1.12. *Let M be a smooth finite-dimensional manifold and let X be a smooth vector field on M. Then X induces a local flow, that is, for every $p \in M$ there exist an open interval $I_p \subset \mathbb{R}$ containing 0 and a smooth map*

$$\Phi_p : I_p \to M$$

such that

$$\Phi_p(0) = p, \qquad \frac{d}{dt}\Phi_p(t) = X(\Phi_p(t)).$$

If X is complete, these local flows combine to a smooth action

$$\Phi : \mathbb{R} \times M \to M.$$

Remark 1.13. *On a Lie group, every left-invariant vector field is complete. In particular, each $A \in \mathfrak{g}$ generates a one-parameter subgroup of G.*

For later use we briefly recall the exponential map.

Definition 1.14. *For $A \in M_n(\mathbb{K})$ the* matrix exponential *is defined by*

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Proposition 1.15. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. For every $A \in \mathfrak{g}$ the curve*

$$\gamma_A(t) := \exp(tA)$$

is a one-parameter subgroup of G, and

$$\gamma_A(0) = e, \qquad \gamma_A'(0) = A.$$

Moreover, γ_A is the integral curve of the left-invariant vector field X_A through the identity.

1.1.3 Matrix-valued differential equations

Since the objects occurring later in the thesis are matrix-valued, we record the basic form of the corresponding differential equations.

Let $\Omega \subset \mathbb{R}^2$ be open with coordinates (x, y), and let $U, V : \Omega \to M_n(\mathbb{K})$ be smooth maps. We consider a matrix-valued map $F : \Omega \to GL(n, \mathbb{K})$ satisfying

$$\partial_x F = FU, \qquad \partial_y F = FV. \tag{1.1}$$

This is the form in which the frame equations appear later on.

Proposition 1.16. *Let $F : \Omega \to GL(n, \mathbb{K})$ be a smooth solution of (1.1). Then U and V satisfy the compatibility condition*

$$\partial_y U - \partial_x V + [U, V] = 0. \tag{1.2}$$

Proof. Since mixed partial derivatives commute, we obtain

$$\partial_y(\partial_x F) = \partial_x(\partial_y F).$$

Using (1.1) this becomes

$$\partial_y(FU) = \partial_x(FV),$$

hence

$$(\partial_y F)U + F(\partial_y U) = (\partial_x F)V + F(\partial_x V).$$

Substituting again (1.1) yields

$$FVU + F(\partial_y U) = FUV + F(\partial_x V).$$

Multiplying from the left by F^{-1}, we obtain

$$VU + \partial_y U = UV + \partial_x V,$$

that is,

$$\partial_y U - \partial_x V + [U, V] = 0. \tag{1.3}$$

□

The previous condition is the coordinate form of the Maurer–Cartan equation.

Definition 1.17. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. For a smooth map $F : \Omega \to G$, the $\mathfrak{g}$-valued 1-form*

$$\alpha := F^{-1} dF$$

is called the Maurer–Cartan form *of F.*

If we write

$$\alpha = U\, dx + V\, dy,$$

then (1.1) is equivalent to $F^{-1} dF = \alpha$, and (1.2) is precisely the Maurer–Cartan equation.

Theorem 1.18 (Maurer–Cartan equation)**.** *Let $\Omega \subset \mathbb{R}^2$ be simply connected and let $\alpha \in \Omega^1(\Omega, \mathfrak{g})$ be a smooth $\mathfrak{g}$-valued 1-form. Then the following are equivalent:*

1. *There exists a smooth map $F : \Omega \to G$ such that*

$$F^{-1} dF = \alpha.$$

2. *The form α satisfies*

$$d\alpha + \alpha \wedge \alpha = 0.$$

Moreover, if a solution exists, it is unique up to left multiplication by a constant element of G.

Remark 1.19. *If $\alpha = U\, dx + V\, dy$, then*

$$d\alpha + \alpha \wedge \alpha = \big(\partial_x V - \partial_y U + [U, V]\big)\, dx \wedge dy.$$

Thus the Maurer–Cartan equation is equivalent to the condition (1.2).

In the thesis we also encounter vector fields of commutator type on a vector space of matrices. We therefore record the corresponding infinitesimal compatibility condition.

Proposition 1.20. *Let $\mathcal{V}$ be a vector subspace of $M_n(\mathbb{K})$, and let $U, V : \mathcal{V} \to M_n(\mathbb{K})$ be smooth maps. Define vector fields X, Y on $\mathcal{V}$ by*

$$X(\zeta) := [\zeta, U(\zeta)], \qquad Y(\zeta) := [\zeta, V(\zeta)].$$

Then their Lie bracket is given by

$$[X, Y](\zeta) = \big[\zeta,\, dU_\zeta(Y(\zeta)) - dV_\zeta(X(\zeta)) - [U(\zeta), V(\zeta)]\big].$$

In particular, the Jacobi identity is exactly what combines the double commutator terms into a single commutator.

Proof. For $\eta \in \mathcal{V}$ we have

$$dX_\zeta(\eta) = [\eta, U(\zeta)] + [\zeta, dU_\zeta(\eta)],$$

and similarly

$$dY_\zeta(\eta) = [\eta, V(\zeta)] + [\zeta, dV_\zeta(\eta)].$$

Hence

$$\begin{aligned} [X, Y](\zeta) &= dY_\zeta(X(\zeta)) - dX_\zeta(Y(\zeta)) \\ &= [X(\zeta), V(\zeta)] - [Y(\zeta), U(\zeta)] + [\zeta, dV_\zeta(X(\zeta)) - dU_\zeta(Y(\zeta))]. \end{aligned}$$

Since $X(\zeta) = [\zeta, U(\zeta)]$ and $Y(\zeta) = [\zeta, V(\zeta)]$, the Jacobi identity implies

$$[X(\zeta), V(\zeta)] - [Y(\zeta), U(\zeta)] = [\zeta, [U(\zeta), V(\zeta)]].$$

Substituting this into the previous expression gives the claim. □

Remark 1.21. *In applications one chooses U and V in such a way that the expression*

$$dU_\zeta(Y(\zeta)) - dV_\zeta(X(\zeta)) - [U(\zeta), V(\zeta)]$$

vanishes. Then the vector fields X and Y commute, and the corresponding flows are compatible.

1.1.4 Differentiation of the determinant

The determinant polynomial plays a central role later on, so we collect the relevant differentiation formulas.

Definition 1.22. *For $A \in M_n(\mathbb{K})$ we denote by $A^\dagger$ the adjugate matrix of A, characterized by*

$$A\,A^\dagger = A^\dagger\,A = \det(A)\,I.$$

Proposition 1.23 (Derivative of the determinant)**.** *Let $A : I \to M_n(\mathbb{K})$ be a differentiable curve, where $I \subset \mathbb{R}$ is an interval. Then*

$$\frac{d}{dt}\det(A(t)) = \operatorname{tr}\bigl(A(t)^\dagger\,A'(t)\bigr).$$

If $A(t)$ is invertible, this becomes

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\ \operatorname{tr}\bigl(A(t)^{-1}A'(t)\bigr).$$

Proof. The determinant is a polynomial in the matrix entries, hence differentiable. Its differential at A is given by the cofactor expansion, which yields

$$d(\det)_A(H) = \operatorname{tr}(\,A^\dagger H)$$

for $H \in M_n(\mathbb{K})$. Applying this to $H = A'(t)$ gives the first formula. If A is invertible, then

$$A^\dagger = \det(A)\,A^{-1},$$

and the second identity follows. □

The next consequence is the one needed most often in the sequel.

Corollary 1.24. *Let $A : I \to M_n(\mathbb{K})$ and $B : I \to M_n(\mathbb{K})$ be differentiable curves satisfying*

$$A'(t) = [A(t), B(t)].$$

Then $\det(A(t))$ is constant on I.

Proof. By the previous proposition,

$$\frac{d}{dt}\det(A(t)) = \operatorname{tr}\bigl(A(t)^\dagger[A(t), B(t)]\bigr).$$

Using $A^\dagger A = AA^\dagger = \det(A)I$, we obtain

$$\operatorname{tr}\bigl(A^\dagger[A,B]\bigr) = \operatorname{tr}\bigl(A^\dagger AB - A^\dagger BA\bigr) = \det(A)\operatorname{tr}(B) - \operatorname{tr}\bigl(BA^\dagger A\bigr).$$

Since $A^\dagger A = \det(A)I$, this equals

$$\det(A)\operatorname{tr}(B) - \det(A)\operatorname{tr}(B) = 0.$$

Hence $\det(A(t))$ is constant. □

In particular, the same argument applies to characteristic polynomials.

Corollary 1.25. *Let $A : I \to M_n(\mathbb{K})$ and $B : I \to M_n(\mathbb{K})$ be differentiable curves satisfying*

$$A'(t) = [A(t), B(t)].$$

Then for every $\lambda \in \mathbb{K}$ the quantity

$$\det(\lambda I - A(t))$$

is independent of t. Equivalently, the characteristic polynomial of $A(t)$ is preserved along the flow.

Proof. For fixed λ we set

$$C_\lambda(t) := \lambda I - A(t).$$

Then

$$C'_\lambda(t) = -A'(t) = -[A(t), B(t)] = [\lambda I - A(t), B(t)] = [C_\lambda(t), B(t)].$$

The claim now follows from the previous corollary. □

Remark 1.26. *This observation is the basic mechanism behind the preservation of determinant polynomials along Lax-type flows.*

Chapter 2

Potentials and Polynomial Killing Fields

In this chapter we introduce the space of potentials and the associated Polynomial Killing Fields that will be studied in the sequel. We restrict ourselves to the case of spectral genus $g = 2$, since this is the relevant setting for the applications considered in this thesis. Our aim at this stage is to formulate the basic objects and to record the corresponding differential equations, while postponing the more substantial analysis to the following chapters.

The sinh–Gordon equation is a fundamental example of an integrable nonlinear partial differential equation. It appears in differential geometry and mathematical physics, for instance in the theory of surfaces with constant negative curvature. A particularly important class of solutions is given by the finite-type, or finite-gap, solutions. These arise from the integrable structure of the equation and are described by algebro-geometric data associated with a spectral curve of finite genus. In this way, the nonlinear equation can be studied by methods from algebraic geometry and spectral theory. This point of view not only provides an explicit description of special families of solutions, but also reveals deep connections between integrable systems, Riemann surfaces, and the theory of periodic and quasi-periodic solutions.

In the present situation, solutions are parametrized by flows on a space of potentials induced by a particular system of ordinary differential equations. Before studying these flows, we first introduce the corresponding space of potentials. In our case, this is the following set of matrix-valued polynomials of degree 3:

$$\mathcal{P}_2 := \left\{ \zeta = \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+ \right\}.$$

On this space, the Lax equations define two vector fields:

$$\frac{\partial \zeta}{\partial x} = [\zeta, U(\zeta)], \qquad \frac{\partial \zeta}{\partial y} = [\zeta, V(\zeta)], \tag{2.1}$$

where the two 2×2 matrices $U(\zeta)$ and $V(\zeta)$ are given by

$$U(\zeta) := \begin{pmatrix} \frac{\alpha - \bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\bar{\alpha} - \alpha}{2} \end{pmatrix} \quad \text{and} \quad V(\zeta) := \begin{pmatrix} \frac{\alpha + \bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & \frac{-(\alpha + \bar{\alpha})}{2} \end{pmatrix}.$$

The two equations in (2.1) form a system of ordinary differential equations, and by standard ODE theory there exists an integral curve for every initial potential $\zeta(0) = \zeta_0 \in \mathcal{P}_2$. It therefore remains to show that these integral curves fit together to define a map

$$(x, y) \mapsto \zeta(x, y)$$

on an open neighbourhood of $(0,0) \in \mathbb{R}^2$. For this, it is enough to verify that the two vector fields defined by (2.1) commute.

This is precisely the situation considered in the preliminaries. For vector fields of the form

$$X(\zeta) = [\zeta, U(\zeta)], \qquad Y(\zeta) = [\zeta, V(\zeta)],$$

Proposition 1.20 shows that their Lie bracket is determined by the expression

$$dU_\zeta(Y(\zeta)) - dV_\zeta(X(\zeta)) - [U(\zeta), V(\zeta)].$$

Along a solution $\zeta(x, y)$ of (2.1), this is exactly the quantity

$$\frac{\partial U(\zeta)}{\partial y} - \frac{\partial V(\zeta)}{\partial x} - [U, V].$$

Hence, by Remark 1.21, the two flows are compatible provided that

$$[U, V] + \frac{\partial V(\zeta)}{\partial x} - \frac{\partial U(\zeta)}{\partial y} = 0. \tag{2.2}$$

This is precisely the Maurer–Cartan equation. A direct computation shows that (2.2) is satisfied for our specific choice of $U(\zeta)$ and $V(\zeta)$. We do not carry out this computation here, since it is straightforward and has already been done in Hoepner [2015], Chapter 5. It follows that the vector fields commute, and consequently the corresponding integral curves combine to a local map $(x, y) \mapsto \zeta(x, y)$.

Definition 2.1. *A local Polynomial Killing Field is a smooth map*

$$\zeta : \Omega \to \mathcal{P}_2,$$

where $\Omega \subseteq \mathbb{R}^2$ is an open set, such that ζ satisfies (2.1) on Ω.

At this point it is not yet clear whether Polynomial Killing Fields exist globally. We will see in the next chapter that this is indeed the case.

Remark 2.2. *The relevance of Polynomial Killing Fields comes from their connection with the sinh–Gordon equation. They may be regarded not only as flows on the space of potentials, but also as flows on the parameter space given by $\alpha, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}^+$. It is shown in Hoepner [2015], Chapter 5, that for a given Polynomial Killing Field the function*

$$u(x, y) := \ln \gamma(x, y)$$

solves the sinh–Gordon equation.

To conclude this chapter, we introduce the objects that will be studied in the next chapter. First, we observe that every $\zeta \in \mathcal{P}_2$ determines a polynomial $a \in \mathbb{C}^4[\lambda]$ such that

$$\det \zeta = \lambda a(\lambda).$$

This motivates the following definition.

Definition 2.3. *Let $a \in \mathbb{C}^4[\lambda]$ be such that $\det \zeta = \lambda a(\lambda)$ for some $\zeta \in \mathcal{P}_2$. We then define the* isospectral set *of a by*

$$I(a) = \{\zeta \in \mathcal{P}_2 \mid \det \zeta = \lambda a(\lambda)\}.$$

Such a polynomial a is called a determinant polynomial. *We denote the space of all determinant polynomials by*

$$\mathcal{M}_2 := \{a \in \mathbb{C}^4[\lambda] \mid \det \zeta = \lambda a(\lambda) \textit{ for some } \zeta \in \mathcal{P}_2\}.$$

These sets are relevant because every Polynomial Killing Field whose initial value lies in an isospectral set remains in that set.

Lemma 2.4. *Let $\zeta(x, y)$ be a Polynomial Killing Field, such that $\det \zeta(x, y) = \lambda a(\lambda)$. Then it holds*

$$\frac{\partial a}{\partial x} = 0 = \frac{\partial a}{\partial y}. \tag{2.3}$$

Proof. We prove only the first identity in (2.3), since the second one is completely analogous. By Proposition 1.23, we have

$$\partial_x(\det \zeta) = \operatorname{tr}\big(\zeta^\dagger\, \partial_x \zeta\big) = \operatorname{tr}\big(\zeta^\dagger(\zeta U - U\zeta)\big).$$

Using the defining property of the adjugate matrix, we obtain

$$\operatorname{tr}\big(\zeta^\dagger \zeta U\big) = \operatorname{tr}\big(\det(\zeta) U\big) = \det(\zeta)\operatorname{tr}(U),$$

and similarly

$$\operatorname{tr}\big(\zeta^\dagger U \zeta\big) = \det(\zeta)\operatorname{tr}(U).$$

Since the trace is linear, it follows that

$$\partial_x(\det \zeta) = 0.$$

This proves the claim. □

Chapter 3

Isospectral Sets

In this chapter we study the behaviour of the isospectral sets $I(a)$ for $a \in \mathcal{M}_2$. To this end, we decompose $\mathcal{M}_2$ into five cases:

$$\mathcal{M}_2^1 := \{a \in \mathcal{M}_2 \mid a \text{ has four pairwise distinct simple roots not on } \mathbb{S}^1\},$$
$$\mathcal{M}_2^2 := \{a \in \mathcal{M}_2 \mid a \text{ has one double root on } \mathbb{S}^1 \text{ and two simple roots not on } \mathbb{S}^1\},$$
$$\mathcal{M}_2^3 := \{a \in \mathcal{M}_2 \mid a \text{ has two distinct double roots on } \mathbb{S}^1\},$$
$$\mathcal{M}_2^4 := \{a \in \mathcal{M}_2 \mid a \text{ has one root of order four on } \mathbb{S}^1\},$$
$$\mathcal{M}_2^5 := \{a \in \mathcal{M}_2 \mid a \text{ has two distinct double roots not on } \mathbb{S}^1\}.$$

We shall see that in all cases the corresponding isospectral set is compact, and we will record some basic facts about its algebraic structure. Since this has already been studied elsewhere, see for instance Knopf et al. [2025], we keep the exposition rather compact, while still including enough detail for the purposes of this thesis.

We begin by observing that a takes a specific form in terms of the parameters α, β, and γ. Indeed, by explicitly computing the determinant of $\zeta \in \mathcal{P}_2$, we obtain

$$a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \bar{a}_1\lambda + 1, \quad \text{where}$$
$$a_1 = -\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma \qquad \text{and} \qquad a_2 = 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2}.$$

In particular, a_1 is complex-valued, whereas a_2 is a strictly positive real number. This allows us to represent the map $\zeta \mapsto a$ as

$$f : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}^+, \qquad (\alpha, \beta, \gamma) \mapsto (a_1, a_2).$$

Before turning to the compactness arguments, we want to describe $\mathcal{M}_2$ more concretely. For this purpose, we analyze the determinant polynomial a arising from $\zeta \in \mathcal{P}_2$.

First, simply evaluating at $\lambda = 0$ gives

$$a(0) = 1. \tag{3.1}$$

For convenience, we write ζ_λ in order to emphasize the value of the spectral parameter. At this point we observe that ζ satisfies the reality condition

$$\lambda^3 \bar{\zeta}^t_{1/\bar{\lambda}} = \lambda^3 \begin{pmatrix} \bar{\alpha}\lambda^{-1} - \alpha\lambda^{-2} & \gamma\lambda^{-1} - \beta\lambda^{-2} + \gamma^{-1}\lambda^{-3} \\ -\gamma^{-1} + \bar{\beta}\lambda^{-1} - \gamma\lambda^{-2} & -\bar{\alpha}\lambda^{-1} + \alpha\lambda^{-2} \end{pmatrix} = -\zeta_\lambda. \tag{3.2}$$

For the next step, we recall a basic fact about skew-Hermitian matrices. A square matrix A is called skew-Hermitian if $\bar{A}^t = -A$. Let a be an eigenvector of such a matrix A with eigenvalue λ. Since $a/\|a\|$ is again an eigenvector, we may assume without loss of generality that $\bar{a}^t a = 1$. Then

$$\bar{\lambda} = \bar{\lambda}\bar{a}^t a = \bar{a}^t \bar{\lambda} a = \overline{\bar{a}^t \lambda a}^{\,t} = \bar{a}^t \bar{A}^t a = \bar{a}^t(-A)a = -\bar{a}^t \lambda a = -\lambda.$$

Thus the eigenvalues of a skew-Hermitian matrix are purely imaginary.

In our case, we are interested in the matrix

$$\lambda^{-3/2}\zeta_\lambda = \begin{pmatrix} \alpha\lambda^{-1/2} - \bar{\alpha}\lambda^{1/2} & -\gamma^{-1}\lambda^{-3/2} + \beta\lambda^{-1/2} - \gamma\lambda^{1/2} \\ \gamma\lambda^{-1/2} - \bar{\beta}\lambda^{1/2} + \gamma^{-1}\lambda^{3/2} & -\alpha\lambda^{-1/2} + \bar{\alpha}\lambda^{1/2} \end{pmatrix}.$$

It is easily checked that this matrix is skew-Hermitian for $\lambda \in \mathbb{S}^1$. Moreover, it is traceless. Since the sum of the eigenvalues is equal to the trace, it follows from the previous discussion that there exists $r \in \mathbb{R}$ such that the eigenvalues are of the form

$$\lambda_{1/2} = \pm ir.$$

If λ_1 and λ_2 denote these two eigenvalues, then

$$\det(\lambda^{-3/2}\zeta_\lambda) = \lambda_1\lambda_2 = r^2 \geq 0 \qquad \text{for } \lambda \in \mathbb{S}^1.$$

Rewriting this yields the second condition:

$$0 \leq \det(\lambda^{-3/2}\zeta_\lambda) = \lambda^{-3}\det\zeta_\lambda = \lambda^{-2}a(\lambda) \qquad \text{for } \lambda \in \mathbb{S}^1. \tag{3.3}$$

The final property relevant for us follows directly from the reality condition (3.2). A short computation gives the corresponding reality condition for the determinant polynomial:

$$\lambda a(\lambda) = \det\zeta_\lambda = \det(-\lambda^3\overline{\zeta^t_{1/\bar{\lambda}}}) = \lambda^6\overline{\det\zeta_{1/\bar{\lambda}}} = \lambda^6\overline{\bar{\lambda}^{-1}a(\bar{\lambda}^{-1})} = \lambda^5\overline{a(\bar{\lambda}^{-1})}. \tag{3.4}$$

We now show that these three conditions completely determine $\mathcal{M}_2$. The proof of the following theorem goes back to Hoepner [2015]. We only sketch it here in order to indicate the main ideas and techniques.

Theorem 3.1. *The conditions* (3.1), (3.3), *and* (3.4) *completely characterize the determinant polynomials $a \in \mathcal{M}_2$, that is,*

$$\mathcal{M}_2 = \left\{ a \in \mathbb{C}^4[\lambda] \,\middle|\, a(0) = 1,\ \lambda^4\overline{a(\bar{\lambda}^{-1})} = a(\lambda),\ \lambda^{-2}a(\lambda) \geq 0 \text{ for } \lambda \in \mathbb{S}^1 \right\}.$$

Proof. We already know that each determinant polynomial $a \in \mathcal{M}_2$ satisfies these conditions. Thus it remains to show that every polynomial a satisfying them is in fact a determinant polynomial.

By condition (3.4), every root of a gives rise to a second root: if λ_1 is a root of a, then so is $\bar{\lambda}_1^{-1}$. Hence a has the structure

$$a(\lambda) = (\lambda - \lambda_1)(\lambda - \bar{\lambda}_1^{-1})(\lambda - \lambda_2)(\lambda - \bar{\lambda}_2^{-1}).$$

Condition (3.1) implies that the product of all four roots is equal to 1, that is,

$$\lambda_1\bar{\lambda}_1^{-1}\lambda_2\bar{\lambda}_2^{-1} = 1.$$

Rewriting this gives

$$\bar{\lambda}_1\lambda_1^{-1} = \lambda_2\bar{\lambda}_2^{-1}. \tag{3.5}$$

The idea is now to find suitable $(\alpha, \beta, \gamma) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ such that the corresponding ζ_λ satisfies $\det\zeta_\lambda = \lambda a(\lambda)$. To do so, we choose $\alpha = 0$ and then try to determine β and γ accordingly. For convenience, we write

$$\zeta_\lambda = \begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda C(\lambda) & -A(\lambda) \end{pmatrix},$$

where A, B, and C are polynomials of degree at most two. In our case, $A(\lambda) \equiv 0$, and therefore

$$\det\zeta_\lambda = -\lambda B(\lambda)C(\lambda).$$

Thus we want to choose γ and β in such a way that

$$a(\lambda) = -B(\lambda)C(\lambda).$$

Since both sides must have the same roots, and the roots of a are already known, we make the ansatz

$$\begin{aligned} B(\lambda) &= \gamma(\lambda - \lambda_1)(\lambda - \bar{\lambda}_2^{-1}), \\ C(\lambda) &= \gamma^{-1}(\lambda - \lambda_2)(\lambda - \bar{\lambda}_1^{-1}). \end{aligned}$$

It is shown in Hoepner [2015] that the following choice of β and γ indeed makes a the determinant polynomial of some $\zeta_\lambda \in \mathcal{P}_2$:

$$\begin{aligned} \gamma &= (\lambda_1 \bar{\lambda}_2^{-1})^{-1/2}, \\ \beta &= (\lambda_1 \bar{\lambda}_2)^{1/2} + (\lambda_1 \bar{\lambda}_2)^{-1/2}. \end{aligned}$$

What remains to be checked is that γ is in fact a positive real number. To prove this, Hoepner [2015] first shows that

$$\tilde{\lambda} := i\sqrt{\lambda_1 \bar{\lambda}_1^{-1}}$$

is an element of $\mathbb{S}^1$. Then condition (3.3) and equation (3.5) imply that

$$\begin{aligned} \tilde{\lambda} a(\tilde{\lambda}) &= -\lambda_1 \bar{\lambda}_1^{-1} - a_1 \tilde{\lambda} + a_2 - \bar{a}_1 \tilde{\lambda}^{-1} - \lambda_2 \bar{\lambda}_2^{-1} \geq 0, \\ \tilde{\lambda} a(-\tilde{\lambda}) &= -\lambda_1 \bar{\lambda}_1^{-1} + a_1 \tilde{\lambda} + a_2 + \bar{a}_1 \tilde{\lambda}^{-1} - \lambda_2 \bar{\lambda}_2^{-1} \geq 0. \end{aligned}$$

Adding these two inequalities is equivalent to

$$\lambda_1 \lambda_2 \left(\frac{1}{|\lambda_1 \lambda_2|^2} + \frac{1}{|\lambda_1|^2} + \frac{1}{|\lambda_2|^2} + 1 \right) \geq 0.$$

Since the expression in parentheses is a positive real number, it follows that $\lambda_1 \lambda_2 \geq 0$. By condition (3.1), both roots are non-zero, and hence

$$\lambda_1 \lambda_2 > 0.$$

Consequently,

$$\gamma = \left(\frac{\lambda_1}{\bar{\lambda}_2} \right)^{-1/2} = \left(\frac{\lambda_1 \lambda_2}{|\lambda_2|^2} \right)^{-1/2} > 0.$$

□

Before we return to the study of the isospectral sets, we need to cite one more statement. A flow through $\mathcal{P}_2$ can equivalently be described by a flow through the parameter space. As shown in Hoepner [2015], the parameters must satisfy differential equations which we will simply cite

Lemma 3.2. *Let zeta$_\lambda$ be a Polynomial Killing Field. The paramters $\alpha, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}^+$ satisfy the following modified Lax equations:*

$$\begin{aligned} \frac{\partial \alpha}{\partial x} &= \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} & \frac{\partial \alpha}{\partial y} &= i(\gamma^{-1}\beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2) \\ \frac{\partial \beta}{\partial x} & - \alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} & \frac{\partial \beta}{\partial y} &= i(-\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1}) \\ \frac{\partial \gamma}{\partial x} &= -\alpha\gamma - \bar{\alpha}\gamma & \frac{\partial \gamma}{\partial y} &= i(\bar{\alpha}\gamma - \alpha\gamma). \end{aligned}$$

Proof. For the explicit calculations, see Hoepner [2015], chapter 5. □

We finally proceed to the main theorem of this chapter. The following result is proved in Knopf et al. [2025]. Our aim is to make the proof more accessible by presenting the argument in greater detail.

Remark 3.3. *The theorem below first shows that in each of the five cases the isospectral sets are compact. We already know that if $\zeta_0 \in I(a)$, then every solution of (2.1) remains inside $I(a)$. By standard ODE theory, compactness of the isospectral set excludes blow-up inside $I(a)$, and no solution can escape the set in finite time. Consequently, the local solution extends to all $(x, y) \in \mathbb{R}^2$, and the two commuting flows from (2.1) define a global $\mathbb{R}^2$-action:*

$$\phi : \mathbb{R}^2 \to \mathrm{End}(\mathcal{P}_2), \qquad (x, y) \mapsto \phi(x, y), \qquad \zeta \mapsto \phi(x, y)\zeta. \tag{3.6}$$

Theorem 3.4. *The vector fields in (2.1) induce the action (3.6) of $\mathbb{R}^2$ on $\mathcal{P}_2$. The isospectral sets $I(a) \subset \mathcal{P}_2$ are compact and have the following properties:*

1. *For $a \in \mathcal{M}_2^1$, the isospectral sets $I(a)$ are two-dimensional compact submanifolds of $\mathcal{P}_2$ with transitive group action (3.6), that is,*
$$I(a) = \{\phi(x, y)\zeta \mid (x, y) \in \mathbb{R}^2\} \qquad \text{for any } \zeta \in I(a).$$
2. *For $a \in \mathcal{M}_2^2$, the isospectral sets $I(a)$ are one-dimensional compact subsets of $\mathcal{P}_2$ with transitive group action (3.6).*
3. *For $a \in \mathcal{M}_2^3 \cup \mathcal{M}_2^4$, the isospectral set $I(a)$ consists of a single fixed point of the group action (3.6).*
4. *For $a \in \mathcal{M}_2^5$, the isospectral set $I(a)$ is the union of two disjoint orbits of the group action (3.6), namely*
$$I(a) = \{\zeta \in I(a) \mid \zeta|_{\lambda \in \{\lambda_1, \bar{\lambda}_1^{-1}\}} \neq 0\} \cup \{\zeta \in I(a) \mid \zeta|_{\lambda \in \{\lambda_1, \bar{\lambda}_1^{-1}\}} = 0\},$$
where λ_1 and $\bar{\lambda}_1^{-1}$ are the double roots of a. We denote the set on the left by K_a and the set on the right by L_a. Then K_a is a two-dimensional non-compact submanifold of $\mathcal{P}_2$ with closure $\overline{K_a} = I(a)$, and L_a consists of a single point.

Proof. **Step 1: Compactness of the isospectral set $I(a)$**

By construction, the map

$$f : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}^+, \qquad (\alpha, \beta, \gamma) \mapsto (a_1, a_2)$$

is continuous. Since (a_1, a_2) is fixed on $I(a)$, the set $I(a)$ is the preimage of a closed set under a continuous map and is therefore closed.

It remains to show boundedness. For $\zeta \in I(a)$ we have

$$a_2 = 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2}. \tag{3.7}$$

Since the left-hand side is constant on $I(a)$, the non-negative quantities $2|\alpha|^2$, $|\beta|^2$, γ^2, and γ^{-2} are uniformly bounded on $I(a)$. In particular, α, β, γ, and γ^{-1} are bounded. Thus $I(a)$ is bounded in the parameter space $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ and therefore compact.

Step 2: The case $a \in \mathcal{M}_2^1$

Let

$$\zeta = \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda C(\lambda) & -A(\lambda) \end{pmatrix} \in I(a).$$

Since $a \in \mathcal{M}_2^1$ has four pairwise distinct simple roots, the matrix polynomial ζ has no root in $\mathbb{C} \setminus \{0\}$. Indeed, if $\zeta(\lambda_0) = 0$ for some $\lambda_0 \neq 0$, then all entries of ζ vanish at λ_0. Hence $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ are all divisible by $(\lambda - \lambda_0)$, and therefore

$$\det \zeta(\lambda) = -A(\lambda)^2 - \lambda B(\lambda)C(\lambda)$$

is divisible by $(\lambda - \lambda_0)^2$. Since $\det \zeta(\lambda) = \lambda a(\lambda)$ and $\lambda_0 \neq 0$, it follows that $(\lambda - \lambda_0)^2$ divides $a(\lambda)$. Thus λ_0 is a multiple root of a, which is impossible. For $\lambda_0 = 0$, ζ cannot have a root, since $B(0) \neq 0$, because $\gamma \neq 0$.

We now study the map f, where

$$a_1 = -\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma, \qquad a_2 = 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2}.$$

In order to show that $I(a)$ is a two-dimensional submanifold, it suffices to prove that the Jacobian of f has full rank 3 on $I(a)$ and then apply the Implicit Function Theorem A.2.

Writing the variables as $(\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)$, the Jacobian is

$$Jf = \begin{pmatrix} 0 & -2\bar{\alpha} & -\gamma^{-1} & -\gamma & \beta\gamma^{-2} - \bar{\beta} \\ -2\alpha & 0 & -\gamma & -\gamma^{-1} & \bar{\beta}\gamma^{-2} - \beta \\ 2\bar{\alpha} & 2\alpha & \bar{\beta} & \beta & 2(\gamma - \gamma^{-3}) \end{pmatrix}.$$

We distinguish two cases.

Case 1: $\alpha \neq 0$. By Gaussian elimination, Jf can be transformed into a matrix of the form

$$\begin{pmatrix} -2\alpha & 0 & -\gamma & -\gamma^{-1} & \bar{\beta}\gamma^{-2} - \beta \\ 0 & -2\bar{\alpha} & -\gamma^{-1} & -\gamma & \beta\gamma^{-2} - \bar{\beta} \\ 0 & 0 & \theta & \iota & \kappa \end{pmatrix},$$

where

$$\theta = \bar{\beta} - \gamma\alpha^{-1}\bar{\alpha} - \gamma^{-1}\bar{\alpha}^{-1}\alpha = -\alpha^{-1}\bar{\alpha}\, C(\bar{\alpha}^{-1}\alpha),$$
$$\iota = \beta - \gamma^{-1}\alpha^{-1}\bar{\alpha} - \gamma\bar{\alpha}^{-1}\alpha = \alpha^{-1}\bar{\alpha}\, B(\bar{\alpha}^{-1}\alpha).$$

If Jf failed to have full rank, then necessarily $\theta = \iota = 0$. But this is equivalent to

$$B(\bar{\alpha}^{-1}\alpha) = 0 = C(\bar{\alpha}^{-1}\alpha).$$

Since moreover

$$A(\bar{\alpha}^{-1}\alpha) = \alpha\bar{\alpha}^{-1}\alpha - \bar{\alpha}(\bar{\alpha}^{-1}\alpha)^2 = 0,$$

the matrix ζ would vanish at $\lambda = \bar{\alpha}^{-1}\alpha$, contradicting the fact that ζ is rootless. Hence Jf has full rank.

Case 2: $\alpha = 0$. Then $A(\lambda) \equiv 0$, so ζ is off-diagonal. In this case, ζ has a root if and only if B and C have a common root. Equivalently, their resultant vanishes. The resultant can be computed using the Sylvester matrix:

$$\operatorname{res}(B, C) = \det \begin{pmatrix} -\gamma & \beta & -\gamma^{-1} & 0 \\ 0 & -\gamma & \beta & -\gamma^{-1} \\ \gamma^{-1} & -\bar{\beta} & \gamma & 0 \\ 0 & \gamma^{-1} & -\bar{\beta} & \gamma \end{pmatrix}.$$

Since ζ is rootless, we have $\operatorname{res}(B, C) \neq 0$.

For $\alpha = 0$, the first two columns of Jf vanish, and we may regard the Jacobian as

$$Jf = \begin{pmatrix} -\gamma^{-1} & -\gamma & \beta\gamma^{-2} - \bar{\beta} \\ -\gamma & -\gamma^{-1} & \bar{\beta}\gamma^{-2} - \beta \\ \bar{\beta} & \beta & 2(\gamma - \gamma^{-3}) \end{pmatrix}.$$

Expanding along the first row, we obtain

$$\det(Jf) = -\gamma^{-1} \begin{vmatrix} -\gamma^{-1} & \bar{\beta}\gamma^{-2} - \beta \\ \beta & 2(\gamma - \gamma^{-3}) \end{vmatrix} + \gamma \begin{vmatrix} -\gamma & \bar{\beta}\gamma^{-2} - \beta \\ \bar{\beta} & 2(\gamma - \gamma^{-3}) \end{vmatrix}$$
$$+ (\beta\gamma^{-2} - \bar{\beta}) \begin{vmatrix} -\gamma & -\gamma^{-1} \\ \bar{\beta} & \beta \end{vmatrix}.$$

The three 2×2 determinants are

$$\begin{vmatrix} -\gamma^{-1} & \bar{\beta}\gamma^{-2} - \beta \\ \beta & 2(\gamma - \gamma^{-3}) \end{vmatrix} = -\gamma^{-1}\, 2(\gamma - \gamma^{-3}) - \beta(\bar{\beta}\gamma^{-2} - \beta)$$
$$= -2(1 - \gamma^{-4}) - \beta\bar{\beta}\gamma^{-2} + \beta^2,$$

$$\begin{vmatrix} -\gamma & \bar{\beta}\gamma^{-2} - \beta \\ \bar{\beta} & 2(\gamma - \gamma^{-3}) \end{vmatrix} = -\gamma\, 2(\gamma - \gamma^{-3}) - \bar{\beta}(\bar{\beta}\gamma^{-2} - \beta)$$
$$= -2(\gamma^2 - \gamma^{-2}) - \bar{\beta}^2\gamma^{-2} + \beta\bar{\beta},$$

$$\begin{vmatrix} -\gamma & -\gamma^{-1} \\ \bar{\beta} & \beta \end{vmatrix} = -\gamma\beta + \gamma^{-1}\bar{\beta}.$$

Substituting these expressions and simplifying yields

$$\det(Jf) = -2\gamma^{-1}\Big(\gamma^2\bar{\beta}^2 + \gamma^{-2}\beta^2 + \beta\bar{\beta} - \gamma^4 - \gamma^{-4} - 2\Big).$$

On the other hand, a direct computation of the resultant using the Sylvester matrix gives

$$\operatorname{res}(B, C) = \gamma^2\bar{\beta}^2 + \gamma^{-2}\beta^2 + \beta\bar{\beta} - \gamma^4 - \gamma^{-4} - 2.$$

Hence

$$\det(Jf) = -2\gamma^{-1}\operatorname{res}(B, C).$$

Since $\operatorname{res}(B, C) \neq 0$, it follows that $\det(Jf) \neq 0$, so Jf has full rank also in the case $\alpha = 0$.

The Implicit Function Theorem A.2 therefore implies that $I(a)$ is a two-dimensional submanifold of $\mathcal{P}_2$.

We now show that the $\mathbb{R}^2$-action is transitive. Recall that the tangent vectors to the orbit through a point $\zeta \in \mathcal{P}_2$ are given by the two vector fields appearing in the Lax equations (2.1). To prove that the orbit through ζ is two-dimensional, it is enough to show that these two vectors are linearly independent over $\mathbb{R}$ at every $\zeta \in I(a)$.

Assume therefore that there exist real numbers $r, s \in \mathbb{R}$ such that

$$r[\zeta, U(\zeta)] + s[\zeta, V(\zeta)] = 0.$$

By linearity of the commutator, this is equivalent to

$$[\zeta, rU(\zeta) + sV(\zeta)] = 0.$$

We now consider this identity pointwise in the spectral parameter λ. For every λ we obtain a 2×2 matrix $\zeta(\lambda)$ satisfying

$$[\zeta(\lambda),\, rU(\zeta)(\lambda) + sV(\zeta)(\lambda)] = 0.$$

For a non-zero traceless 2×2 matrix A, the set of traceless matrices commuting with A is one-dimensional and consists precisely of scalar multiples of A. Indeed, all matrices from the above equation are traceless. Hence for every λ there exists a scalar g such that

$$rU(\zeta)(\lambda) + sV(\zeta)(\lambda) = g\zeta(\lambda).$$

In this way we obtain a meromorphic function $g(\lambda)$ satisfying

$$rU(\zeta) + sV(\zeta) = g(\lambda)\zeta. \tag{3.8}$$

We next analyse the possible singularities of g. The matrices $U(\zeta)$ and $V(\zeta)$ have at most a simple pole at $\lambda = 0$, coming from the terms involving λ^{-1}. Since ζ itself is meromorphic at $\lambda = 0$ and has no pole there, any singular behaviour on the right-hand side of (3.8) must come from g. Thus g can have at most a simple pole at $\lambda = 0$.

Any additional poles of g could only occur at points where $\zeta(\lambda) = 0$. However, since ζ is rootless, no such points exist. Thus g has no poles away from $\lambda = 0$. Finally, we examine the behaviour at $\lambda = \infty$. From the explicit formulas for $U(\zeta)$ and $V(\zeta)$, the left-hand side of (3.8) grows at most linearly in λ, whereas $\zeta(\lambda)$ grows like λ^3. Hence (3.8) implies that

$$g(\lambda) = \mathbf{O}(\lambda^{-2}) \qquad \text{as } \lambda \to \infty,$$

and in particular g has no pole at infinity. We conclude that g is a meromorphic function on the Riemann sphere without poles and therefore, by Liouville's Theorem A.3, must be constant. Since then $g(\lambda)\zeta(\lambda)$ grows like λ^3, whereas the left-hand side grows at most linearly in λ, this constant must in fact be zero. Thus

$$g(\lambda) \equiv 0,$$

and hence

$$rU(\zeta) + sV(\zeta) = 0.$$

Inspecting the explicit formulas for $U(\zeta)$ and $V(\zeta)$, we see that the coefficients of λ^{-1} and λ occurring in these matrices are linearly independent over $\mathbb{R}$. Therefore

$$r = s = 0.$$

This shows that the tangent vectors $[\zeta, U(\zeta)]$ and $[\zeta, V(\zeta)]$ are linearly independent. Hence the orbit through ζ is two-dimensional. It follows that the differential of the map

$$\phi : \mathbb{R}^2 \to I(a), \qquad (x, y) \mapsto \phi(x, y)\zeta,$$

has full rank 2 at every point. Since $\dim I(a) = 2$, the map ϕ is a local submersion and therefore open onto its image. Consequently, the orbit

$$\{\phi(x, y)\zeta \mid (x, y) \in \mathbb{R}^2\}$$

is an open subset of $I(a)$.

Since the orbit is the continuous image of the connected set $\mathbb{R}^2$, it is connected. Because $I(a)$ is compact, every open orbit is also closed in its connected component. It follows that the orbits coincide with the connected components of $I(a)$.

It therefore remains to prove that $I(a)$ is connected. To this end, we consider the function

$$I(a) \to \mathbb{R}^+, \qquad \zeta \mapsto \gamma(\zeta). \tag{3.9}$$

Restricted to any orbit, this is a smooth real-valued function. Since the orbit is compact, the function attains both a maximum and a minimum.

By using the equations from Lemma 3.2, we see that the derivatives of γ vanish precisely when $\alpha = 0$. Thus at the critical points of γ, one has $A(\lambda) \equiv 0$, and hence the critical points of γ are exactly the off-diagonal potentials.

Such off-diagonal potentials correspond to factorizations of the determinant polynomial a into two quadratic factors coming from the product $B(\lambda)C(\lambda)$.

Indeed, let the four simple roots of $a \in \mathcal{M}_2^1$ be

$$\lambda_1, \bar{\lambda}_1^{-1}, \lambda_2, \bar{\lambda}_2^{-1}.$$

If $\alpha = 0$, then

$$a(\lambda) = -B(\lambda)C(\lambda).$$

Thus the roots of a must be distributed among the roots of B and the roots of C.

Because of the reality condition, the roots of B and C occur in paired form: if λ is a root of B, then $\bar{\lambda}^{-1}$ is a root of C, and conversely. Therefore, for each reality pair

$$\{\lambda_1, \bar{\lambda}_1^{-1}\}, \qquad \{\lambda_2, \bar{\lambda}_2^{-1}\},$$

exactly one element must be chosen as a root of B, and the other is then forced to be a root of C. Once the roots of B are fixed, the roots of C are uniquely determined as the corresponding inverse-conjugate partners. Hence exactly four off-diagonal possibilities occur. Therefore $I(a)$ contains precisely four off-diagonal elements.

A direct computation of the Hessian shows that these critical points are non-degenerate, that is, the Hessian is invertible at each of them. Consequently, γ has exactly four critical points on each orbit, namely one maximum, one minimum, and two saddle points. In order to keep the proof reasonably transparent, we do not carry out the Hessian computation here, since it is straightforward.

In particular, there is exactly one off-diagonal element that can serve as a maximum. Therefore, (3.9) restricted to any orbit, must achieve this value. Uniqueness of the maximum then proves that all orbits must be the same, i.e. there is only one orbit and consequently, $I(a)$ is connected.

Since the orbits of the $\mathbb{R}^2$-action coincide with the connected components of $I(a)$, it follows that there is only one orbit. Hence the action is transitive. This proves part 1.

Step 3: The case $a \in \mathcal{M}_2^2$ Now a has one double root on $\mathbb{S}^1$ and two simple roots away from $\mathbb{S}^1$. Let λ_0 be the double root on the unit circle. We have already seen that for such λ_0, the matrix $\lambda^{-3/2}\zeta_\lambda$ is skew-Hermitian, and since $a(\lambda_0) = 0$, its determinant vanishes as well. By the structure of its eigenvalues, this can only happen if both eigenvalues are equal to 0. In particular, a skew-Hermitian matrix is normal:

$$A^\dagger A = -AA = A(-A) = AA^\dagger.$$

Hence the Spectral Theorem A.4 implies that

$$\lambda_0^{-3/2}\zeta_{\lambda_0} = 0,$$

and therefore also

$$\zeta_{\lambda_0} = 0.$$

Thus every unitary root of a is necessarily a root of ζ. Since the double root lies on $\mathbb{S}^1$, every $\zeta \in I(a)$ has a common polynomial factor corresponding to this root. The idea is therefore to work with

$$\hat{\zeta}_\lambda := (\lambda - \lambda_0)^{-1}\zeta_\lambda$$

instead. This reduces a from a polynomial of degree four to a polynomial $\hat{a}$ of degree two, defined by

$$\det \hat{\zeta}_\lambda = \lambda \hat{a}(\lambda).$$

In this way, the problem reduces to genus–one.

We therefore introduce the genus–one potentials

$$\mathcal{P}_1 := \left\{ \hat{\zeta} = \begin{pmatrix} 0 & -\hat{\beta}^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} i\hat{\alpha} & -\hat{\beta} \\ \hat{\beta} & -i\hat{\alpha} \end{pmatrix} \hat{\lambda} + \begin{pmatrix} 0 & 0 \\ \hat{\beta}^{-1} & 0 \end{pmatrix} \hat{\lambda}^2 \;\middle|\; \hat{\alpha} \in \mathbb{R},\ \hat{\beta} \in \mathbb{R}^+ \right\}.$$

We proceed in the same way as in the genus–two case, but only record the resulting calculations, since the methods are identical. A direct computation yields

$$\det(\hat{\zeta}) = \hat{\lambda}\big(\hat{\lambda}^2 + \hat{a}_1\hat{\lambda} + 1\big), \qquad \hat{a}_1 = \hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2}.$$

Hence the map $(\hat{\alpha}, \hat{\beta}) \mapsto \hat{a}_1$ has one-dimensional target. Its gradient is

$$\begin{pmatrix} 2\hat{\alpha} \\ 2\hat{\beta} - 2\hat{\beta}^{-3} \end{pmatrix}.$$

This gradient vanishes exactly at
$$\hat{\alpha} = 0, \qquad \hat{\beta} = 1,$$
corresponding to the unique degenerate case in which the determinant polynomial has a double root. Therefore, as long as the two roots of $\hat{a}$ are distinct, the Implicit Function Theorem A.2 implies that the level set
$$\hat{I}(\hat{a}) := \{\hat{\zeta} \in \mathcal{P}_1 \mid \det \hat{\zeta} = \hat{\lambda}\hat{a}(\hat{\lambda})\}$$
is a compact one-dimensional submanifold. Compactness follows for the same reasons as in the genus–two case.

The Lax equations for $\hat{\zeta}$ are
$$\frac{\partial \hat{\zeta}}{\partial \hat{x}} = [\hat{\zeta}, \hat{U}(\hat{\zeta})], \qquad \frac{\partial \hat{\zeta}}{\partial \hat{y}} = [\hat{\zeta}, \hat{V}(\hat{\zeta})],$$
with
$$\hat{U}(\hat{\zeta}) = \begin{pmatrix} i\hat{\alpha} & -\hat{\beta}^{-1}\hat{\lambda}^{-1} - \hat{\beta} \\ \hat{\beta} + \hat{\beta}^{-1}\hat{\lambda} & -i\hat{\alpha} \end{pmatrix}, \qquad \hat{V}(\hat{\zeta}) = i\begin{pmatrix} 0 & -\hat{\beta}^{-1}\hat{\lambda}^{-1} + \hat{\beta} \\ \hat{\beta} - \hat{\beta}^{-1}\hat{\lambda} & 0 \end{pmatrix}.$$
An explicit computation gives
$$\partial_{\hat{x}}\hat{\alpha} = 0, \qquad \partial_{\hat{x}}\hat{\beta} = 0,$$
$$\partial_{\hat{y}}\hat{\alpha} = 2(\hat{\beta}^{-2} - \hat{\beta}^2), \qquad \partial_{\hat{y}}\hat{\beta} = 2\hat{\alpha}\hat{\beta}.$$
Thus the $\hat{x}$-flow is trivial and the dynamics are effectively one-dimensional.

Now let $e^{2i\varphi}$ be the double root of a, and let $\hat{\lambda}_1 e^{-2i\varphi}$ and $\hat{\lambda}_1^{-1}e^{-2i\varphi}$ be the two simple roots of a, where $\hat{\lambda}_1 \in (-1, 0)$. Define
$$\hat{\lambda} = -e^{2i\varphi}\lambda, \qquad \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
and
$$\zeta(x, y) := e^{3i\varphi} \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\varphi} \end{pmatrix} \hat{\zeta}(\hat{x}, \hat{y}) \begin{pmatrix} 1 & 0 \\ 0 & -ie^{-i\varphi} \end{pmatrix} (\lambda - e^{2i\varphi}).$$

We now verify that this transformation identifies $\hat{I}(\hat{a})$ with $I(a)$ and intertwines the corresponding flows.

Since $\hat{\zeta}_{\hat{\lambda}}$ is a polynomial of degree two in $\hat{\lambda}$, and $\hat{\lambda} = -e^{2i\varphi}\lambda$, it follows that $\hat{\zeta}_{\hat{\lambda}}$ is a polynomial of degree two in λ. Multiplication by $(\lambda - e^{2i\varphi})$ therefore produces a polynomial of degree three. A direct computation of the entries shows that ζ_λ has the form
$$\zeta_\lambda = \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix},$$
for suitable parameters $\alpha, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}^+$. Hence $\zeta \in \mathcal{P}_2$.

Next we examine how the determinant transforms. By multiplicativity of the determinant,
$$\det \zeta_\lambda = e^{6i\varphi}(\lambda - e^{2i\varphi})^2 \det(\hat{\zeta}_{\hat{\lambda}}).$$
Since $\det(\hat{\zeta}_{\hat{\lambda}}) = \hat{\lambda}\,\hat{a}(\hat{\lambda})$ and $\hat{\lambda} = -e^{2i\varphi}\lambda$, we obtain
$$\begin{aligned} \det \zeta_\lambda &= e^{6i\varphi}(\lambda - e^{2i\varphi})^2(-e^{2i\varphi}\lambda)\,\hat{a}(-e^{2i\varphi}\lambda) \\ &= \lambda a(\lambda), \end{aligned}$$
where
$$a(\lambda) = -e^{8i\varphi}(\lambda - e^{2i\varphi})^2\hat{a}(-e^{2i\varphi}\lambda).$$
Thus $\zeta_\lambda \in I(a)$.

Finally, we show that the flows are intertwined. Let $\hat{\zeta}$ satisfy the genus–one Lax equations

$$\partial_{\hat{x}}\hat{\zeta} = [\hat{\zeta}, \hat{U}(\hat{\zeta})], \qquad \partial_{\hat{y}}\hat{\zeta} = [\hat{\zeta}, \hat{V}(\hat{\zeta})].$$

By the chain rule and the linear change of variables between (x, y) and $(\hat{x}, \hat{y})$, we get

$$\partial_x \zeta = e^{3i\varphi} M\big(\cos\varphi\, \partial_{\hat{x}}\hat{\zeta} - \sin\varphi\, \partial_{\hat{y}}\hat{\zeta}\big) M^{-1}(\lambda - e^{2i\varphi}),$$

and similarly for $\partial_y\zeta$, where M denotes the constant diagonal matrix.

Substituting the Lax equations for $\hat{\zeta}$ shows that ζ satisfies

$$\partial_x\zeta = [\zeta, U(\zeta)], \qquad \partial_y\zeta = [\zeta, V(\zeta)],$$

that is, the genus–two Lax equations.

Thus the transformation maps $\hat{I}(\hat{a})$ into $I(a)$ and intertwines the corresponding flows. Since the construction is reversible, it defines a bijection between the two isospectral sets.

Therefore $I(a)$ is a compact one-dimensional manifold. Since it is the continuous image of a connected one-dimensional flow and the function $\hat{\beta}$ has exactly two critical points on $\hat{I}(\hat{a})$, the set $\hat{I}(\hat{a})$ is connected. Consequently, $I(a)$ is connected as well, and the induced action is transitive. This proves part 2.

Step 4: The case $a \in \mathcal{M}_2^3 \cup \mathcal{M}_2^4$

In this case,

$$a(\lambda) = (\lambda - \lambda_1)^2(\lambda - \bar{\lambda}_1)^2, \qquad |\lambda_1| = 1.$$

Arguing exactly as at the beginning of part 2, every $\zeta \in I(a)$ must vanish at the unitary roots of a, and hence must be divisible by $(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)$. Because of the special form of the elements of $\mathcal{P}_2$, this forces ζ to be off-diagonal. Moreover, the determinant condition yields

$$\gamma^2 = \lambda_1\bar{\lambda}_1 = 1,$$

so $\gamma = 1$. Therefore there is only one possible element, namely

$$\zeta = \left(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\lambda\right)(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1).$$

Hence $I(a)$ consists of a single point, and the flow (3.6) acts trivially on $I(a)$. Consequently, the vector fields in (2.1) vanish at this point, so it is a fixed point. This proves part 3.

Step 5: The case $a \in \mathcal{M}_2^5$

Here

$$a(\lambda) = (\lambda - \lambda_1)^2(\lambda - \bar{\lambda}_1^{-1})^2, \qquad |\lambda_1| \neq 1.$$

The condition $a(0) = 1$ implies

$$\begin{aligned} \lambda_0\bar{\lambda}_0^{-1} = 1 \Leftrightarrow \left(\frac{\lambda_0}{|\lambda_0|}\right)^2 &= \frac{\lambda_0}{\bar{\lambda}_0} \in \{-1, 1\} \\ \Leftrightarrow \frac{\lambda_0}{|\lambda_0|} &\in \{-1, 1, i, -i\} \\ \Leftrightarrow \lambda_0 &\in (\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}. \end{aligned}$$

We can exclude the unitary values, since $\lambda_1 \notin \mathbb{S}^1$. Suppose now that $\lambda_1 = ir$ with $r \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then

$$\lambda^{-2}a(\lambda) = \lambda^2 - (r + r^{-1})2i\lambda - (4 + r^2 + r^{-2}) + (r + r^{-1})2i\lambda^{-1} + \lambda^{-2}.$$

Evaluating at $\lambda = 1$ gives

$$a(1) = -(2 + r^2 + r^{-2}) < 0.$$

This contradicts the condition $\lambda^{-2}a(\lambda) \geq 0$ for all $\lambda \in \mathbb{S}^1$. Therefore,

$$\lambda_1 \in \mathbb{R} \setminus \{-1, 0, 1\}.$$

We first describe L_a. If $\zeta \in L_a$, then ζ vanishes at both double roots of a. As in part 3, this forces ζ to be divisible by $(\lambda - \lambda_1)(\lambda - \lambda_1^{-1})$. In particular,

$$B(\lambda) = c(\lambda - \lambda_1)(\lambda - \lambda_1^{-1})$$

for some constant $c \neq 0$. Comparing this with

$$B(\lambda) = -\gamma^{-1} + \beta\lambda - \gamma\lambda^2$$

shows, by looking at the highest-order coefficient, that

$$c = -\gamma, \qquad \text{and hence} \qquad -\gamma^{-1} = -\gamma\lambda_1\lambda_1^{-1} = -\gamma.$$

Since $\gamma > 0$, it follows that

$$\gamma = 1.$$

Therefore there is exactly one such potential:

$$\zeta_L = \left(\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \lambda \right) (\lambda - \lambda_1)(\lambda - \lambda_1^{-1}).$$

Thus $L_a = \{\zeta_L\}$ consists of a single point. Since the roots of ζ are preserved along the Lax flow, L_a is itself an orbit.

We now consider

$$K_a = I(a) \setminus L_a.$$

As in part 1, the Jacobian argument shows that K_a is a two-dimensional submanifold of $\mathcal{P}_2$. The same commutator argument as before shows that the two vector fields are linearly independent on K_a, and hence every orbit in K_a is open in K_a. Since the orbits are connected, they are precisely the connected components of K_a.

In contrast to part 1, these orbits need not be compact. We therefore study their closures in the compact set $I(a)$. The closure of any orbit is compact and is either the orbit itself or the union of the orbit with L_a. Consider again the function

$$\zeta \mapsto \gamma(\zeta).$$

On the closure of each orbit, this function attains a maximum and a minimum. These extrema are either interior critical points in K_a or the boundary point in L_a. In either case they must be off-diagonal potentials.

In the present case there are exactly three off-diagonal elements in $I(a)$: the point in L_a and two further off-diagonal elements in K_a. We know that on L_a one has $\gamma = 1$, whereas at the two off-diagonal elements in K_a the corresponding values of γ are respectively larger and smaller than 1. The latter follows from the symmetry of the polynomials $B(\lambda)$ and $C(\lambda)$: whenever (β, γ) is a solution, so is $(\bar{\beta}, \gamma^{-1})$.

To analyse the structure more concretely, we study the slice $\gamma = 1$. Expanding

$$a(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_1^{-1})^2$$

gives

$$a(\lambda) = \lambda^4 - 2(\lambda_1 + \lambda_1^{-1})\lambda^3 + (\lambda_1^2 + 4 + \lambda_1^{-2})\lambda^2 - 2(\lambda_1 + \lambda_1^{-1})\lambda + 1.$$

On the other hand, for $\gamma = 1$, the formula (3.7) and the expression for a_1 become

$$-2(\lambda_1 + \lambda_1^{-1}) = -\bar{\alpha}^2 - \beta - \bar{\beta},$$
$$\lambda_1^2 + 2 + \lambda_1^{-2} = 2\alpha\bar{\alpha} + \beta\bar{\beta}.$$

We distinguish according to the sign of λ_1.

If $\lambda_1 < 0$, we write

$$\alpha = \sqrt{2}\, ix, \qquad \beta = -z + iy, \qquad x, y, z \in \mathbb{R}.$$

If $\lambda_1 > 0$, we write

$$\alpha = \sqrt{2}\, x, \qquad \beta = z + iy, \qquad x, y, z \in \mathbb{R}.$$

In both cases the equations above reduce to

$$|\lambda_1 + \lambda_1^{-1}| = x^2 + z,$$
$$|\lambda_1 + \lambda_1^{-1}|^2 = 4x^2 + y^2 + z^2.$$

Eliminating z yields

$$x^4 - \big(|\lambda_1 + \lambda_1^{-1}| - 2\big)x^2 + y^2 = 0. \tag{3.10}$$

Equivalently,

$$y = \pm x\sqrt{|\lambda_1 + \lambda_1^{-1}| - 2 - x^2}.$$

Since $|\lambda_1 + \lambda_1^{-1}| > 2$, the real solution set of (3.10) is a figure-eight curve with an ordinary double point at $(x, y) = (0, 0)$. This double point corresponds exactly to the unique element of L_a, while the two remaining branches correspond to elements of K_a.

In particular, every orbit in K_a intersects the slice $\gamma = 1$. Since at the two off-diagonal points of K_a the function γ takes values on opposite sides of 1, every orbit in K_a must contain both of these points. Therefore all points of K_a lie in one and the same orbit. Hence K_a is connected and consists of a single orbit.

This orbit cannot be compact. Indeed, otherwise, as in part 1, the restriction of γ to it would be a Morse function on a compact surface whose critical set would force the third off-diagonal point to lie on the same compact orbit, contradicting the fact that this point is precisely the isolated orbit L_a. Thus K_a is non-compact. Since its closure in the compact set $I(a)$ contains L_a, we obtain

$$\overline{K_a} = I(a).$$

This proves part 4 and completes the proof. □

We want to close this chapter with an observation we will need again later.

Remark 3.5. *Let $a \in M_2^2$. By definition, a has one double root on $\mathbb{S}^1$ and two simple roots away from $\mathbb{S}^1$. Denote the double root by $\lambda_0 \in \mathbb{S}^1$. Since λ_0 lies on the unit circle, there exists $\varphi \in [0, \pi)$ such that*

$$\lambda_0 = e^{2i\varphi}.$$

Moreover, by the reality condition from the Theorem above, the roots of a occur in inverse-conjugate pairs. Hence the two simple roots are of the form

$$\mu, \qquad \bar{\mu}^{-1}, \qquad \mu \in \mathbb{C}^\times \setminus \mathbb{S}^1.$$

Therefore a admits the factorisation

$$a(\lambda) = (\lambda - \lambda_0)^2(\lambda - \mu)(\lambda - \bar{\mu}^{-1}).$$

We now determine the argument of μ. Write

$$\mu = \rho e^{i\theta}, \qquad \rho > 0.$$

Since $a(0) = 1$ and a is a polynomial of degree 4 and has highest coefficient 1, the product of its four roots is equal to 1. Thus

$$1 = \lambda_0^2\, \mu\, \bar{\mu}^{-1} = e^{4i\varphi} e^{2i\theta}.$$

It follows that

$$e^{i\theta} = \pm e^{-2i\varphi}.$$

Equivalently, there exists $\xi \in \mathbb{R} \setminus \{0, \pm 1\}$ *such that*

$$\mu = \xi e^{-2i\varphi}.$$

Hence every polynomial $a \in M_2^2$ *can be written in the form*

$$a(\lambda) = (\lambda - e^{2i\varphi})^2(\lambda - \xi e^{-2i\varphi})(\lambda - \xi^{-1}e^{-2i\varphi}), \qquad \xi \in \mathbb{R} \setminus \{0, \pm 1\}.$$

After interchanging the two simple roots, we may assume $|\xi| < 1$. *Thus the stratum* M_2^2 *is parametrised by the angle* φ *of the double unitary root and one real radial parameter. In Chapter 5, after fixing the sign convention used in the reduced spectral coordinate, we denote this radial parameter by* $r \in (0, 1)$.

Chapter 4

Lattices of Periods

In this chapter we study the period lattices associated with the isospectral sets introduced in the previous chapter. For determinant polynomials $a \in \mathcal{M}_2^1$, the corresponding isospectral set $I(a)$ is a single compact orbit of the $\mathbb{R}^2$-action, and hence its isotropy group defines a lattice in $\mathbb{C}$. Our aim is to describe these lattices in terms of the spectral data and to investigate how their conformal classes behave under degeneration. To this end, we introduce frames and monodromies, relate the period lattice to meromorphic differentials on the spectral curve, and finally show that the resulting lattice map extends continuously on $\mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, while degenerating as one approaches $\mathcal{M}_2^4 \cup \mathcal{M}_2^5$.

By Theorem 3.4, the isospectral set $I(a)$ of every $a \in \mathcal{M}_2 \setminus \mathcal{M}_2^5$ consists of exactly one orbit. Hence the isotropy group

$$\Gamma_\zeta = \{(x, y) \in \mathbb{R}^2 \mid \phi(x, y)(\zeta) = \zeta\}$$

does not depend on the choice of $\zeta \in I(a)$. Identifying $(x, y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$, we therefore define

$$\Gamma_a := \{x + iy \in \mathbb{C} \mid \forall \zeta \in I(a) : \ \phi(x, y)(\zeta) = \zeta\}.$$

Lemma 4.1. *For every $a \in \mathcal{M}_2 \setminus \mathcal{M}_2^5$, the set $\Gamma_a \subset \mathbb{C}$ is an additive subgroup.*

Proof. Let $\omega_1 = x_1 + iy_1$ and $\omega_2 = x_2 + iy_2$ be elements of Γ_a. Then, for every $\zeta \in I(a)$, the flow property gives

$$\phi(\omega_1 + \omega_2)(\zeta) = \phi(x_1 + x_2, y_1 + y_2)(\zeta) = \phi(x_1, y_1)\big(\phi(x_2, y_2)(\zeta)\big) = \phi(x_1, y_1)(\zeta) = \zeta.$$

Hence $\omega_1 + \omega_2 \in \Gamma_a$.

Moreover,

$$\phi(-\omega_1)(\zeta) = \phi(-x_1, -y_1)\big(\phi(x_1, y_1)(\zeta)\big) = \zeta,$$

so $-\omega_1 \in \Gamma_a$. Since $\mathbb{C}$ is abelian under addition, it follows that Γ_a is an additive subgroup of $\mathbb{C}$. $\square$

4.1 Period Lattice for $a \in \mathcal{M}_2^1$

For $a \in \mathcal{M}_2^1$, Theorem 3.4 shows that $I(a)$ is a two-dimensional compact orbit of the $\mathbb{R}^2$-action. Since the $\mathbb{R}^2$-action is transitive on $I(a)$, the orbit-stabilizer theorem A.7 yields a diffeomorphism

$$\mathbb{R}^2/\Gamma_a \longrightarrow I(a).$$

As $I(a)$ is a compact two-dimensional manifold, it follows that $\mathbb{R}^2/\Gamma_a$ is compact and two-dimensional. Hence Γ_a must be a discrete subgroup of $\mathbb{R}^2 \cong \mathbb{C}$ with compact quotient.

We now recall the relevant notion of lattice.

Definition 4.2. *Let $b_1, \dots, b_m \in \mathbb{R}^n$ be linearly independent over $\mathbb{R}$. The subgroup*

$$\Gamma = \langle b_1, \dots, b_m \rangle_{\mathbb{Z}} := \left\{ \sum_{j=1}^{m} n_j b_j \;\middle|\; n_j \in \mathbb{Z} \right\}$$

is called a lattice in $\mathbb{R}^n$.

Lemma 4.3. *Let $\Gamma \subset \mathbb{C}$ be a non-trivial discrete subgroup such that $\mathbb{C}/\Gamma$ is compact. Then Γ is a lattice of rank 2. More precisely, there exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, such that*

$$\Gamma = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}.$$

Proof. Since Γ is discrete and non-trivial, there exists an element $\omega_1 \in \Gamma \setminus \{0\}$ of minimal modulus:

$$|\omega_1| = \min\{|\gamma| : \gamma \in \Gamma \setminus \{0\}\}.$$

Then $\omega_1 \mathbb{Z} \subset \Gamma$.

We claim that $\Gamma \not\subset \mathbb{R}\omega_1$. Indeed, if $\Gamma \subset \mathbb{R}\omega_1$, then by minimality of ω_1 one has $\Gamma = \omega_1\mathbb{Z}$, and hence $\mathbb{C}/\Gamma$ would be non-compact, since only one real direction is quotiented out. This contradicts the compactness of $\mathbb{C}/\Gamma$.

Therefore there exists

$$\omega_2 \in \Gamma \setminus \mathbb{R}\omega_1.$$

Choose ω_2 such that its distance to the line $\mathbb{R}\omega_1$ is minimal among all elements of $\Gamma \setminus \mathbb{R}\omega_1$. Then ω_1 and ω_2 are linearly independent over $\mathbb{R}$, and hence form an $\mathbb{R}$-basis of $\mathbb{C}$.

It remains to show that every $\gamma \in \Gamma$ lies in $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Let

$$\gamma = a\omega_1 + b\omega_2, \qquad a, b \in \mathbb{R}.$$

Choose integers $m, n \in \mathbb{Z}$ such that

$$\gamma' = \gamma - m\omega_1 - n\omega_2$$

lies in the fundamental parallelogram

$$P = \{s\omega_1 + t\omega_2 \mid 0 \le s, t < 1\}.$$

Since $\gamma' \in \Gamma$, every element of Γ is congruent modulo $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ to a point of the bounded set P. If

$$\Gamma \neq \omega_1\mathbb{Z} + \omega_2\mathbb{Z},$$

then there would exist infinitely many distinct elements of $\Gamma \cap P$, contradicting discreteness. Hence

$$\Gamma = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}.$$

□

Thus, for $a \in \mathcal{M}_2^1$, the isotropy group Γ_a is a lattice in $\mathbb{C}$:

$$\Gamma_a = \omega_1\mathbb{Z} + \omega_2\mathbb{Z},$$

for suitable $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$.

The choice of generators is not unique. We therefore identify lattices up to rotation and dilation.

Definition 4.4. *Two lattices $\Gamma, \Gamma' \subset \mathbb{C}$ are called isomorphic if there exists $c \in \mathbb{C}^\times$ such that*

$$\Gamma' = c\,\Gamma.$$

In Freitag and Busam [2005] it is proven, that every lattice in $\mathbb{C}$ is, up to a rotation dilation isomorphic to a lattice

$$\Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z},$$

where τ is in

$$\tau \in \left\{ \tau \in \mathbb{C} \;\middle|\; \Im(\tau) > 0,\ |\Re(\tau)| \leq \frac{1}{2},\ |\tau| \geq 1 \right\}, \tag{4.1}$$

and becomes unique with the boundary identifications

$$-\frac{1}{2} + iy \sim \frac{1}{2} + iy \qquad \text{for } y \in \left[\frac{\sqrt{3}}{2}, \infty\right),$$

and

$$-x + i\sqrt{1-x^2} \sim x + i\sqrt{1-x^2} \qquad \text{for } x \in \left[0, \frac{1}{2}\right].$$

We denote the resulting quotient space by $\mathcal{F}$ and call every $\tau \in \mathcal{F}$ the conformal class of a, where we impose the quotient topology of the set in (4.1).

4.2 Frames, monodromy and the spectral curve

The natural question is whether the conformal class of Γ_a extends continuously from $\mathcal{M}_2^1$ to the limiting cases $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$.

Remark 4.5. *We note very briefly that it is not possible to define a lattice for $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$ as we did for the generic case. Theorem 3.4 already showed, that the Isospectral sets are just one-dimensional. This fact alone lets us only hope to find a meaningful extension from the generic case to this degenerate case. It turns out that we can do that.*

We first want to figure out, how the conformal class depends on the determinant polynomia a. For this purpose we introduce frames and monodromies.

Let $\zeta : \mathbb{R}^2 \to \mathcal{P}_2$ be a Polynomial Killing Field with initial value $\zeta_0 \in \mathcal{P}_2$. Consider the system

$$\frac{\partial F}{\partial x} = F\,U(\zeta), \qquad \frac{\partial F}{\partial y} = F\,V(\zeta), \qquad F(0,0) = 1. \tag{4.2}$$

Its fundamental solution F is called the *frame*.

Lemma 4.6. *The Maurer–Cartan equation*

$$[U(\zeta), V(\zeta)] + \frac{\partial V(\zeta)}{\partial x} - \frac{\partial U(\zeta)}{\partial y} = 0$$

implies that the frame F is well-defined as a function of $(x, y) \in \mathbb{R}^2$.

Proof. We must show that the mixed partial derivatives commute. Differentiating the first equation in (4.2) with respect to y yields

$$F_{xy} = F_y U + F U_y = (FV)U + FU_y = F(VU + U_y).$$

Similarly, differentiating the second equation with respect to x gives

$$F_{yx} = F_x V + F V_x = (FU)V + FV_x = F(UV + V_x).$$

Hence the identity $F_{xy} = F_{yx}$ is equivalent to

$$VU + U_y = UV + V_x,$$

or, after rearranging,

$$V_x - U_y + [U, V] = 0.$$

This is precisely the Maurer–Cartan equation. Therefore the system is integrable, and F is well-defined on $\mathbb{R}^2$.

Further, by Theorem 1.18 F exists and is unique. □

By Theorem 3.4, for $a \in \mathcal{M}_2^3$, the Polynomial Killing Field is constant, and therefore also $U(\zeta)$ and $V(\zeta)$ are constant. In this case one obtains

$$F(x, y) = \exp\big(xU(\zeta) + yV(\zeta)\big).$$

We now identify $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$, and thus regard F as a function of $z \in \mathbb{C}$.

For $\omega \in \Gamma_a$, we define the *monodromy* M_ω by

$$M_\omega := F(\omega).$$

Remark 4.7. *The frame recovers the Polynomial Killing Field from its initial value:*

$$\zeta = F^{-1}\zeta_0 F.$$

Indeed,

$$\frac{\partial}{\partial x}\big(F^{-1}\zeta_0 F\big) = \frac{\partial F^{-1}}{\partial x}\zeta_0 F + F^{-1}\zeta_0\frac{\partial F}{\partial x} = -U\,F^{-1}\zeta_0 F + F^{-1}\zeta_0 F\,U = [F^{-1}\zeta_0 F, U],$$

and analogously,

$$\frac{\partial}{\partial y}\big(F^{-1}\zeta_0 F\big) = [F^{-1}\zeta_0 F, V].$$

Hence $F^{-1}\zeta_0 F$ satisfies the same Lax equations and the same initial condition as ζ, and uniqueness therefore implies the claim.

In particular, for every $\omega \in \Gamma_a$, the monodromy M_ω commutes with ζ_0. Indeed, we obtain, for $z = \omega \in \Gamma_a$,

$$\zeta_0 = \zeta(\omega) = F(\omega)^{-1}\zeta_0 F(\omega) = M_\omega^{-1}\zeta_0 M_\omega.$$

Multiplying by M_ω from the left yields

$$M_\omega \zeta_0 = \zeta_0 M_\omega.$$

Thus M_ω lies in the centralizer of ζ_0.

Hence M_ω preserves the eigenspaces of ζ_0. More precisely, let

$$E_{\lambda,\nu} := \ker\big(\zeta_0(\lambda) - \nu 1\big)$$

be the eigenspace of $\zeta_0(\lambda)$ corresponding to the eigenvalue ν. If $v \in E_{\lambda,\nu}$, then

$$\zeta_0(\lambda)v = \nu v.$$

Using the commutation relation, we compute

$$\zeta_0(\lambda)\big(M_\omega(\lambda)v\big) = M_\omega(\lambda)\zeta_0(\lambda)v = M_\omega(\lambda)(\nu v) = \nu\, M_\omega(\lambda)v.$$

Therefore

$$M_\omega(\lambda)v \in E_{\lambda,\nu}.$$

Now assume that $a \in \mathcal{M}_2^1$. Recall from the proof of Theorem 3.4 that every $\zeta_0 \in I(a)$ is rootless on $\mathbb{C}^\times$.

Since all potentials in $\mathcal{P}_2$ are traceless, the eigenvalues of $\zeta_0(\lambda)$ are $\pm\nu$, where ν is determined by

$$\det(\nu\mathbb{1} - \zeta_0(\lambda)) = \nu^2 + \lambda a(\lambda) = 0.$$

Thus the corresponding spectral curve is

$$\Sigma^* = \left\{(\lambda, \nu) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \;\middle|\; \nu^2 + \lambda a(\lambda) = 0\right\}. \tag{4.3}$$

Lemma 4.8. *Assume that $a \in \mathcal{M}_2^1$. Let $\bar{\Sigma}$ be the smooth compactification of Σ^*, i.e. the set obtained by adding the points lying over $\lambda = 0$ and $\lambda = \infty$. Then $\bar{\Sigma}$ is a hyperelliptic curve of genus–two.*

Proof. We rewrite the defining equation as $\nu^2 = -\lambda a(\lambda)$. Since $a(\lambda)$ is a polynomial of degree 4, the polynomial

$$P(\lambda) := -\lambda a(\lambda)$$

has degree 5. Moreover, because $a \in \mathcal{M}_2^1$, the four roots of a are pairwise distinct and non-zero, and since $a(0) = 1$, the factor λ contributes an additional simple root at $\lambda = 0$. Hence P has exactly five distinct simple roots.

Consider the projection

$$\pi : \bar{\Sigma} \to \mathbb{P}^1, \qquad (\lambda, \nu) \mapsto \lambda.$$

This is a holomorphic map of degree 2, since for every λ with $P(\lambda) \neq 0$ the equation $\nu^2 = P(\lambda)$ has exactly two solutions $\nu = \pm\sqrt{P(\lambda)}$.

The map π is branched exactly at the zeros of $P(\lambda)$ and at $\lambda = \infty$. Indeed, the five simple zeros of P are branch points, and because $\deg P = 5$ is odd, there is one additional branch point at infinity. Hence π has in total 6 branch points.

We now apply the Riemann–Hurwitz formula from Hartshorne [2013] section 4 to the double covering $\pi : \bar{\Sigma} \to \mathbb{P}^1$:

$$2g(\bar{\Sigma}) - 2 = 2\big(2g(\mathbb{P}^1) - 2\big) + 6.$$

Since $g(\mathbb{P}^1) = 0$, this becomes

$$2g(\bar{\Sigma}) - 2 = 2(-2) + 6 = 2.$$

Therefore

$$g(\bar{\Sigma}) = 2.$$

□

For $\lambda \in \mathbb{C}^\times$, the matrix $\zeta_0(\lambda)$ is traceless and non-zero. Hence it cannot have a two-dimensional eigenspace. Indeed, if one eigenspace had dimension 2, then $\zeta_0(\lambda)$ would be a scalar matrix, and since it is traceless, it would have to be the zero matrix, which is impossible. It follows that both eigenspaces are one-dimensional. In particular, the points of Σ^* parametrize the one-dimensional eigenspaces of ζ_0.

The curve Σ^* carries two natural involutions. The first one is

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu),$$

which simply exchanges the two eigenvalues of the traceless matrix $\zeta_0(\lambda)$.

The second involution ρ is induced by the reality condition

$$\bar{\lambda}^4 a(\bar{\lambda}^{-1}) = \overline{a(\lambda)}.$$

Indeed, let $(\lambda, \nu) \in \Sigma^*$, then $\nu^2 + \lambda a(\lambda) = 0$. Taking complex conjugates, we obtain

$$\bar{\nu}^{\,2} + \bar{\lambda}\,\overline{a(\lambda)} = 0.$$

Using the reality condition for a, this becomes

$$\bar{\nu}^{\,2} + \bar{\lambda} \cdot \bar{\lambda}^4 a(\bar{\lambda}^{-1}) = 0, \qquad \text{i.e.} \qquad \bar{\nu}^{\,2} + \bar{\lambda}^5 a(\bar{\lambda}^{-1}) = 0.$$

Multiplying by $\bar{\lambda}^{-6}$, we obtain

$$\bar{\lambda}^{-6}\bar{\nu}^{\,2} + \bar{\lambda}^{-1} a(\bar{\lambda}^{-1}) = 0.$$

Hence

$$\big(-\bar{\lambda}^{-3}\bar{\nu}\big)^2 + \bar{\lambda}^{-1} a(\bar{\lambda}^{-1}) = 0,$$

which shows that

$$\rho : (\lambda, \nu) \mapsto \left(\bar{\lambda}^{-1}, -\bar{\lambda}^{-3}\bar{\nu}\right)$$

maps Σ^* to itself and is indeed an involution.

Since each eigenspace $E_{\lambda,\nu}$ is one-dimensional and $M_\omega(\lambda)$ preserves it, the restriction

$$M_\omega(\lambda)\big|_{E_{\lambda,\nu}} : E_{\lambda,\nu} \to E_{\lambda,\nu}$$

is a linear endomorphism of a one-dimensional complex vector space. Therefore there exists a function

$$\mu_\omega : \Sigma^* \to \mathbb{C}^\times$$

such that

$$M_\omega(\lambda)v = \mu_\omega(\lambda, \nu)\, v \qquad \text{for all } v \in E_{\lambda,\nu}.$$

To see that the value $\mu_\omega(\lambda, \nu)$ is non-zero, it is enough to show that $M_\omega(\lambda) \in \mathrm{SL}(2, \mathbb{C})$, since then it is in particular invertible. Since M_ω is obtained from the frame, using 1.23 we compute

$$\frac{\partial}{\partial x} \det F = \det F \cdot \mathrm{tr}(U(\zeta)) = 0.$$

Analogously, the same holds for the derivative in the y-direction. Hence the determinant of the frame remains constant, and the initial condition implies that it is equal to 1 for all (x, y). In particular, for $\omega \in \Gamma_a$,

$$\det M_\omega = \det F(\omega) = 1.$$

It remains to justify the transformation rules

$$\sigma^*\mu_\omega = \mu_\omega^{-1}, \qquad \rho^*\mu_\omega = \overline{\mu_\omega}^{\,-1}. \tag{4.4}$$

The points (λ, ν) and $(\lambda, -\nu)$ correspond to the two eigenlines of $\zeta_0(\lambda)$. Since $M_\omega(\lambda)$ commutes with $\zeta_0(\lambda)$, it is simultaneously diagonalizable with $\zeta_0(\lambda)$ whenever the eigenvalues are distinct. Let the two eigenvalues of $M_\omega(\lambda)$ on these eigenlines be

$$\mu_\omega(\lambda, \nu) \qquad \text{and} \qquad \mu_\omega(\lambda, -\nu).$$

Because $M_\omega(\lambda) \in \mathrm{SL}(2, \mathbb{C})$, the product of its two eigenvalues is equal to 1. Hence

$$\mu_\omega(\lambda, \nu)\, \mu_\omega(\lambda, -\nu) = 1 \quad \Leftrightarrow \quad \mu_\omega(\lambda, -\nu) = \mu_\omega(\lambda, \nu)^{-1}.$$

This is exactly

$$\sigma^*\mu_\omega = \mu_\omega^{-1}.$$

By construction, M_ω belongs to the loop group G_- (the definition follows in the proof of Lemma 4.9), and therefore for $\lambda \in \mathbb{C}^\times$ it satisfies the SU(2) reality condition 1.8 for $\lambda \in \mathbb{S}^1$

$$M_\omega(\lambda) = \left(M_\omega(\bar{\lambda}^{-1})^{\,T}\right)^{-1}.$$

Equivalently, the eigenvalues of $M_\omega(\bar{\lambda}^{-1})$ are the inverses of the complex conjugates of the eigenvalues of $M_\omega(\lambda)$. Since the involution ρ maps (λ, ν) to the point lying over $\bar{\lambda}^{-1}$, we obtain

$$\mu_\omega\left(\rho(\lambda, \nu)\right) = \overline{\mu_\omega(\lambda, \nu)}^{\,-1}.$$

Thus

$$\rho^*\mu_\omega = \overline{\mu_\omega}^{\,-1}.$$

Therefore the monodromy eigenvalue is a holomorphic function

$$\mu_\omega : \Sigma^* \to \mathbb{C}^\times$$

satisfying the symmetry relations (4.4). These functions will be the key tool for describing the period lattice Γ_a.

Lemma 4.9. *Let $a \in \mathcal{M}_2^1$. Then $\omega \in \Gamma_a$ if and only if the function*

$$\exp(\omega\lambda^{-1}\nu)$$

on Σ^* can be factorized as the product of

- *a holomorphic function $\mu_\omega : \Sigma^* \to \mathbb{C}^\times$ satisfying (4.4), and*
- *a holomorphic function on Σ^* which extends holomorphically to $\lambda = 0$ and takes the value 1 there.*

Proof. We follow the standard Iwasawa factorization argument as in Dorfmeister et al. [1998], Theorem 2.2.

Let G denote the group of holomorphic maps

$$g : \mathbb{C}^\times \to \mathrm{SL}(2, \mathbb{C}),$$

and define the subgroups

$$G_- := \{g \in G \mid g(\lambda) \in \mathrm{SU}(2) \text{ for } \lambda \in \mathbb{S}^1\},$$

and

$$G_+ := \{g : \mathbb{C} \to \mathrm{SL}(2, \mathbb{C}) \mid g \text{ holomorphic and } g(0) \in \mathrm{SL}_+(2, \mathbb{C})\},$$

where $\mathrm{SL}_+(2, \mathbb{C})$ denotes the subgroup of upper triangular matrices with positive real diagonal entries. Let $\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+$ be the corresponding Lie algebras.

We first show that

$$\lambda \longmapsto F(z)^{-1} \exp(z\lambda^{-1}\zeta_0)$$

belongs to G_+. Indeed, differentiating with respect to z gives

$$d\big(F^{-1}(z) \exp(z\lambda^{-1}\zeta_0)\big) \exp(-z\lambda^{-1}\zeta_0)F(z) = -U(\zeta)\, dx - V(\zeta)\, dy + F^{-1}\lambda^{-1}\zeta_0 F\, dz.$$

Using $\zeta = F^{-1}\zeta_0 F$, we obtain

$$\lambda^{-1}\zeta\, dz - U(\zeta)\, dx - V(\zeta)\, dy = \begin{pmatrix} \frac{\alpha}{2}\, dz + \frac{\bar\alpha}{2}\, d\bar z - \bar\alpha\lambda\, dz & \beta\, dz + \gamma\, d\bar z - \gamma\lambda\, dz \\ -\bar\beta\lambda\, dz - \gamma^{-1}\lambda\, d\bar z + \gamma^{-1}\lambda^2\, dz & -\frac{\alpha}{2}\, dz - \frac{\bar\alpha}{2}\, d\bar z + \bar\alpha\lambda\, dz \end{pmatrix}.$$

This one-form takes values in $\mathfrak{g}_+$. Hence

$$\lambda \mapsto F(z)^{-1} \exp(z\lambda^{-1}\zeta_0) \in G_+.$$

Consequently, $F(z) \in G_-$, and

$$\exp(z\lambda^{-1}\zeta_0) = g_-(z, \lambda)\, g_+(z, \lambda)$$

is the corresponding factorization with

$$g_-(z, \lambda) = F(z).$$

Next we interpret the condition $\omega \in \Gamma_a$. By definition,

$$\omega \in \Gamma_a \iff F(\omega)\zeta_0 = \zeta_0 F(\omega).$$

Since

$$\exp(\omega\lambda^{-1}\zeta_0) = F(\omega)\, g_+(\omega, \lambda),$$

this is equivalent to both factors commuting with ζ_0.

Since $a \in \mathcal{M}_2^1$, the matrix ζ_0 has no roots in $\mathbb{C}^\times$. Therefore its eigenspaces are one-dimensional at every point of Σ^*. Any holomorphic matrix-valued function commuting with ζ_0 must then be of the form

$$f(\lambda)\, 1 + g(\lambda)\, \zeta_0$$

with holomorphic functions

$$f, g : \mathbb{C}^\times \to \mathbb{C}.$$

On the eigenspace corresponding to $(\lambda, \nu) \in \Sigma^*$, such an endomorphism acts by multiplication with

$$f(\lambda) + g(\lambda)\nu.$$

Thus the G_--factor acts by a holomorphic function μ_ω on Σ^*, and the G_+-factor acts by another holomorphic function h on Σ^*, such that

$$\exp(\omega\lambda^{-1}\nu) = \mu_\omega\, h.$$

Lastly, we identify the two conditions. The G_--factor takes values in $\mathrm{SU}(2)$ on $\mathbb{S}^1$, and this is equivalent to the transformation laws

$$\sigma^*\mu_\omega = \mu_\omega^{-1}, \qquad \rho^*\mu_\omega = \overline{\mu_\omega}^{\,-1}.$$

Likewise, the defining property of G_+ is equivalent to the statement that h extends holomorphically to $\lambda = 0$ and satisfies $h(0) = 1$.

This proves the claimed characterization. □

The preceding proof shows that μ_ω is uniquely determined. Since $\mu_\omega : \Sigma^* \to \mathbb{C}^\times$ is holomorphic and nowhere vanishing, its logarithmic derivative

$$d\ln\mu_\omega = \frac{d\mu_\omega}{\mu_\omega}$$

is holomorphic on Σ^*. Hence possible poles can only occur at the points added in the compactification of Σ^*, i.e. at the points over $\lambda = 0$ and $\lambda = \infty$. Near $\lambda = 0$, Lemma 4.9 gives the asymptotic behavior

$$\ln\mu_\omega = \omega\lambda^{-1}\nu + \text{holomorphic terms},$$

so $d\ln\mu_\omega$ may indeed have a pole there. By the symmetry relations, the same is true in $\lambda = \infty$.By the symmetry relations (4.4), the same type of singular behavior occurs in $\lambda = \infty$. Consequently, $d\ln\mu_\omega$ is a meromorphic differential of the second kind on Σ^* with poles only at $\lambda = 0$ and $\lambda = \infty$, and therefore it is of the form

$$d\ln\mu_\omega = \frac{b_\omega(\lambda)}{2\nu\lambda}\,d\lambda, \qquad b_\omega \in \mathbb{C}^3[\lambda], \qquad \bar\lambda^3 b_\omega(\bar\lambda^{-1}) = -\overline{b_\omega(\lambda)}. \tag{4.5}$$

We now determine the constant coefficient of b_ω. Since $a(0) = 1$, one has near $\lambda = 0$

$$\nu^2 = -\lambda a(\lambda) = -\lambda + \mathcal{O}(\lambda^2).$$

Hence near $\lambda = 0$,

$$\nu = \mathcal{O}(\lambda^{1/2}), \qquad \text{and} \qquad 2\nu\, d\nu = \big(-1 + \mathcal{O}(\lambda)\big)\, d\lambda.$$

As just seen, Lemma 4.9 implies that one branch of $\ln\mu_\omega$ has the expansion

$$\ln\mu_\omega = \omega\lambda^{-1}\nu + \mathcal{O}(\nu) \qquad \text{near } \lambda = 0.$$

Differentiating, we obtain

$$d\ln\mu_\omega = d(\omega\lambda^{-1}\nu) + \mathcal{O}(d\nu) = \omega\nu^{-2}\, d\nu + \text{lower order terms}.$$

On the other hand, (4.5) gives

$$d\ln\mu_\omega = \frac{b_\omega(0) + \mathcal{O}(\lambda)}{2\nu\lambda}\, d\lambda = \frac{b_\omega(0)}{\nu^2}\, d\nu + \text{lower order terms}$$

near $\lambda = 0$. Comparing the leading terms yields

$$b_\omega(0) = \omega.$$

The reality condition in (4.5) then implies that b_ω has the form

$$b_\omega(\lambda) = a_1\lambda^3 + a_2\lambda^2 + \bar a_2\lambda + \bar a_1. \tag{4.6}$$

Therefore the coefficient of λ^3 in b_ω is equal to $-\bar\omega$.

Lemma 4.10. *Let $a \in \mathcal{M}_2^1$ and let*

$$\eta_\omega = \frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda$$

be a meromorphic 1-form on Σ^*. Then the following are equivalent:

1. *There exists a holomorphic function*

$$\mu_\omega : \Sigma^* \to \mathbb{C}^\times$$

such that $d \ln \mu_\omega = \eta_\omega$.

2. *For every closed cycle γ on Σ^*,*

$$\int_\gamma \eta_\omega \in 2\pi i\, \mathbb{Z}.$$

Proof. **(1) ⇒ (2):** Assume that there exists a holomorphic function μ_ω such that

$$\eta_\omega = d \ln \mu_\omega.$$

Let γ be a closed cycle on Σ^*. Then

$$\int_\gamma \eta_\omega = \int_\gamma d \ln \mu_\omega = \ln \mu_\omega(\gamma(1)) - \ln \mu_\omega(\gamma(0)).$$

Although $\gamma(1) = \gamma(0)$, the logarithm is multivalued. Hence the difference of the logarithms is an integer multiple of $2\pi i$, and therefore

$$\int_\gamma \eta_\omega \in 2\pi i\, \mathbb{Z}.$$

(2) ⇒ (1): Suppose that

$$\int_\gamma \eta_\omega \in 2\pi i\, \mathbb{Z} \qquad \text{for all closed cycles } \gamma.$$

Fix a base point $p_0 \in \Sigma^*$ and define

$$\ln \mu_\omega(p) := \int_{p_0}^{p} \eta_\omega.$$

If two paths from p_0 to p differ by a closed cycle γ, then their integrals differ by

$$\int_\gamma \eta_\omega \in 2\pi i\, \mathbb{Z}.$$

Hence $\ln \mu_\omega$ is well-defined up to addition of multiples of $2\pi i$.

Therefore the function

$$\mu_\omega(p) := \exp\left(\int_{p_0}^{p} \eta_\omega\right)$$

is globally well-defined and holomorphic on Σ^*. By construction, it satisfies

$$d \ln \mu_\omega = \eta_\omega.$$

This proves the equivalence. ☐

We have thus obtained the following corollary.

Corollary 4.11. *Let $a \in \mathcal{M}_2^1$. Then the elements of Γ_a are precisely the values $\omega = b_\omega(0)$ for which the differential*

$$\frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda$$

is the logarithmic derivative of a holomorphic function on Σ^*.

4.3 Extension of the Lattice map

For $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, the spectral curve becomes singular at the double roots of a. In order to obtain a meaningful limiting notion, we impose the additional condition

$$\mu_\omega = f_\omega(\lambda) + g_\omega(\lambda)\nu, \qquad f_\omega, g_\omega : \mathbb{C}^\times \to \mathbb{C} \text{ holomorphic.} \tag{4.7}$$

This condition guarantees regularity at the singular points, since it gives a well-defined finite value at $\nu = 0$.

In particular, at a root λ_0 of a, one has $\nu = 0$, hence

$$\sigma(\lambda_0, 0) = (\lambda_0, 0),$$

and therefore

$$\mu_\omega(\lambda_0, 0) = \mu_\omega(\lambda_0, 0)^{-1}.$$

Thus

$$\mu_\omega(\lambda_0, 0) \in \{\pm 1\}.$$

For every $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, let Σ° denote the variety (4.3) with the singular points over the double roots of a removed.

Definition 4.12. *An* A-cycle *on* Σ° *is a closed cycle which surrounds in the* λ*-plane exactly two roots* α *and* $\bar{\alpha}^{-1}$ *of* a*.*

For $a \in \mathcal{M}_2^1$, there are two such A-cycles. Each of them crosses the unit circle in exactly two points and is therefore reversed by ρ. Moreover, ρ preserves the intersection points on the unit circle. In the limiting case where two simple roots coalesce to a unitary double root, the corresponding A-cycle converges to a cycle surrounding that double root.

The next result is the central existence statement for the polynomials b_ω.

Lemma 4.13. *For every* $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ *and every* $\omega \in \mathbb{C}$*, there exists a unique polynomial* $b_\omega \in \mathbb{C}^3[\lambda]$ *such that:*

1. $b_\omega(0) = \omega$,
2. $\bar{\lambda}^3 b_\omega(\bar{\lambda}^{-1}) = -\overline{b_\omega(\lambda)}$ *for all* $\lambda \in \mathbb{C}$,
3. *the integral of*
$$\frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda$$
along each A*-cycle vanishes.*

Proof. Set

$$\eta_\omega := \frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda.$$

By (4.6), the general form of b_ω is

$$b_\omega(\lambda) = \omega - \bar{\omega}\,\lambda^3 + \beta_1(\lambda - \lambda^2) + i\beta_2(\lambda + \lambda^2), \qquad \beta_1, \beta_2 \in \mathbb{R}.$$

First we show that the A-periods of η_ω are real. We already know that the involutions satisfy

$$\sigma^*\eta_\omega = -\eta_\omega, \qquad \rho^*\eta_\omega = -\overline{\eta_\omega}.$$

Let A be an A-cycle. Since ρ reverses the orientation of A, we obtain

$$\int_A \eta_\omega = \int_{-A} \rho^*\eta_\omega = -\int_A \rho^*\eta_\omega = -\int_A (-\overline{\eta_\omega}) = \overline{\int_A \eta_\omega}.$$

Hence the integrals over the A-cycles are real.

next we reduce the vanishing of the A-periods to a real linear system in β_1 and β_2. The conditions

$$\int_{A_j} \eta_\omega = 0, \qquad j = 1, 2,$$

become, using the above expression for b_ω,

$$\mathbf{A}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = -\begin{pmatrix}\displaystyle\int_{A_1} \frac{\omega - \bar{\omega}\lambda^3}{2\nu\lambda}\, d\lambda \\ \displaystyle\int_{A_2} \frac{\omega - \bar{\omega}\lambda^3}{2\nu\lambda}\, d\lambda\end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix}\displaystyle\int_{A_1} \frac{\lambda - \lambda^2}{2\nu\lambda}\, d\lambda & \displaystyle\int_{A_1} \frac{i(\lambda + \lambda^2)}{2\nu\lambda}\, d\lambda \\ \displaystyle\int_{A_2} \frac{\lambda - \lambda^2}{2\nu\lambda}\, d\lambda & \displaystyle\int_{A_2} \frac{i(\lambda + \lambda^2)}{2\nu\lambda}\, d\lambda\end{pmatrix}.$$

By Step 1, the entries of $\mathbf{A}$ are real.

It remains to show that $\mathbf{A}$ is invertible.

If $a \in \mathcal{M}_2^1$, then the compactification $\bar{\Sigma}$ of Σ^* is a smooth hyperelliptic curve of genus–two. The two differentials

$$\frac{\lambda - \lambda^2}{\nu\lambda}\, d\lambda \qquad \text{and} \qquad \frac{i(\lambda + \lambda^2)}{\nu\lambda}\, d\lambda$$

extend holomorphically to $\bar{\Sigma}$, and they are linearly independent. Since the space of holomorphic 1-forms on a compact Riemann surface of genus–two is two-dimensional, these two differentials form a basis. Their A-period matrix is therefore invertible, and hence $\mathbf{A}$ is invertible.

If $a \in \mathcal{M}_2^2$, then $\bar{\Sigma}$ has genus–one. The differentials corresponding to those b_ω which vanish at the double root of a and at $\lambda = 0$ extend holomorphically to $\bar{\Sigma}$. The integral along the A-cycle surrounding the simple roots is in general non-zero, and therefore imposes a non-trivial condition on β_1 and β_2. The integral along the A-cycle surrounding the double root is proportional to the value of b_ω at that double root and therefore imposes a second condition. We shall show after this discussion that this is indeed the case. These two conditions again determine β_1, β_2 uniquely. Hence $\mathbf{A}$ is invertible.

If $a \in \mathcal{M}_2^3$, both A-cycles surround double roots, and the corresponding integrals are proportional to the values of b_ω at these roots. Since the two roots are distinct, the resulting linear system is again non-degenerate. Thus $\mathbf{A}$ is invertible.

Therefore the linear system has a unique solution $(\beta_1, \beta_2) \in \mathbb{R}^2$, and hence b_ω exists uniquely.

To complete the proof, we show that at double roots of a, the integral along the A-cycle surrounding that double root is indeed proportional to the value of b_ω there. In the case of a double root λ_0, we can write locally

$$a(\lambda) = (\lambda - \lambda_0)^2 \tilde{a}(\lambda), \qquad \tilde{a}(\lambda_0) \neq 0.$$

This implies $\nu^2 = -(\lambda - \lambda_0)^2 \lambda \tilde{a}(\lambda)$, so ν vanishes to first order at λ_0. Writing $t = \lambda - \lambda_0$, we have locally

$$\nu = ct + O(t^2), \qquad c \neq 0.$$

Expanding b_ω at λ_0 gives

$$b_\omega(\lambda) = b_\omega(\lambda_0) + O(t).$$

It follows that

$$\frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda = \left(\frac{b_\omega(\lambda_0)}{2c\lambda_0}\frac{1}{t} + O(1)\right) dt.$$

Thus the integral along a small loop around $t = 0$, and hence along A_{λ_0}, is

$$\int_{A_{\lambda_0}} \frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda = \frac{2\pi i}{2c\lambda_0}\, b_\omega(\lambda_0).$$

This completes the proof. □

The A-cycles can be chosen continuously under small deformations of $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$. Since the entries of the matrix $\mathbf{A}$ and the right-hand side of the above linear system depend continuously on (ω, a), it follows that b_ω depends continuously on (ω, a) on this set.

In addition to the A-cycles, we now choose B-cycles.

Definition 4.14. *Assume $a \in \mathcal{M}_2^1$, and let α_1, α_2 denote the two simple roots of a inside the unit disc, numbered so that the j-th A-cycle surrounds α_j and $\bar{\alpha}_j^{-1}$.*

1. *The B_1-cycle is chosen as a cycle in the λ-plane which surrounds $\lambda = 0$, passes through α_1, and does not surround α_2.*
2. *The B_2-cycle is chosen as a cycle which passes through $\bar{\alpha}_2^{-1}$ and surrounds all remaining roots.*

Let $H_1(\bar{\Sigma}, \mathbb{Z})$ denote the first singular homology group of $\bar{\Sigma}$ with integer coefficients, that is, the abelian group of homology classes of closed oriented curves on $\bar{\Sigma}$.

Lemma 4.15. *Let $a \in \mathcal{M}_2^1$, and let $\bar{\Sigma}$ denote the compactification of the spectral curve Σ^*. Then the two A-cycles and the two B-cycles constructed above form a canonical homology basis of $H_1(\bar{\Sigma}, \mathbb{Z})$.*

Proof. We already know that $\bar{\Sigma}$ is a compact Riemann surface of genus–two. We choose the orientations of the cycles in such a way that every intersection of A_i with B_i contributes positively.

By construction, the A-cycles are disjoint, and likewise the B-cycles are disjoint. Hence

$$A_i \cdot A_j = 0, \qquad B_i \cdot B_j = 0.$$

Here "$\cdot$" denotes the intersection number of homology classes in $H_1(\bar{\Sigma}, \mathbb{Z})$. Moreover, B_j intersects A_j exactly once and does not intersect the other A-cycle. With the chosen orientations, this gives

$$A_i \cdot B_j = \delta_{ij}.$$

It remains to show that the homology classes of these four cycles form a basis of $H_1(\bar{\Sigma}, \mathbb{Z})$. Since $\bar{\Sigma}$ has genus–two, it is enough to prove that

$$[A_1], [A_2], [B_1], [B_2] \in H_1(\bar{\Sigma}, \mathbb{Z})$$

are linearly independent over $\mathbb{Z}$.

Suppose that

$$m_1[A_1] + m_2[A_2] + n_1[B_1] + n_2[B_2] = 0 \qquad \text{in } H_1(\bar{\Sigma}, \mathbb{Z})$$

for some integers m_1, m_2, n_1, n_2. Intersecting this relation with A_1 yields

$$0 = \big(m_1[A_1] + m_2[A_2] + n_1[B_1] + n_2[B_2]\big) \cdot A_1 = n_1(B_1 \cdot A_1).$$

Since

$$B_1 \cdot A_1 = -A_1 \cdot B_1 = -1,$$

it follows that $n_1 = 0$. In the same way, intersecting with A_2 gives $n_2 = 0$.

Thus we are left with

$$m_1[A_1] + m_2[A_2] = 0.$$

Now intersecting with B_1 gives

$$0 = \big(m_1[A_1] + m_2[A_2]\big) \cdot B_1 = m_1(A_1 \cdot B_1) = m_1,$$

hence $m_1 = 0$. Likewise, intersecting with B_2 gives $m_2 = 0$.

Therefore all coefficients vanish, so the four classes are linearly independent. Since $\bar{\Sigma}$ has genus–two, these four classes form a canonical homology basis of $H_1(\bar{\Sigma}, \mathbb{Z})$. □

Lemma 4.16. *The involution ρ preserves the B-cycles up to addition of A-cycles. Moreover, for every $\omega \in \mathbb{C}$, the integrals of η_ω along the B-cycles are purely imaginary.*

Proof. Since ρ is an orientation-reversing diffeomorphism, it preserves the intersection pairing up to sign. In particular,

$$A_i \cdot \rho(B_j) = \pm\delta_{ij}.$$

Thus $\rho(B_j)$ has the same intersection properties with the A-cycles as B_j. Such a cycle is uniquely determined up to addition of A-cycles. Indeed, let $\gamma, \tilde{\gamma} \in H_1(\bar{\Sigma}, \mathbb{Z})$ be cycles satisfying

$$A_i \cdot \gamma = A_i \cdot \tilde{\gamma} = \delta_{ij}.$$

Then their difference $\delta := \gamma - \tilde{\gamma}$ satisfies

$$A_i \cdot \delta = A_i \cdot \gamma - A_i \cdot \tilde{\gamma} = 0 \qquad \text{for all } i.$$

Since we already derived the basis of $H_1(\bar{\Sigma}, \mathbb{Z})$, we write

$$\delta = \sum_k \alpha_k A_k + \sum_k \beta_k B_k.$$

Since the intersection numbe rof two A-cycles is zero, we obtain

$$0 = A_i \cdot \delta = \sum_k \beta_k (A_i \cdot B_k) = \beta_i,$$

and hence $\beta_i = 0$ for all i. Therefore $\delta = \sum_k \alpha_k A_k$, and thus

$$\gamma \sim \tilde{\gamma} + \sum_k \alpha_k A_k.$$

It follows that

$$\rho(B_j) \sim \pm B_j + \sum_k n_k A_k, \qquad n_k \in \mathbb{Z}.$$

Now recall that

$$\rho^* \eta_\omega = -\overline{\eta_\omega}.$$

Hence

$$\int_{\rho(B_j)} \eta_\omega = \int_{B_j} \rho^* \eta_\omega = -\int_{B_j} \overline{\eta_\omega} = -\overline{\int_{B_j} \eta_\omega}.$$

On the other hand, since the A-periods vanish by construction of b_ω, addition of A-cycles does not change the value of the integral. Thus, up to sign,

$$\int_{\rho(B_j)} \eta_\omega = \pm \int_{B_j} \eta_\omega.$$

To determine the sign, note that ρ reverses the A-cycles, so

$$\rho(A_i) = -A_i.$$

Since ρ is orientation-reversing, the intersection pairing changes sign under ρ. Hence

$$A_i \cdot \rho(B_j) = -\rho(A_i) \cdot \rho(B_j) = A_i \cdot B_j = \delta_{ij}.$$

Therefore the cycle $\rho(B_j) - B_j$ has zero intersection number with both A_1 and A_2.

Because A_1, A_2, B_1, B_2 form a canonical basis, any cycle with zero intersection number with both A_1 and A_2 is homologous to an integral linear combination of A_1 and A_2. Thus

$$\rho(B_j) \sim B_j + \sum_k n_{jk} A_k$$

for suitable integers n_{jk}.

Since the A-periods of η_ω vanish by construction of b_ω, it follows that

$$\int_{\rho(B_j)} \eta_\omega = \int_{B_j} \eta_\omega.$$

Combining this with the previous identity gives

$$\int_{B_j} \eta_\omega = -\overline{\int_{B_j} \eta_\omega},$$

so the integral over the B-cycle is purely imaginary. □

At $\lambda = 0$, one has as before $\nu^2 = -\lambda + O(\lambda^2)$, hence ν is a local parameter and

$$\eta_\omega = \frac{b_\omega(\lambda)}{2\nu\lambda} d\lambda \sim \frac{b_\omega(0)}{t^2} dt,$$

where $t = \nu$. Thus η_ω has a pole of order two and no term of the form $t^{-1}dt$, so the residue vanishes. The same holds at $\lambda = \infty$.

Since η_ω is holomorphic, it is closed, and therefore

$$d(\Re(\eta_\omega)) = \Re(d\eta_\omega) = 0.$$

Hence $\Re(\eta_\omega)$ is a closed real 1-form on Σ°. Moreover, for every closed cycle γ,

$$\int_\gamma \Re(\eta_\omega) = \Re\left(\int_\gamma \eta_\omega \right).$$

Since all periods of η_ω are purely imaginary, the right-hand side vanishes. Thus $\Re(\eta_\omega)$ has vanishing periods. Since the real part of a holomorphic function is harmonic, it follows that there exists a globally defined harmonic function

$$h_\omega : \Sigma^\circ \to \mathbb{R}$$

such that $dh_\omega = \Re(\eta_\omega)$. This function is unique up to an additive constant, which can be fixed by imposing

$$\sigma^* h_\omega = -h_\omega.$$

We are now ready to extend the map $(\omega, a) \mapsto b_\omega$ to all of $\mathcal{M}_2$.

Lemma 4.17. *The map $(\omega, a) \longmapsto b_\omega$ extends continuously to $(\omega, a) \in \mathbb{C} \times \mathcal{M}_2$.*

Proof. We only need to consider limits

$$a_n \to a$$

with

$$a_n \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3, \qquad a \in \mathcal{M}_2^4 \cup \mathcal{M}_2^5.$$

Let $\| \cdot \|$ be any norm on the finite-dimensional vector space $\mathbb{C}^3[\lambda]$. For a bounded sequence ω_n, consider the normalized polynomials

$$\tilde{b}_n := \frac{b_{\omega_n}}{\|b_{\omega_n}\|}.$$

Since the unit sphere in $\mathbb{C}^3[\lambda]$ is compact, there exists a convergent subsequence of $(\tilde{b}_n)$.

The corresponding normalized harmonic functions

$$\tilde{h}_n := \frac{h_{\omega_n}}{\|b_{\omega_n}\|}$$

converge uniformly on compact subsets of Σ°. Indeed, their differentials are the real parts of the normalized forms associated with $\tilde{b}_n$, and these forms converge on compact subsets away from the singular points.

The singular parts of

$$dh_{\omega_n}$$

at $\lambda = 0$ and $\lambda = \infty$ are determined solely by ω_n. Hence, for bounded ω_n, the functions h_{ω_n} are uniformly bounded on the complement of two small disjoint discs around $\lambda = 0$ and $\lambda = \infty$. By the strong maximum principle A.5 and the removable singularity theorem for harmonic functions A.6, the limit extends harmonically across the remaining points of the limiting two-sheeted cover.

In the limit, the two-sheeted covering is branched only at the simple roots of a. Therefore the limiting differential must vanish at the double roots of a, and if a has a fourth-order root, then the limiting polynomial has a second-order zero there. It follows that the coefficients β_1, β_2 in the representation

$$b_\omega(\lambda) = \omega - \bar{\omega}\,\lambda^3 + \beta_1(\lambda - \lambda^2) + i\beta_2(\lambda + \lambda^2)$$

remain bounded for bounded ω.

Since β_1 and β_2 remain bounded, the polynomials b_{ω_n} themselves remain bounded. Thus every subsequence admits a further subsequence converging to the same limit, which proves continuity of

$$(\omega, a) \mapsto b_\omega$$

at points of $\mathbb{C} \times \mathcal{M}_2$.

Hence the map extends continuously to all of $\mathbb{C} \times \mathcal{M}_2$. □

We are now ready to state the main theorem of this chapter.

Theorem 4.18. *For every $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ the values*

$$b_\omega(0) = \omega$$

of all polynomials b_ω in (4.5) with the following property form a lattice $\tilde{\Gamma}_a \subset \mathbb{C}$: they define the logarithmic derivative

$$d \ln \mu_\omega = \frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda$$

of a function μ_ω on Σ^* satisfying (4.4) and (4.7). The map

$$T : \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3 \to \mathcal{F}, \qquad a \mapsto [\tilde{\Gamma}_a],$$

which assigns to a the corresponding isomorphism class of lattices, is continuous.

Moreover, for every compact subset $\mathcal{K} \subset \mathcal{F}$ and every $a \in \mathcal{M}_2^4 \cup \mathcal{M}_2^5$ there exists a neighbourhood $O \subset \mathcal{M}_2$ of a such that $T^{-1}(\mathcal{K}) \cap O = \emptyset$.

Proof. For $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ and $\omega \in \mathbb{C}$, let

$$\eta_\omega := \frac{b_\omega(\lambda)}{2\nu\lambda}\, d\lambda,$$

where b_ω is the unique polynomial from Lemma 4.13.

We first show that $\tilde{\Gamma}_a$ is indeed a lattice. By Corollary 4.11 in the smooth case $a \in \mathcal{M}_2^1$, and by the discussion following (4.7) in the singular cases $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, an element ω belongs to $\tilde{\Gamma}_a$ if and only if η_ω is the logarithmic derivative of a function μ_ω satisfying (4.4) and (4.7). Since the A-periods of η_ω vanish by construction of b_ω, Lemma 4.10 shows that this is equivalent to requiring that both B-periods of η_ω lie in $2\pi i\,\mathbb{Z}$.

We now show that the map

$$\omega \longmapsto \left(\int_{B_1} \eta_\omega, \int_{B_2} \eta_\omega\right)$$

has trivial kernel. Assume that both B-periods vanish. Then all periods of η_ω vanish, because the A-periods vanish by Lemma 4.13. Hence

$$\eta_\omega = dh$$

for a meromorphic function h on the normalization of the compactified spectral curve $\tilde{\Sigma}$.

In the cases $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, the curve $\bar{\Sigma}$ is singular at the points lying over the double roots of a.

We therefore consider the normalization

$$\pi : \tilde{\Sigma} \to \bar{\Sigma},$$

which is a smooth compact Riemann surface obtained by resolving these singularities. All differential-geometric arguments are carried out on $\tilde{\Sigma}$.

Suppose $\omega \neq 0$. Then $\eta_\omega \neq 0$, since $b_\omega(0) = \omega$. Because $\sigma^*\eta_\omega = -\eta_\omega$, the function

$$\psi := h - \sigma^* h$$

satisfies

$$d\psi = 2\eta_\omega,$$

so ψ is non-constant.

If $a \in \mathcal{M}_2^1$, then since η_ω has at most simple poles at the two points over $\lambda = 0$ and $\lambda = \infty$, the same is true for ψ. At every branch point over a simple root of a, the involution σ fixes the point, and hence ψ vanishes there. Thus ψ has at least four zeros and at most two simple poles. This is impossible for a non-zero meromorphic function on a compact Riemann surface, since the orders of poles and zeros must have the same total degree.

If $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, then the vanishing of the A-period around each double root implies

$$b_\omega(\lambda_0) = 0$$

at every double root λ_0 of a; see the last part of the proof of Lemma 4.13. Hence η_ω is holomorphic at the points over the double roots on the normalization. Again ψ has at most simple poles at the points over $\lambda = 0$ and $\lambda = \infty$, while it vanishes at all points over the roots of a: at the simple roots because they are branch points, and at the points over the double roots because η_ω is holomorphic there and anti-invariant under σ. This again contradicts the degree of the divisor of a non-zero meromorphic function.

Therefore $\omega = 0$.

So the B-period map has trivial kernel. By the lemma on the B-cycles, both B-periods are purely imaginary. Since both domain and codomain are two-dimensional real vector spaces, the map

$$\mathbb{C} \longrightarrow i\mathbb{R}^2, \qquad \omega \longmapsto \left(\int_{B_1} \eta_\omega, \int_{B_2} \eta_\omega\right)$$

is an $\mathbb{R}$-linear isomorphism.

Indeed, by Lemma 4.13, the polynomial b_ω depends $\mathbb{R}$-linearly on ω. Hence also η_ω depends $\mathbb{R}$-linearly on ω, and therefore the period map is $\mathbb{R}$-linear.

Moreover, by the lemma on the B-cycles, both periods are purely imaginary, so the image is contained in $i\mathbb{R}^2$, It remains to show that this map is injective. Suppose that

$$\left(\int_{B_1} \eta_\omega, \int_{B_2} \eta_\omega\right) = 0.$$

Then all B-periods of η_ω vanish. Since the A-periods vanish by construction of b_ω, all periods of η_ω vanish. By the previous argument, this implies $\omega = 0$. Thus the map is injective.

Now $\mathbb{C}$ is a real vector space of dimension 2, and $i\mathbb{R}^2$ is also a real vector space of dimension 2. Hence an injective $\mathbb{R}$-linear map between them is automatically surjective. Therefore the period map is an $\mathbb{R}$-linear isomorphism.

It follows immediately that

$$\tilde{\Gamma}_a = \left\{\omega \in \mathbb{C} \,\middle|\, \int_{B_1} \eta_\omega, \int_{B_2} \eta_\omega \in 2\pi i\,\mathbb{Z}\right\}$$

is the inverse image of the lattice

$$(2\pi i\,\mathbb{Z})^2 \subset i\mathbb{R}^2$$

under an $\mathbb{R}$-linear isomorphism. Hence $\tilde{\Gamma}_a \subset \mathbb{C}$ is a lattice itself.

For the continuity statement, note that for the same reasons for which the A-cycles vary continuously, the B-cycles vary continuously on $\mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$, and that $(\omega, a) \mapsto b_\omega$ depends continuously on (ω, a) by Lemma 4.17. Therefore the B-period map depends continuously on a. Since $T(a)$ is the isomorphism class of the inverse image of the fixed lattice $(2\pi i\,\mathbb{Z})^2$ under this continuously varying $\mathbb{R}$-linear isomorphism, the map

$$T : \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3 \to \mathcal{F}$$

is continuous.

It remains to prove the last statement. Let

$$a \in \mathcal{M}_2^4 \cup \mathcal{M}_2^5, \qquad a_n \to a, \qquad a_n \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3.$$

Write the two B-periods of η_ω as

$$\Pi_1(\omega, a_n), \qquad \Pi_2(\omega, a_n),$$

and introduce

$$\Pi_+ := \Pi_1 + \Pi_2, \qquad \Pi_- := \Pi_1 - \Pi_2.$$

By Lemma 4.17, bounded sequences ω_n give rise to convergent subsequences of b_{ω_n}. In the limit $a_n \to a$, the cycle corresponding to Π_- collapses. Hence, if $\Pi_-(\omega_n, a_n)$ is a fixed non-zero element of $2\pi i\,\mathbb{Z}$, then ω_n cannot remain bounded; otherwise the integrals over the collapsing cycles would converge to 0, contradicting the fact that they are equal to $\Pi_-(\omega_n, a_n)$.

Thus any sequence with fixed non-zero Π_- must satisfy $|\omega_n| \to \infty$.

On the other hand, if $\Pi_-(\omega_n, a_n) = 0$, then the corresponding local primitives on the two-sheeted covering near the collapsing roots are single-valued. After normalization, the resulting sequence has a convergent subsequence, and therefore ω_n remains bounded.

Consequently, near $a \in \mathcal{M}_2^4 \cup \mathcal{M}_2^5$, one period direction remains bounded whereas the independent period direction becomes unbounded. After rescaling by the bounded direction, the corresponding lattice basis therefore degenerates, and the conformal class leaves every compact subset of the fundamental domain $\mathcal{F}$. Hence for every compact subset

$$\mathcal{K} \subset \mathcal{F}$$

there exists a neighbourhood O of a such that

$$T^{-1}(\mathcal{K}) \cap O = \emptyset.$$

This proves the theorem. $\square$

Remark 4.19. *The last part of Theorem 4.18 shows that the conformal classes of the lattices do not accumulate in a compact region of the moduli space $\mathcal{F}$ as a approaches $\mathcal{M}_2^4 \cup \mathcal{M}_2^5$. In other words, the corresponding lattices degenerate in the limit. This is precisely the statement needed in order to extend the map*

$$T : a \mapsto [\tilde{\Gamma}_a]$$

continuously to all of $\mathcal{M}_2$ by assigning the value ∞ on $\mathcal{M}_2^4 \cup \mathcal{M}_2^5$, that is, by viewing T as a map into the one-point compactification $\mathcal{F} \cup \{\infty\}$.

Chapter 5

Whitham Deformations

We now introduce Whitham deformations and explain why they provide the natural framework for defining flows on constant mean curvature tori.

The Whitham-deformation viewpoint and its interpretation as an isoperiodic deformation of spectral data are standard in the integrable-systems approach to CMC tori; compare Kilian et al. [2010] and Carberry and Schmidt [2012].

A finite-type CMC torus is encoded by spectral data consisting of a hyperelliptic spectral curve together with logarithmic differentials and Sym points. From the integrable-systems point of view, a deformation of the surface is therefore described by a deformation of these spectral data. Among all such deformations, the distinguished ones are the *Whitham deformations*: they vary the spectral curve and the associated differentials in such a way that the relevant periods remain unchanged. For this reason they are also called *isoperiodic deformations*. The preservation of periods is essential, because the closing conditions of a CMC torus are expressed precisely in terms of period and monodromy data. Consequently, a Whitham deformation moves the spectral data inside the moduli space of periodic solutions, and therefore defines a flow through CMC tori rather than merely through arbitrary finite-type solutions.

In the present thesis we use them to describe how spectral data move while preserving the torus closing conditions.

The chapter *Lattices of Periods* already provides the basic periodicity data needed for this discussion. There we constructed, for

$$a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3,$$

a lattice

$$\tilde{\Gamma}_a \subset \mathbb{C}$$

from the periods of the logarithmic differentials. This lattice is the spectral expression of the double periodicity of the corresponding finite-type solution. In particular, if such a lattice exists, then the associated monodromy closes with respect to two independent periods, and the spectral data satisfy exactly the closing conditions required for a CMC torus.

For the boundary strata

$$a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3,$$

this interpretation has a particularly clear geometric meaning. In these cases the genus–two spectral curve develops double roots on the unit circle, and after normalisation the geometric genus drops to 1 or 0, respectively. Hence the spectral data lying over $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$ should be understood as the lower-genus boundary objects of the genus–two theory, and geometrically they correspond to equivariant CMC tori. The lattices constructed in the previous chapter are therefore not auxiliary objects: in these limiting cases they are exactly the period lattices of the corresponding equivariant CMC tori.

This also explains why Whitham deformations can be used on these strata. Since Whitham deformations preserve the periods of the logarithmic differentials, they preserve the lattice data that encode the closing conditions. Therefore they are tangent to the space of periodic spectral data and allow us to move inside the family of CMC tori determined by a given period structure.

The aim of the present chapter is therefore twofold. First, we formulate the Whitham deformation problem in the notation of this thesis and derive the corresponding Whitham equations. Second, we explain how these equations describe deformations that preserve the period lattice, and hence define flows on the associated CMC tori. This will make precise the role of Whitham deformations as the infinitesimal motion of spectral data inside the moduli space of periodic finite-type solutions.

Closing condition. The *closing condition* is the condition that the spectral data give rise to a doubly periodic immersion, hence to a torus rather than merely to a local or singly periodic surface. In the spectral-curve description this is expressed by requiring that the monodromy of the associated family is trivial, or equivalently has the prescribed period lattice, at the relevant values of the spectral parameter. Thus the closing condition is the global periodicity condition that ensures that the reconstructed CMC surface closes to a torus.

5.1 The period bundle, the Sym point condition, and the Whitham equations in spectral genus–one

In this section we formulate the Whitham deformation theory in the special case of spectral genus–one. Thus the compactified spectral curve is elliptic, and there is exactly one A-cycle and one B-cycle. In particular, for this case, the relevant periods that are preserved by the Whitham Deformations are exactly the periods over those cycles.

The genus–one Whitham formalism used in this section is adapted from the general local Whitham theory in [Carberry et al., 2021, Section 3] and from the spectral-curve description of Whitham deformations and Sym points in [Kilian et al., 2010, Sections 1–3].

As in the general Whitham theory, it is convenient to write the spectral curve in the form $\nu^2 = \lambda a(\lambda)$, which is equivalent to the convention $\nu^2 + \lambda a(\lambda) = 0$ used earlier, after replacing ν by $i\nu$.

Definition 5.1. *Let $\mathcal{H}^1$ denote the set of all polynomials $a \in \mathbb{C}^2[\lambda]$ such that*

1. $a(0) = 1$,
2. *a satisfies the reality condition*
$$\lambda^2 \overline{a(\bar{\lambda}^{-1})} = a(\lambda),$$
3. *a has two simple roots,*
4. *and the hyperelliptic curve*
$$\Sigma_a := \left\{ (\lambda, \nu) \in \mathbb{C}^\times \times \mathbb{C} \;\middle|\; \nu^2 = \lambda a(\lambda) \right\}$$
is the spectral curve of spectral genus–one.
5. $a(1) > 0$.

Remark 5.2. *The last condition is usually not assumed, but since we want to show the Whitham equations specifically for the case where $a \in \mathcal{M}_2$, we add this assumption because it makes the proof of Lemma 5.12 easier.*

We now define the corresponding space of period-normalised differentials.

Definition 5.3. *For $a \in \mathcal{H}^1$, let $\mathcal{B}_a$ denote the set of all polynomials $b \in \mathbb{C}^2[\lambda]$ such that*

1. *b satisfies the reality condition*
$$\lambda^2 \overline{b(\bar{\lambda}^{-1})} = b(\lambda),$$

2. and the meromorphic differential

$$\Theta_b := \frac{b(\lambda)}{\nu}\frac{d\lambda}{\lambda}$$

has vanishing A-period.

Lemma 5.4. *For every $a \in \mathcal{H}^1$, the space $\mathcal{B}_a$ is a real vector space of dimension 2. More precisely, the evaluation map*

$$\mathrm{ev}_0 : \mathcal{B}_a \to \mathbb{C}, \qquad b \mapsto b(0),$$

is an $\mathbb{R}$-linear isomorphism.

Proof. The defining conditions of $\mathcal{B}_a$ are linear over $\mathbb{R}$, hence $\mathcal{B}_a$ is an $\mathbb{R}$-vector space.

Lemma 4.13 states that the following problem has a unique solution: for every $\omega \in \mathbb{C}$ there exists a unique polynomial $b_\omega \in \mathbb{C}^2[\lambda]$ such that $b_\omega(0) = \omega$, b_ω satisfies the reality condition, and the differential

$$\Theta_{b_\omega} = \frac{b_\omega(\lambda)}{\nu}\frac{d\lambda}{\lambda}$$

has vanishing A-period.

Thus the map

$$\mathbb{C} \to \mathcal{B}_a, \qquad \omega \mapsto b_\omega$$

is well-defined and $\mathbb{R}$-linear, since $\mathcal{B}_a$ is $\mathbb{R}$-linear. By construction, $\mathrm{ev}_0(b_\omega) = \omega$, so ev_0 is surjective.

To prove injectivity, let $b \in \mathcal{B}_a$ satisfy $b(0) = 0$. Then b satisfies the same defining conditions as b_0. By uniqueness, it follows that

$$b = b_0 \equiv 0.$$

Hence ev_0 is injective. Therefore it is an $\mathbb{R}$-linear isomorphism. Since $\mathbb{C}$ has real dimension 2, the same is true for $\mathcal{B}_a$. □

Corollary 5.5. *For every $a \in \mathcal{H}^1$, there exists a unique basis (b_1, b_2) of $\mathcal{B}_a$ such that*

$$b_1(0) = 1, \qquad b_2(0) = i.$$

Proof. By Lemma 5.4, the inverse images of 1 and i under ev_0 exist and are unique. Since 1 and i are linearly independent over $\mathbb{R}$, these two elements form a basis. □

From now on we fix $a \in \mathcal{H}^1$ and denote by (b_1, b_2) the normalised basis of $\mathcal{B}_a$.

Definition 5.6. *A point $\lambda_0 \in \mathbb{S}^1$ is called a* Sym point *of the spectral data (a, b_1, b_2) if*

$$b_1(\lambda_0) = 0 \qquad \text{and} \qquad b_2(\lambda_0) = 0.$$

Equivalently, the logarithmic differentials

$$\Theta_l = \frac{b_l(\lambda)}{\nu}\frac{d\lambda}{\lambda}, \qquad l = 1, 2,$$

vanish at $\lambda = \lambda_0$. We say that the sym point condition is satisfied, if there exists such a point.

Remark 5.7. *The Sym point condition is the spectral form of the closing condition for constant mean curvature tori. Since the rotation*

$$\lambda \mapsto e^{i\theta}\lambda, \qquad \theta \in \mathbb{R},$$

preserves $\mathbb{S}^1$ and all reality conditions, every Sym point may be moved to 1. We therefore fix the distinguished Sym point

$$\lambda_0 = 1.$$

We define

$$\mathcal{S}^1_1 := \{a \in \mathcal{H}^1 \mid b_1(1) = 0 = b_2(1)\}.$$

Definition 5.8. *Let $a \in \mathcal{S}^1_1$. An* infinitesimal Whitham deformation *at a is a smooth family*

$$t \mapsto \big(a_t, b_{1,t}, b_{2,t}\big)$$

with $a_0 = a$ such that the periods of the logarithmic differentials

$$\Theta_{l,t} = \frac{b_{l,t}(\lambda)}{\nu_t}\frac{d\lambda}{\lambda}, \qquad l = 1, 2,$$

are independent of t.

We denote the infinitesimal variations by

$$\dot{a} = \frac{d}{dt}\bigg|_{t=0} a_t, \qquad \dot{b}_l = \frac{d}{dt}\bigg|_{t=0} b_{l,t}, \qquad l = 1, 2.$$

Remark 5.9. *The Whitham problem describes isoperiodic deformations of the spectral data, that is, deformations for which all period integrals remain fixed.*

Let q_l be a local primitive of Θ_l, $l = 1, 2$. Since Θ_l is odd under the hyperelliptic involution σ, i.e. $\sigma^*\Theta_l = -\Theta_l$, we may choose q_l locally so that

$$\sigma^* q_l = -q_l.$$

Lemma 5.10. *Let $a \in \mathcal{S}^1_1$, and let*

$$t \mapsto (a_t, b_{1,t}, b_{2,t})$$

be a smooth period-preserving deformation. Then there exist unique polynomials $c_1, c_2 \in \mathbb{C}^2[\lambda]$ satisfying the same reality condition as the b_l, namely

$$\lambda^2\overline{c_l(\bar{\lambda}^{-1})} = c_l(\lambda),$$

such that

$$\dot{q}_l = \frac{i\,c_l(\lambda)}{\nu}, \qquad l = 1, 2.$$

Proof. Since the periods of dq_l remain fixed, the derivative $\dot{q}_l$ is a globally defined meromorphic function on the compactified spectral curve away from the poles over $\lambda = 0$ and $\lambda = \infty$. Moreover, it is odd under σ. Therefore $\dot{q}_l$ must be of the form

$$\dot{q}_l = \frac{r_l(\lambda)}{\nu}$$

for a rational function r_l of λ.

Because $\dot{q}_l$ may only have poles at the points over 0 and ∞, and because in spectral genus–one the admissible pole behaviour is of second-kind type with no additional poles, r_l must in fact be a polynomial of degree at most 2. Indeed, $r_l = ic_l$ is such a function and we obtain

$$\dot{q}_l = \frac{i\,c_l(\lambda)}{\nu}$$

with $c_l \in \mathbb{C}^2[\lambda]$.

The reality condition can be verified explicitly. Recall that the spectral curve carries the anti-holomorphic involution ρ.

Now let q_l be a local primitive of Θ_l. Using the reality condition for b_l we obtain

$$\rho^*\Theta_l = -\,\overline{\Theta_l}.$$

We may therefore assume that $\overline{q_l \circ \rho} = -q_l$. Differentiating with respect to t yields $\overline{\dot{q}_l \circ \rho} = -\dot{q}_l$. Substituting $\dot{q}_l$, we obtain

$$\overline{\dot{q}_l\big(\bar{\lambda}^{-1}, \bar{\lambda}^{-2}\bar{\nu}\big)} = \frac{\overline{i\, c_l(\bar{\lambda}^{-1})}}{\bar{\lambda}^{-2}\bar{\nu}} = -\frac{i\,\lambda^2\overline{c_l(\bar{\lambda}^{-1})}}{\nu},$$

where we used the reality condition for c_l. On the other hand,

$$-\dot{q}_l(\lambda, \nu) = -\frac{i\, c_l(\lambda)}{\nu}.$$

Comparing the two expressions gives

$$\lambda^2\overline{c_l(\bar{\lambda}^{-1})} = c_l(\lambda),$$

which is exactly the required reality condition for c_l.

Uniqueness is immediate: if

$$\frac{i\, c_l}{\nu} = \frac{i\,\widetilde{c}_l}{\nu},$$

then

$$\frac{c_l - \widetilde{c}_l}{\nu} = 0$$

as a meromorphic function on Σ_a. Since $\nu \neq 0$ away from the finitely many branch points, it follows that

$$c_l(\lambda) = \widetilde{c}_l(\lambda)$$

on a dense open subset of $\mathbb{C}^\times$. As both sides are polynomials, we conclude that

$$c_l \equiv \widetilde{c}_l.$$

Thus c_l is uniquely determined. □

Lemma 5.11 (First Whitham equation)**.** *Let $a \in \mathcal{S}_1^1$, let (b_1, b_2) be the normalised basis of $\mathcal{B}_a$, and let*

$$t \mapsto (a_t, b_{1,t}, b_{2,t})$$

be a smooth period-preserving deformation. If c_l is defined by

$$\dot{q}_l = \frac{ic_l}{\nu}, \qquad l = 1, 2,$$

then

$$2a(\lambda)\dot{b}_l(\lambda) - \dot{a}(\lambda)b_l(\lambda) = 2i\lambda a(\lambda)c_l'(\lambda) - ic_l(\lambda)\big(a(\lambda) + \lambda a'(\lambda)\big), \qquad l = 1, 2. \tag{5.1}$$

Proof. We start from

$$dq_l = \Theta_l = \frac{b_l}{\nu}\frac{d\lambda}{\lambda}.$$

Differentiating with respect to t, we obtain

$$d\dot{q}_l = \partial_t\left(\frac{b_l}{\nu}\frac{d\lambda}{\lambda}\right) = \left(\frac{\dot{b}_l}{\nu} - \frac{b_l\dot{\nu}}{\nu^2}\right)\frac{d\lambda}{\lambda}.$$

Recall that $\nu^2 = \lambda a$, then differentiation with respect to t gives

$$2\nu\dot{\nu} = \lambda\dot{a}, \qquad \text{hence} \qquad \dot{\nu} = \frac{\lambda\dot{a}}{2\nu}.$$

Thus

$$d\dot{q}_l = \left(\frac{\dot{b}_l}{\nu} - \frac{b_l\lambda\dot{a}}{2\nu^3}\right)\frac{d\lambda}{\lambda}.$$

On the other hand, differentiating $\dot q_l$ with respect to λ implies

$$d\dot q_l = d\left(\frac{ic_l}{\nu}\right) = \frac{ic_l'}{\nu}\,d\lambda - \frac{ic_l}{\nu^2}\,d\nu.$$

Differentiating $\nu^2 = \lambda a$ with respect to λ, we obtain

$$2\nu\,d\nu = (a + \lambda a')\,d\lambda.$$

Hence

$$d\dot q_l = \left(\frac{ic_l'}{\nu} - \frac{ic_l(a+\lambda a')}{2\nu^3}\right)d\lambda.$$

Comparing the two expressions for $d\dot q_l$ and multiplying by $2\nu^3$, we find

$$2\nu^2\frac{\dot b_l}{\lambda} - b_l\dot a = 2i\nu^2 c_l' - ic_l(a+\lambda a').$$

Using once again $\nu^2 = \lambda a$, this becomes

$$2a\dot b_l - \dot a\,b_l = 2i\lambda a c_l' - ic_l(a+\lambda a'),$$

which is exactly (5.1). □

Lemma 5.12 (Compatibility equation)**.** *In the situation of Lemma 5.11, there exists a unique polynomial*

$$Q \in \mathbb{C}^2[\lambda]$$

such that

$$b_2(\lambda)c_1(\lambda) - b_1(\lambda)c_2(\lambda) = Q(\lambda)a(\lambda). \tag{5.2}$$

Moreover,

$$Q(1) = 0.$$

Proof. Consider the meromorphic differential

$$\dot q_1\Theta_2 - \dot q_2\Theta_1.$$

Substituting the definitions of $\dot q_l$ and Θ_l, we obtain

$$\dot q_1\Theta_2 - \dot q_2\Theta_1 = \frac{i(c_1b_2 - c_2b_1)}{\nu^2}\frac{d\lambda}{\lambda} = \frac{i(c_1b_2 - c_2b_1)}{\lambda a}\frac{d\lambda}{\lambda}.$$

This differential is globally defined on the spectral curve and may only have poles at the points over $\lambda = 0$ and $\lambda = \infty$. Therefore the numerator must vanish at every zero of a. Hence a divides $c_1b_2 - c_2b_1$, and there exists a polynomial Q such that

$$c_1b_2 - c_2b_1 = Qa.$$

Since $\deg b_l \le 2$, $\deg c_l \le 2$, and $\deg a = 2$, the quotient Q has degree at most 2. Uniqueness is immediate because $a \not\equiv 0$.

Finally, because $a \in \mathcal{S}_1^1$, we have $b_1(1) = 0 = b_2(1)$. Evaluating (5.2) at $\lambda = 1$ gives

$$Q(1)a(1) = 0.$$

Since $a(1) > 0$, it follows that

$$Q(1) = 0.$$

□

Lemma 5.13 (Jets at the Sym point)**.** *Let $a \in \mathcal{S}_1^1$ and let $\big(a_t, b_{1,t}, b_{2,t}\big)$ be a deformation through $\mathcal{S}_1^1$. Then*

$$\dot{b}_1(1) = 0 = \dot{b}_2(1),$$

and the associated polynomials c_1, c_2, Q satisfy

$$c_l'(1) = c_l(1)\left(\frac{1}{2} + \frac{a'(1)}{2a(1)}\right), \qquad l = 1, 2, \tag{5.3}$$

$$a(1)Q'(1) = b_2'(1)c_1(1) - b_1'(1)c_2(1), \tag{5.4}$$

and

$$\begin{aligned} a(1)Q''(1) = b_2''(1)c_1(1) + 2b_2'(1)c_1'(1) - b_1''(1)c_2(1) - 2b_1'(1)c_2'(1) \\ - 2a'(1)Q'(1). \end{aligned} \tag{5.5}$$

In particular, the two values $c_1(1)$ and $c_2(1)$ determine Q uniquely.

Proof. Because the deformation stays inside $\mathcal{S}_1^1$, the identities

$$b_{1,t}(1) = 0 = b_{2,t}(1)$$

hold for all t. Differentiating with respect to t yields

$$\dot{b}_1(1) = 0 = \dot{b}_2(1).$$

Now evaluate (5.1) at $\lambda = 1$. Since $b_l(1) = 0$ and $\dot{b}_l(1) = 0$, we obtain

$$0 = 2ia(1)c_l'(1) - ic_l(1)\big(a(1) + a'(1)\big),$$

which is equivalent to (5.3).

Next differentiate (5.2). Since $Q(1) = 0$, evaluation at $\lambda = 1$ gives (5.4). Differentiating once more and evaluating again at $\lambda = 1$ yields (5.5).

Finally, a quadratic polynomial with $Q(1) = 0$ is uniquely determined by $Q'(1)$ and $Q''(1)$. Hence Q is uniquely determined by $c_1(1)$ and $c_2(1)$. □

Lemma 5.14 (Bézout identity for polynomials)**.** *Let $p, q \in \mathbb{C}[\lambda]$, not both zero. Then the following are equivalent:*

1. $\gcd(p, q) = 1$,
2. *there exist $u, v \in \mathbb{C}[\lambda]$ such that*

$$up + vq = 1.$$

Proof. If $up + vq = 1$, then every common divisor of p and q divides 1, hence is constant. Therefore p and q are coprime.

Conversely, let

$$I := \{up + vq \mid u, v \in \mathbb{C}[\lambda]\}.$$

Since $\mathbb{C}[\lambda]$ is a principal ideal domain, there exists $d \in \mathbb{C}[\lambda]$ such that

$$I = (d).$$

Then d is a greatest common divisor of p and q, up to multiplication by a non-zero constant. If $\gcd(p, q) = 1$, then d is constant, and after normalisation we may assume $d = 1$. Hence there exist $u, v \in \mathbb{C}[\lambda]$ such that

$$up + vq = 1.$$

□

Finally we can state the genus–one version of the local Whitham existence and uniqueness theorem.

Theorem 5.15. *Let $a \in \mathcal{S}_1^1$, and let (b_1, b_2) be the normalised basis of $\mathcal{B}_a$. Assume that the greatest common divisor of b_1 and b_2 has degree 1, so that their only common zero is the Sym point $\lambda = 1$. Then for every prescribed pair of real values $\gamma_1, \gamma_2 \in \mathbb{R}$ there exist unique polynomials*

$$c_1, c_2 \in \mathbb{C}^2[\lambda], \qquad Q \in \mathbb{C}^2[\lambda], \qquad \dot{a} \in \mathbb{C}^2[\lambda], \qquad \dot{b}_1, \dot{b}_2 \in \mathbb{C}^2[\lambda]$$

satisfying the reality conditions, the values

$$c_1(1) = \gamma_1, \qquad c_2(1) = \gamma_2,$$

and the Whitham equations

$$2a\dot{b}_l - \dot{a}\, b_l = 2i\lambda a c_l' - i c_l(a + \lambda a'), \qquad l = 1, 2, \tag{5.6}$$

$$b_2 c_1 - b_1 c_2 = Qa. \tag{5.7}$$

Equivalently, the infinitesimal Whitham deformation is uniquely determined by the two values $c_1(1)$ and $c_2(1)$.

Proof. By Lemma 5.13, the prescribed values

$$c_1(1) = \gamma_1, \qquad c_2(1) = \gamma_2$$

determine $c_1'(1)$ and $c_2'(1)$ through (5.3). The same lemma then determines $Q'(1)$ and $Q''(1)$. Since $Q(1) = 0$, the polynomial Q is uniquely determined.

Because $\deg \gcd(b_1, b_2) = 1$, and their zeros coincide at $\lambda = 1$, the common divisor is $\lambda - 1$. Set

$$\widetilde{b}_1 := \frac{b_1}{\lambda - 1}, \qquad \widetilde{b}_2 := \frac{b_2}{\lambda - 1}.$$

Then $\widetilde{b}_1$ and $\widetilde{b}_2$ are coprime. Since $Q(1) = 0$, the polynomial

$$R := \frac{Qa}{\lambda - 1}$$

is well-defined, and (5.7) is equivalent to

$$\widetilde{b}_2 c_1 - \widetilde{b}_1 c_2 = R.$$

By Bézout's identity there exist $u, v \in \mathbb{C}[\lambda]$ such that

$$\widetilde{b}_2 u - \widetilde{b}_1 v = 1.$$

Hence, multiplying both sides by R gives

$$c_1^0 := uR, \qquad c_2^0 := vR$$

as one polynomial solution. Every other polynomial solution is of the form

$$c_1 = c_1^0 + p\widetilde{b}_1, \qquad c_2 = c_2^0 + p\widetilde{b}_2,$$

for a polynomial p. Indeed, let (c_1, c_2) be any other polynomial solution of

$$\widetilde{b}_2 c_1 - \widetilde{b}_1 c_2 = R.$$

Subtracting the equation for (c_1^0, c_2^0), we obtain

$$\widetilde{b}_2(c_1 - c_1^0) - \widetilde{b}_1(c_2 - c_2^0) = 0 \Leftrightarrow \widetilde{b}_2(c_1 - c_1^0) = \widetilde{b}_1(c_2 - c_2^0).$$

Since $\widetilde{b}_1$ and $\widetilde{b}_2$ are coprime, it follows that $\widetilde{b}_1$ divides $c_1 - c_1^0$ and $\widetilde{b}_2$ divides $c_2 - c_2^0$. Hence there exists a polynomial p such that

$$c_1 - c_1^0 = p\widetilde{b}_1, \qquad c_2 - c_2^0 = p\widetilde{b}_2.$$

Conversely, every pair of this form satisfies

$$\widetilde{b}_2 c_1 - \widetilde{b}_1 c_2 = \widetilde{b}_2(c_1^0 + p\widetilde{b}_1)\widetilde{b}_1(c_2^0 + p\widetilde{b}_2) = \widetilde{b}_2 c_1^0 - \widetilde{b}_1 c_2^0 = R,$$

since the additional terms cancel identically. Therefore these are exactly all polynomial solutions. Since $\deg c_l \leq 2$ and $\deg \widetilde{b}_l = 1$, it follows that p has degree at most 1. Thus the freedom in (c_1, c_2) is a two-real-dimensional affine space.

There exists $l \in \{1, 2\}$ such that

$$b_l'(1) \neq 0,$$

since otherwise both b_1 and b_2 would vanish to order at least 2 at 1, contradicting $\deg \gcd(b_1, b_2) = 1$. The two real conditions

$$c_l(1) = \gamma_l, \qquad c_l'(1) = \gamma_l \left(\frac{1}{2} + \frac{a'(1)}{2a(1)} \right)$$

therefore determine the linear polynomial p uniquely. Thus c_1 and c_2 are uniquely determined.

Once c_1 and c_2 are known, define

$$F_l := 2i\lambda a c_l' - i c_l (a + \lambda a'), \qquad l = 1, 2.$$

Then (5.6) becomes

$$2a\dot{b}_l - \dot{a}\, b_l = F_l.$$

Let α_1, α_2 be the two simple roots of a. Since b_1 and b_2 have no common root away from $\lambda = 1$, for each α_j there exists an index $l(j) \in \{1, 2\}$ such that

$$b_{l(j)}(\alpha_j) \neq 0.$$

Evaluating the corresponding equation at $\lambda = \alpha_j$, we obtain

$$-\dot{a}(\alpha_j) b_{l(j)}(\alpha_j) = F_{l(j)}(\alpha_j),$$

hence

$$\dot{a}(\alpha_j) = -\frac{F_{l(j)}(\alpha_j)}{b_{l(j)}(\alpha_j)}.$$

Thus the values of $\dot{a}$ at the two roots of a are uniquely determined.

Since $a(0) = 1$ for all t, every infinitesimal variation satisfies

$$\dot{a}(0) = 0.$$

A polynomial of degree at most 2 is uniquely determined by its values at three distinct points. Therefore the values at α_1, α_2, and 0 determine $\dot{a}$ uniquely.

Finally, once $\dot{a}$ is known, the equations

$$2a\dot{b}_l = F_l + \dot{a}\, b_l$$

determine $\dot{b}_l$ uniquely. The right-hand side vanishes at the roots of a by construction, hence it is divisible by a, and division by $2a$ yields a polynomial $\dot{b}_l \in \mathbb{C}^2[\lambda]$.

This proves existence. Uniqueness follows from the same steps in reverse: Q is unique, then (c_1, c_2), then $\dot{a}$, and finally $(\dot{b}_1, \dot{b}_2)$. □

Corollary 5.16. *At every point $a \in \mathcal{S}_1^1$ for which the common divisor of b_1 and b_2 has degree 1, the space of infinitesimal Whitham deformations is two-dimensional over $\mathbb{R}$.*

Proof. By Theorem 5.15, every infinitesimal Whitham deformation is uniquely determined by the two real numbers

$$c_1(1), \qquad c_2(1),$$

and every prescribed pair occurs. Hence the space of infinitesimal solutions is naturally identified with $\mathbb{R}^2$. □

5.2 Reduction of the boundary case $a \in M_2^2$ to a genus–one Whitham problem

In this section we fix $a \in \mathcal{M}_2^2$. By Remark 3.5, every such polynomial admits a parametrisation in terms of an angle $\varphi \in [0, \pi)$ and a radial parameter $r \in (0, 1)$. With the sign convention used below, we may therefore write

$$a(\lambda) = (\lambda - re^{-2i\varphi})(\lambda - r^{-1}e^{-2i\varphi})(\lambda - e^{2i\varphi})^2. \tag{5.8}$$

Thus a has a double root on the unit circle at $\lambda_0 = e^{2i\varphi}$, while the remaining two roots form the inverse-conjugate pair $\lambda_1 = re^{-2i\varphi}$ and $\bar{\lambda}_1^{-1} = r^{-1}e^{-2i\varphi}$.

We rotate the spectral parameter and set

$$\hat{\lambda} := -e^{-2i\varphi}\lambda. \tag{5.9}$$

Under this change of variable, the double root $\lambda = e^{2i\varphi}$ is moved to $\hat{\lambda} = -1$, and the remaining two roots are transformed to $-r$ and $-r^{-1}$. After passing to the normalisation and removing the double factor corresponding to the singular point, one obtains the reduced determinant polynomial

$$\hat{a}(\hat{\lambda}) = (\hat{\lambda} + r)(\hat{\lambda} + r^{-1}) = \hat{\lambda}^2 + \alpha\hat{\lambda} + 1, \qquad \alpha = r + r^{-1} > 2. \tag{5.10}$$

The corresponding normalised curve is therefore

$$\hat{\Sigma} = \left\{ (\hat{\lambda}, \hat{\nu}) \in \mathbb{C}^\times \times \mathbb{C} \;\middle|\; \hat{\nu}^2 = \hat{\lambda}\,\hat{a}(\hat{\lambda}) \right\}.$$

For notational simplicity, we now drop the hat on the spectral parameter and write again λ instead of $\hat{\lambda}$. Thus in the remainder of this section the reduced determinant polynomial is

$$\hat{a}(\lambda) = \lambda^2 + \alpha\lambda + 1, \qquad \alpha > 2. \tag{5.11}$$

After the reduction to genus–one, the real two-dimensional space of admissible logarithmic differentials is generated by polynomials b satisfying the reality condition. In the boundary situation under consideration, this space admits a distinguished decomposition

$$b(\lambda) = y(\lambda^2 + 1) + x\lambda - iz(\lambda^2 - 1), \qquad x, y, z \in \mathbb{R},$$

into a real part

$$y(\lambda^2 + 1) + x\lambda$$

and a purely imaginary part

$$-iz(\lambda^2 - 1).$$

Accordingly, one chooses b_1 in the purely imaginary direction and b_2 in the complementary real direction:

$$b_1(\lambda) = i\gamma(\lambda^2 - 1), \qquad b_2(\lambda) = \gamma(\lambda^2 + \beta\lambda + 1).$$

Remark 5.17. *Note that the appearance of the same factor γ in b_1 and b_2 is a normalization convention, not a consequence of the reality condition. Since b_1 and b_2 form a distinguished ordered basis of a two-dimensional real space, one may choose them with a common overall scale. This is convenient because the associated coordinates q_1 and q_2 are later combined into the conformal parameter $\tau = q_2 + iq_1$. Thus the common factor γ ensures that both directions are expressed in the same unit.*

The zeros of b_2 are determined by the vanishing A–period condition. As in the generic genus–one case, the reality condition implies that the zeros are reciprocal to one another.

Lemma 5.18. *There exists a unique number s with $r < s < 1$ such that*

$$b_2(\lambda) = \gamma(\lambda + s)(\lambda + s^{-1}).$$

Equivalently,

$$b_2(\lambda) = \gamma(\lambda^2 + \beta\lambda + 1), \qquad \beta = s + s^{-1},$$

and the zeros satisfy the ordering

$$-r^{-1} < -s^{-1} < -1 < -s < -r < 0.$$

In particular $2 < \beta < \alpha$.

Proof. Since the A-cycle projects to the interval $[-r^{-1}, -r]$, the condition that

$$\int_A \Theta_{b_2} = 0$$

is equivalent to

$$\int_{-r^{-1}}^{-r} \frac{b_2(\lambda)}{\hat{\nu}(\lambda)} \frac{d\lambda}{\lambda} = 0.$$

On $(-r^{-1}, -r)$ we have $\lambda < 0$ and $\hat{\nu}(\lambda) \in \mathbb{R} \setminus \{0\}$, so after fixing the sign of $\hat{\nu}$ on this interval, the weight

$$\frac{1}{\lambda\hat{\nu}(\lambda)}$$

has constant sign. Since $b_2 \not\equiv 0$, the vanishing of the integral therefore implies that b_2 changes sign on $(-r^{-1}, -r)$. Hence b_2 has a zero in $(-r^{-1}, -r)$.

By the reality condition, the zeros of b_2 occur in reciprocal pairs. Therefore the second zero is the reciprocal of the first one, and since b_2 changes sign on $(-r^{-1}, -r)$, the two zeros are distinct and lie on opposite sides of -1. Thus there exists a unique $s \in (r, 1)$ such that

$$-r^{-1} < -s^{-1} < -1 < -s < -r < 0.$$

Hence

$$b_2(\lambda) = \gamma(\lambda + s)(\lambda + s^{-1}) = \gamma(\lambda^2 + \beta\lambda + 1), \qquad \beta = s + s^{-1}.$$

Finally, since the function $x \mapsto x + x^{-1}$ is strictly decreasing on $(0, 1)$, the inequality $r < s < 1$ yields

$$\beta = s + s^{-1} < r + r^{-1} = \alpha.$$

Moreover, $s \in (0, 1)$ implies $\beta > 2$. Therefore

$$2 < \beta < \alpha.$$

This proves the claim. □

By the preceding Lemma, β is uniquely fixed by the requirement that Θ_{b_2} have vanishing A-period.

Once β is fixed, the differential Θ_{b_2} is determined up to multiplication by a positive real constant. We remove this remaining freedom by the normalisation

$$\int_{\mathbb{S}^1} \Theta_{b_2} = 2. \tag{5.12}$$

This determines $\gamma > 0$ uniquely. We then use the same factor γ also in $b_1(\lambda) = i\gamma(\lambda^2 - 1)$, so that (b_1, b_2) becomes a distinguished ordered basis written in a common scale.

At this point it is important to distinguish between the periods of the logarithmic differentials and the values of their primitives at the distinguished point over $\lambda = 1$. By definition, a Whitham deformation preserves the periods of Θ_{b_1} and Θ_{b_2}, and therefore preserves this normalisation. The conformal class, however, is encoded by the values of the corresponding primitives on the real part of the reduced curve.

Let $\hat{\Sigma}_R := \mathrm{Fix}(\rho) \subset \hat{\Sigma}$ denote the fixed point set of the anti-holomorphic involution ρ on the reduced curve. Note that its projection to the λ-plane is the unit circle $\mathbb{S}^1$. Let y be the point over $\lambda = 1$ for which $\hat{\nu} > 0$. For $j = 1, 2$, let

$$\widetilde{q}_j : \hat{\Sigma}_R \to i\mathbb{R}$$

be the unique primitive satisfying

$$d\widetilde{q}_j = \Theta_{b_j}\big|_{\hat{\Sigma}_R}, \qquad \sigma^*\widetilde{q}_j = -\widetilde{q}_j.$$

We then define

$$q_j := -i\,\widetilde{q}_j(y) \in \mathbb{R}, \qquad j = 1, 2.$$

These are the two real parameters singled out by the chosen ordered basis.

Remark 5.19. *Recall that in Section 5.1 we fixed the distinguished Sym point of the unreduced spectral data at $\lambda = 1$ by rotating the original spectral parameter. In the boundary reduction $a \in M_2^2$, we then pass to the reduced coordinate*

$$\hat{\lambda} = -e^{-2i\varphi}\lambda.$$

Hence the distinguished point $\lambda = 1$ is mapped to $\hat{\lambda} = -e^{-2i\varphi}$. To analyse how the evaluation point varies with φ, we therefore consider the point

$$y(r, \varphi) \in \hat{\Sigma}_R$$

lying over

$$\hat{\lambda} = -e^{-2i\varphi},$$

chosen by the condition $\hat{\nu} > 0$. Why this matters becomes more clear in chapter 6.

The normalisation above gives these coordinates a definite meaning. Since b_1 is chosen as the distinguished imaginary generator with $\gamma > 0$, the corresponding coordinate is taken as the imaginary part. The condition (5.12) shows that changing the integration path by one turn around $\mathbb{S}^1$ changes q_2 by an even integer. Hence q_2 is naturally defined modulo $2\mathbb{Z}$, and there is a unique representative satisfying

$$-1 \leq q_2 \leq 1.$$

We therefore define the conformal parameter by

$$\tau := q_2 + iq_1.$$

With the chosen orientation of the imaginary direction we have $q_1 > 0$, hence $\Im(\tau) > 0$. As imposed in Chapter 4, we can always choose a representative, such that

$$|\tau| \geq 1,$$

we can therefore choose a representative such that

$$\tau \in \mathcal{F}_3 := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0,\ -1 \leq \Re(\tau) \leq 1,\ |\tau| \geq 1\}.$$

Thus the conformal class is not introduced artificially. It is exactly the complex number obtained from the two real coordinates q_1 and q_2 determined by the normalised ordered basis. We will use the identification $\tau \mapsto \tau + 2$ to identify the boundary strips with each other and $\tau \mapsto -\tau^{-1}$ to identify the two quarter circles, where $\Re(\tau) < 0$ and $\Re(\tau) > 0$ respectively, with each other. Lastly we equip the space with the quotient topology.

The reality type of this basis is reflected by the quotient

$$\frac{b_2}{b_1}(e^{i\theta}) = \frac{\lambda^2 + \beta\lambda + 1}{i(\lambda^2 - 1)} = -\frac{2\cos\theta + \beta}{2\sin\theta} \in \mathbb{R} \cup \{\infty\}, \qquad \lambda = e^{i\theta},$$

The reason for considering this quotient is that it compares the two generators pointwise along the real part of the reduced curve, whose projection is the unit circle. Since the quotient is real for $\lambda \in \mathbb{S}^1$, the two differentials differ there only by a real scalar factor. Thus the chosen basis is adapted to the real structure and does not mix the distinguished real and imaginary directions by a complex phase.

We want to summarize the above results.

Corollary 5.20. *After reduction to genus–one, the boundary data $a \in M_2^2$ determine a reduced determinant polynomial*

$$\hat{a}(\lambda) = \lambda^2 + \alpha\lambda + 1, \qquad \alpha = r + r^{-1} > 2,$$

and a distinguished ordered pair of logarithmic differentials

$$\Theta_{b_1} = \frac{b_1(\lambda)}{\hat{\nu}}\frac{d\lambda}{\lambda}, \qquad \Theta_{b_2} = \frac{b_2(\lambda)}{\hat{\nu}}\frac{d\lambda}{\lambda},$$

where

$$b_1(\lambda) = i\gamma(\lambda^2 - 1), \qquad b_2(\lambda) = \gamma(\lambda^2 + \beta\lambda + 1),$$

with

$$\gamma > 0, \qquad 2 < \beta < \alpha.$$

The coefficient β is uniquely determined by the vanishing A–period condition, and γ is uniquely determined by the chosen normalisation of the imaginary period direction. The associated Sym–point coordinates $q_1, q_2 \in \mathbb{R}$ define a unique conformal class

$$\tau = q_2 + iq_1 \in \mathcal{F}_3.$$

Thus the reduced genus–one data determine a unique conformal class, and this is the quantity that will be studied in Chapter 6.

To close this chapter we briefly want to state the sign conventions used in this reduction.

Remark 5.21. *The sign convention used in the reduced boundary problem consists of two choices.*

First, in the rotated spectral parameter we set

$$\hat{\lambda} = -e^{-2i\varphi}\lambda$$

rather than $e^{-2i\varphi}\lambda$. With this convention, the double root $\lambda = e^{2i\varphi}$ is moved to $\hat{\lambda} = -1$, and the remaining two roots are placed on the negative real axis. After removing the double factor, the reduced determinant polynomial therefore takes the standard form

$$\hat{a}(\hat{\lambda}) = (\hat{\lambda} + r)(\hat{\lambda} + r^{-1}) = \hat{\lambda}^2 + \alpha\hat{\lambda} + 1, \qquad \alpha > 2.$$

Second, the distinguished imaginary direction is fixed by choosing

$$b_1(\lambda) = i\gamma(\lambda^2 - 1), \qquad \gamma > 0.$$

Since replacing b_1 by $-b_1$ would only reverse the corresponding direction, the condition $\gamma > 0$ fixes its orientation. The complementary real generator b_2 is then written with the same positive scale,

$$b_2(\lambda) = \gamma(\lambda^2 + \beta\lambda + 1),$$

so that (b_1, b_2) becomes a distinguished ordered basis. With this convention, the coordinate determined by b_1 is taken as the imaginary part and the one determined by b_2 as the real part of the conformal parameter $\tau = q_2 + iq_1$.

5.3 Solutions of the Whitham Equations for $a \in \mathcal{M}_2^2$

We now solve the genus–one Whitham equations for the boundary case $a \in \mathcal{M}_2^2$. As explained in the previous section, after reduction to genus–one the determinant polynomial has the form

$$a(\lambda) = \lambda^2 + \alpha\lambda + 1, \qquad \alpha = r + r^{-1} > 2. \tag{5.13}$$

Moreover, we have chosen the basis of logarithmic differentials in the form

$$b_1(\lambda) = i\gamma(\lambda^2 - 1), \qquad b_2(\lambda) = \gamma(\lambda^2 + \beta\lambda + 1), \tag{5.14}$$

where

$$\alpha = \alpha(t), \qquad \beta = \beta(t), \qquad \gamma = \gamma(t)$$

depend smoothly on the Whitham parameter t.

Our goal is to determine the polynomials c_1 and c_2 from the compatibility equation and then to derive the differential equations for α, β, and γ from the first Whitham equation.

Recall from Lemma 5.12 that the Whitham equations contain a polynomial $Q \in \mathbb{C}^2[\lambda]$ satisfying equation 5.7. At this stage Q is determined only up to an overall non-zero real factor, because a rescaling of the deformation parameter t rescales the corresponding Whitham vector field. Thus one may normalise Q conveniently.

In the present genus-one boundary problem we choose

$$Q(\lambda) = \lambda. \tag{5.15}$$

This fixes the scale of the Whitham parameter t. No geometric information is lost, since a different non-zero multiple of λ would merely correspond to a reparametrisation of the same flow.

We now solve the compatibility equation

$$b_2 c_1 - b_1 c_2 = \lambda a. \tag{5.16}$$

Substituting (5.13) and (5.14), we obtain

$$\gamma(\lambda^2 + \beta\lambda + 1)c_1(\lambda) - i\gamma(\lambda^2 - 1)c_2(\lambda) = \lambda(\lambda^2 + \alpha\lambda + 1). \tag{5.17}$$

The polynomials c_1 and c_2 have degree at most 2 and satisfy the same reality condition as b_1 and b_2. As in the previous section, this suggests the equivalent ansatz

$$c_1(\lambda) = u(\lambda^2 + 1) + v\lambda, \qquad c_2(\lambda) = iw(\lambda^2 - 1), \tag{5.18}$$

with real parameters $u, v, w \in \mathbb{R}$.

We insert (5.18) into (5.17). A direct expansion yields

$$\begin{aligned} \gamma(\lambda^2 + \beta\lambda + 1)\big(u(\lambda^2 + 1) + v\lambda\big) + \gamma(\lambda^2 - 1)w(\lambda^2 - 1) \\ = \lambda(\lambda^2 + \alpha\lambda + 1). \end{aligned}$$

Expanding both sides and collecting coefficients of powers of λ, we obtain

$$\begin{aligned} 0 = \gamma(u + w)\lambda^4 + \gamma(\beta u + v)\lambda^3 + \gamma(2u + \beta v - 2w)\lambda^2 + \gamma(\beta u + v)\lambda + \gamma(u + w) \\ - \lambda^3 - \alpha\lambda^2 - \lambda. \end{aligned}$$

Hence the coefficients must satisfy the system

$$u + w = 0, \tag{5.19}$$

$$\gamma(\beta u + v) = 1, \tag{5.20}$$

$$\gamma(2u + \beta v - 2w) = \alpha. \tag{5.21}$$

From (5.19) we obtain $w = -u$. Substituting this into (5.21), we get

$$\gamma(4u + \beta v) = \alpha.$$

Together with (5.20), this is a linear system in u and v. Solving it yields

$$u = \frac{\beta - \alpha}{\gamma(\beta^2 - 4)}, \qquad v = \frac{\alpha\beta - 4}{\gamma(\beta^2 - 4)}, \qquad w = \frac{\alpha - \beta}{\gamma(\beta^2 - 4)}.$$

Therefore

$$c_1(\lambda) = \frac{\beta - \alpha}{\gamma(\beta^2 - 4)}(\lambda^2 + 1) + \frac{\alpha\beta - 4}{\gamma(\beta^2 - 4)}\,\lambda, \tag{5.22}$$

and

$$c_2(\lambda) = i\,\frac{\alpha - \beta}{\gamma(\beta^2 - 4)}(\lambda^2 - 1). \tag{5.23}$$

Thus the compatibility equation uniquely determines c_1 and c_2 for the chosen normalisation $Q = \lambda$.

We now substitute the polynomials a, b_1, b_2, c_1, and c_2 into the first Whitham equation

$$2a\dot{b}_l - \dot{a}\,b_l = 2i\lambda a c_l' - ic_l(a + \lambda a'), \qquad l = 1, 2. \tag{5.24}$$

Since the parameters α, β, γ depend on t, we have

$$\dot{a}(\lambda) = \dot{\alpha}\,\lambda, \qquad \dot{b}_1(\lambda) = i\dot{\gamma}(\lambda^2 - 1), \qquad \dot{b}_2(\lambda) = \dot{\gamma}(\lambda^2 + \beta\lambda + 1) + \gamma\dot{\beta}\,\lambda. \tag{5.25}$$

We proceed in two steps.

First we look at the equations obtained from $l = 1$. From (5.22) we compute

$$c_1'(\lambda) = 2\frac{\beta - \alpha}{\gamma(\beta^2 - 4)}\,\lambda + \frac{\alpha\beta - 4}{\gamma(\beta^2 - 4)}.$$

Moreover,

$$a(\lambda) + \lambda a'(\lambda) = (\lambda^2 + \alpha\lambda + 1) + \lambda(2\lambda + \alpha) = 3\lambda^2 + 2\alpha\lambda + 1.$$

Substituting these expressions together with (5.25) into (5.24) for $l = 1$, we obtain a polynomial identity in λ. After expansion

$$i\Big(2\dot{\gamma}\,\lambda^4 + (2\alpha\dot{\gamma} - \gamma\dot{\alpha})\lambda^3 + (\gamma\dot{\alpha} - 2\alpha\dot{\gamma})\lambda - 2\dot{\gamma}\Big) =$$
$$\frac{i}{\gamma(\beta^2 - 4)}\Big((\beta - \alpha)\lambda^4 + (-2\alpha^2 + \alpha\beta + 4)\lambda^3 + (2\alpha^2 - \alpha\beta - 4)\lambda + (\alpha - \beta)\Big).$$

The coefficient of λ^4 gives

$$2\gamma(\beta^2 - 4)\dot{\gamma} + \alpha - \beta = 0.$$

Hence

$$\dot{\gamma} = \frac{\beta - \alpha}{2\gamma(\beta^2 - 4)}. \tag{5.26}$$

Next, the coefficient of λ^3 yields

$$\gamma^2(\beta^2 - 4)\dot{\alpha} - \alpha^2 + 4 = 0,$$

and therefore

$$\dot{\alpha} = \frac{\alpha^2 - 4}{\gamma^2(\beta^2 - 4)}. \tag{5.27}$$

Thus the equation for $l = 1$ already determines $\dot{\gamma}$ and $\dot{\alpha}$.

Secondly, we use the first Whitham equation for $l = 2$. From (5.23) we have

$$c_2'(\lambda) = 2i\,\frac{\alpha - \beta}{\gamma(\beta^2 - 4)}\,\lambda.$$

Substituting this together with (5.25), (5.26), and (5.27) into (5.24) for $l = 2$, we again obtain a polynomial identity in λ, in particular.

$$\frac{1}{\gamma(\beta^2-4)}\Big((\beta-\alpha)\lambda^4 + \big(-2\alpha^2 + 2\gamma^2(\beta^2-4)\dot{\beta} + \beta^2 + 4\big)\lambda^3 +$$
$$\big(-2\alpha^2\beta + \alpha\beta^2 + 2\alpha\gamma^2(\beta^2-4)\dot{\beta} - 2\alpha + 6\beta\big)\lambda^2 + \big(-2\alpha^2 + 2\gamma^2(\beta^2-4)\dot{\beta} + \beta^2 + 4\big)\lambda + (\beta-\alpha)\Big) =$$
$$\frac{1}{\gamma(\beta^2-4)}\Big((\beta-\alpha)\lambda^4 + (2\alpha\beta - 2\alpha^2)\lambda^3 + (6\beta - 6\alpha)\lambda^2 + (2\alpha\beta - 2\alpha^2)\lambda + (\beta-\alpha)\Big).$$

Comparing the coefficient of λ^3, we find

$$2\gamma^2(\beta^2-4)\dot{\beta} - 2\alpha\beta + \beta^2 + 4 = 0.$$

Hence

$$\dot{\beta} = \frac{2\alpha\beta - \beta^2 - 4}{2\gamma^2(\beta^2-4)}. \tag{5.28}$$

This determines the remaining evolution equation.

We have therefore proved the following result.

Corollary 5.22. *Let*

$$a(\lambda) = \lambda^2 + \alpha\lambda + 1, \qquad b_1(\lambda) = i\gamma(\lambda^2 - 1), \qquad b_2(\lambda) = \gamma(\lambda^2 + \beta\lambda + 1),$$

and normalise the compatibility equation by choosing

$$Q(\lambda) = \lambda.$$

Then the genus–one Whitham equations admit the unique solution

$$c_1(\lambda) = \frac{\beta-\alpha}{\gamma(\beta^2-4)}(\lambda^2+1) + \frac{\alpha\beta-4}{\gamma(\beta^2-4)}\lambda,$$

$$c_2(\lambda) = i\,\frac{\alpha-\beta}{\gamma(\beta^2-4)}(\lambda^2-1).$$

Moreover, the corresponding evolution of the parameters α, β, and γ is governed by the system

$$\dot{\alpha} = \frac{\alpha^2-4}{\gamma^2(\beta^2-4)}, \qquad \dot{\beta} = \frac{2\alpha\beta-\beta^2-4}{2\gamma^2(\beta^2-4)}, \qquad \dot{\gamma} = \frac{\beta-\alpha}{2\gamma(\beta^2-4)}. \tag{5.29}$$

Proof. The formulas for c_1 and c_2 follow from the coefficient comparison in the compatibility equation (5.17). The evolution equations (5.29) are obtained by substituting these expressions into the first Whitham equation and comparing coefficients, as carried out above. □

Remark 5.23. *The system (5.29) will be the starting point for the further analysis of the Whitham flow on the boundary stratum $\mathcal{M}_2^2$. In particular, it describes the deformation of the reduced genus–one spectral data, and hence the induced flow through the corresponding equivariant CMC tori.*

5.4 The limiting stratum $a \in M_2^3$

We finally consider the limiting case $r \to 1$, which corresponds to the boundary stratum $a \in M_2^3$. By the discussion of the previous subsection, for $a \in M_2^2$ the reduced data determine a conformal class

$$\tau(r,\varphi) = q_2(r,\varphi) + iq_1(r,\varphi) \in \mathcal{F}_3.$$

Our aim is to identify the limiting value of this conformal class when $r = 1$.

By Lemma 5.18, we know that $r < s < 1$ and therefore as $r \to 1$

$$\alpha \to 2, \qquad \beta \to 2.$$

Therefore, on the limiting stratum M_2^3, the reduced data take the form

$$a(\lambda) = (\lambda+1)^2, \qquad b_1(\lambda) = i\gamma(\lambda^2-1), \qquad b_2(\lambda) = \gamma(\lambda+1)^2.$$

The reduced curve is now singular,

$$\hat{\Sigma}_0 = \{(\lambda, \hat{\nu}) \in \mathbb{C}^\times \times \mathbb{C} \mid \hat{\nu}^2 = \lambda(\lambda+1)^2\},$$

and its normalisation is rational. We introduce a normalisation parameter $u \in \mathbb{P}^1$ by

$$\lambda = -u^2, \qquad \hat{\nu} = i\,u(1-u^2).$$

Then indeed

$$\hat{\nu}^2 = -u^2(1-u^2)^2 = \lambda(\lambda+1)^2.$$

The evaluation point on the reduced curve is the point lying over $\lambda = -e^{-2i\varphi}$, hence in the normalisation it is represented by

$$u = e^{-i\varphi}, \qquad 0 < \varphi < \pi.$$

We now compute the limiting primitives explicitly.

Proposition 5.24. *On the limiting stratum M_2^3, the conformal class is given by*

$$\tau(\varphi) = e^{i\varphi}, \qquad 0 < \varphi < \pi.$$

Equivalently, after the reparametrisation

$$x = \tan\Big(\frac{\varphi}{2} - \frac{\pi}{4}\Big) \in (-1,1), \qquad x = \tanh t,$$

one obtains

$$\tau(t) = \frac{i - \tanh t}{1 - i\tanh t}, \qquad t \in \mathbb{R}.$$

Proof. We first compute the limiting differentials on the normalisation.

Using

$$\lambda = -u^2, \qquad \frac{d\lambda}{\lambda} = 2\,\frac{du}{u}, \qquad \hat{\nu} = i\,u(1-u^2),$$

we obtain

$$\Theta_{b_1} = \frac{i\gamma(\lambda^2-1)}{\hat{\nu}}\frac{d\lambda}{\lambda} = \frac{i\gamma(u^4-1)}{i\,u(1-u^2)}\,2\frac{du}{u} = -2\gamma\Big(1+u^{-2}\Big)\,du.$$

Hence

$$\Theta_{b_1} = d\big(-2\gamma(u-u^{-1})\big).$$

Similarly,

$$\Theta_{b_2} = \frac{\gamma(\lambda+1)^2}{\hat{\nu}}\frac{d\lambda}{\lambda} = \frac{\gamma(1-u^2)^2}{i\,u(1-u^2)}\,2\frac{du}{u} = -2i\gamma\Big(u^{-2}-1\Big)\,du,$$

and therefore

$$\Theta_{b_2} = d\big(2i\gamma(u+u^{-1})\big).$$

These primitives are odd under the involution $u \mapsto -u$, which corresponds to the hyperelliptic involution. Hence they are the primitives relevant for the definition of q_1 and q_2.

For $u = e^{-i\varphi}$, we obtain

$$-2\gamma(u-u^{-1}) = -2\gamma(e^{-i\varphi} - e^{i\varphi}) = 4i\gamma\sin\varphi,$$

and

$$2i\gamma(u + u^{-1}) = 2i\gamma(e^{-i\varphi} + e^{i\varphi}) = 4i\gamma\cos\varphi.$$

Thus

$$q_1(\varphi) = 4\gamma\sin\varphi, \qquad q_2(\varphi) = 4\gamma\cos\varphi,$$

and therefore

$$\tau(\varphi) = q_2(\varphi) + iq_1(\varphi) = 4\gamma e^{i\varphi}.$$

So what is left to do, is to calculate γ.

For $r < 1$, the conformal class lies in

$$\mathcal{F}_3 = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0,\ -1 \le \Re(\tau) \le 1,\ |\tau| \ge 1\}.$$

Hence, in the limit $r \to 1$, we must have

$$\tau(\varphi) \in \overline{\mathcal{F}_3} \qquad \text{for all } 0 < \varphi < \pi.$$

Since $|\tau(\varphi)| = 4\gamma$, the condition $|\tau| \ge 1$ yields $4\gamma \ge 1$. On the other hand,

$$\Re(\tau(\varphi)) = 4\gamma\cos\varphi.$$

As $\varphi \to 0$, this approaches 4γ, while the condition $-1 \le \Re(\tau) \le 1$ forces

$$4\gamma \le 1.$$

Therefore

$$4\gamma = 1, \qquad \text{hence} \qquad \gamma = \frac{1}{4}.$$

Substituting this into the formula for $\tau(\varphi)$, we obtain

$$\tau(\varphi) = e^{i\varphi}.$$

Lastly we rewrite this in the parameter t.

$$x = \tan\Big(\frac{\varphi}{2} - \frac{\pi}{4}\Big) \in (-1, 1).$$

Then the standard half-angle identities give

$$\cos\varphi = -\frac{2x}{1+x^2}, \qquad \sin\varphi = \frac{1-x^2}{1+x^2}.$$

Hence

$$e^{i\varphi} = \frac{-2x + i(1-x^2)}{1+x^2} = \frac{i-x}{1-ix}.$$

Since $\tanh : \mathbb{R} \to (-1, 1)$ is bijective, there exists a unique $t \in \mathbb{R}$ such that

$$x = \tanh t.$$

Thus

$$\tau(t) = \frac{i - \tanh t}{1 - i\tanh t}.$$

This proves the claim. □

We can now identify the image of the limiting stratum.

Proposition 5.25. *The conformal class map extends continuously from M_2^2 to M_2^3. Moreover,*

$$\tau(M_2^3) = \{\tau \in \mathbb{C} \mid |\tau| = 1,\ \Im(\tau) > 0\}.$$

In particular, the limiting stratum M_2^3 corresponds exactly to the unit-circle boundary of $\mathcal{F}_3$.

Proof. By Theorem 4.18, the lattice data depend continuously on the determinant polynomial a. In the boundary situation considered here, the normalization from Corollary 5.20 and Remark 5.21 fixes a distinguished ordered basis, so the associated quantities $q_1(a)$ and $q_2(a)$ vary continuously with a as well. Consequently,

$$\tau(a) = q_2(a) + i\, q_1(a)$$

extends continuously from $\mathcal{M}_2^2$ to $\mathcal{M}_2^3$.

By Proposition 5.24, for $0 < \varphi < \pi$ we have

$$\tau(\varphi) = e^{i\varphi}.$$

Therefore

$$|\tau(\varphi)| = 1, \qquad \Im(\tau(\varphi)) = \sin\varphi > 0.$$

Hence

$$\tau(M_2^3) \subset \{\tau \in \mathbb{C} \mid |\tau| = 1,\ \Im(\tau) > 0\}.$$

Conversely, every point of the upper semicircle may be written uniquely as $e^{i\varphi}$ with $0 < \varphi < \pi$. Therefore

$$\tau(M_2^3) = \{\tau \in \mathbb{C} \mid |\tau| = 1,\ \Im(\tau) > 0\}.$$

Finally,

$$\mathcal{F}_3 = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0,\ -1 \leq \Re(\tau) \leq 1,\ |\tau| \geq 1\},$$

so the set

$$\{\tau \in \mathbb{C} \mid |\tau| = 1,\ \Im(\tau) > 0\}$$

is exactly the unit-circle part of the boundary of $\mathcal{F}_3$. This proves the last statement. $\square$

Chapter 6

The map S and the conformal classes on the boundary

In this chapter we show that the boundary strata

$$\mathcal{M}_2^2 \cup \mathcal{M}_2^3$$

are parametrized by the conformal class of the corresponding reduced genus–one data. More precisely, we prove that the map

$$S : \mathcal{M}_2^2 \cup \mathcal{M}_2^3 \to \mathcal{F}_3, \qquad a \mapsto q_2(a) + iq_1(a)$$

is a bijection, where

$$\mathcal{F}_3 := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0,\ -1 \leq \Re(\tau) \leq 1,\ |\tau| \geq 1\}.$$

Here the quantities q_1 and q_2 are the imaginary and real part of the conformal class determined by the reduced genus–one spectral data.

The proof is organized as follows. First we explain why $\mathcal{F}_3$ is the correct fundamental domain in the boundary situation. Then we describe the conformal class attached to the reduced data and study, for fixed r, the corresponding curve in the τ–plane. We show that along such a curve the real part runs from -1 to 1, while the imaginary part attains its minimum at the boundary. After that we analyze the boundary values as $r \to 0$ and compare them with the limiting stratum $\mathcal{M}_2^3$, which gives the lower boundary of $\mathcal{F}_3$. Finally, we use the ordering of the curves and their non–intersection to conclude that every point of $\mathcal{F}_3$ is attained exactly once.

In Chapter 3 we represented lattices up to rotation and dilation by points in the standard fundamental domain

$$\mathcal{F} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0,\ |\Re(\tau)| \leq \tfrac{1}{2},\ |\tau| \geq 1\}.$$

This is the correct choice if one only wants to remember the abstract similarity class of a lattice. In the present boundary situation, however, we have more structure. After the reduction to genus–one we do not only obtain a lattice, but a distinguished ordered pair of period directions: one direction is singled out by the purely imaginary differential and the other by the real complementary differential. Thus the full modular ambiguity from Chapter 3 is no longer present. Instead we keep track of the decomposition into the imaginary and real period direction. This leads naturally to the larger fundamental domain

$$\mathcal{F}_3 = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0,\ -1 \leq \Re(\tau) \leq 1,\ |\tau| \geq 1\}.$$

The difference between $\mathcal{F}$ and $\mathcal{F}_3$ is therefore the following: the set $\mathcal{F}$ parametrizes lattices only up to arbitrary change of basis, whereas $\mathcal{F}_3$ is adapted to our distinguished basis coming from the reduced Whitham problem. In particular, the horizontal strip has width 2 instead of 1, because changing the real period direction by an even multiple of the imaginary one does not alter the chosen normalization, while the condition $|\tau| \geq 1$ still removes the remaining duplication coming from swapping the two directions.

For $a \in \mathcal{M}_2^2$, as in Chapter 5.2, we use the parametrization

$$a(\lambda) = \big(\lambda - re^{-2i\varphi}\big)\big(\lambda - r^{-1}e^{-2i\varphi}\big)(\lambda - e^{2i\varphi})^2, \qquad r \in (0,1),\ \varphi \in [0,\pi].$$

Hence on $\mathcal{M}_2^2$ the map S can be regarded as

$$S : (0,1) \times [0,\pi] \to \mathcal{F}_3, \qquad (r,\varphi) \mapsto S(r,\varphi).$$

The limiting case $r \to 1$ yields the stratum $\mathcal{M}_2^3$. By Proposition 5.25, the conformal class extends continuously to this limiting stratum, and its image is exactly the upper semicircle

$$S(\mathcal{M}_2^3) = \{\tau \in \mathbb{C} \mid |\tau| = 1,\ \Im(\tau) > 0\}.$$

Thus the stratum $\mathcal{M}_2^3$ already accounts for the circular part of the boundary of $\mathcal{F}_3$.

We next fix $r \in (0,1)$ and study the map

$$\varphi \mapsto S(r,\varphi) = q_2(r,\varphi) + iq_1(r,\varphi).$$

For fixed r, this defines a continuous curve in the τ–plane. Since $a(r,0) = a(r,\pi)$, the values $\varphi = 0$ and $\varphi = \pi$ do not define two different points of $\mathcal{M}_2^2$. Rather, they correspond to the two limiting orientations of the distinguished real period direction. In the normalization $\tau = q_2 + iq_1 \in \mathcal{F}_3$, these two limiting configurations are represented by the two vertical boundary lines of $\mathcal{F}_3$. Hence

$$\lim_{\varphi \to 0^+} q_2(r,\varphi) = -1, \qquad \lim_{\varphi \to \pi^-} q_2(r,\varphi) = 1.$$

Lemma 6.1. *For every fixed $r \in (0,1)$, the function $\varphi \mapsto q_2(r,\varphi)$ is strictly increasing.*

Proof. For the reduced genus–one data, the φ–dependence enters through the point

$$\hat\lambda_\varphi := -e^{-2i\varphi}, \qquad \varphi \in (0,\pi).$$

As φ varies from 0 to π, the point $\hat\lambda_\varphi$ runs once around the unit circle. Moreover,

$$\frac{d\hat\lambda_\varphi}{d\varphi} = -2i\,\hat\lambda_\varphi,$$

and therefore

$$\frac{d\hat\lambda_\varphi}{d\varphi} \neq 0 \qquad \text{for all } \varphi \in (0,\pi).$$

By definition,

$$dq_2 = \Theta_{b_2} = \frac{b_2(\hat\lambda)}{\hat\nu}\,\frac{d\hat\lambda}{\hat\lambda}.$$

Hence, along the curve $\varphi \mapsto (\hat\lambda_\varphi, \hat\nu_\varphi)$ on the spectral curve, we obtain

$$\frac{d}{d\varphi} q_2(r,\varphi) = \Theta_{b_2}\left(\frac{d}{d\varphi}\right) = \frac{b_2(\hat\lambda_\varphi)}{\hat\nu_\varphi}\,\frac{1}{\hat\lambda_\varphi}\,\frac{d\hat\lambda_\varphi}{d\varphi} = -2i\,\frac{b_2(\hat\lambda_\varphi)}{\hat\nu_\varphi}.$$

Thus the derivative can vanish only if either $b_2(\hat\lambda_\varphi) = 0$ or $\hat\nu_\varphi = 0$.

The reduced curve is given by

$$\hat\nu^2 = -\hat\lambda\,\hat a(\hat\lambda) = -\hat\lambda(\hat\lambda + r)(\hat\lambda + r^{-1}).$$

Therefore the branch points in the $\hat\lambda$–plane are

$$\hat\lambda = 0, \qquad \hat\lambda = -r, \qquad \hat\lambda = -r^{-1}.$$

Since $0 < r < 1$, none of these points lies on the unit circle $|\hat{\lambda}| = 1$. It follows that $\hat{\nu}_\varphi \neq$ for all $\varphi \in (0, \pi)$.

Recall

$$b_2(\hat{\lambda}) = \gamma(\hat{\lambda} + s)(\hat{\lambda} + s^{-1}),$$

where, by the description of the zeros of b_2, one has $r < s < 1$. The zeros of b_2 are therefore real negative points with moduli s and s^{-1}, respectively. Since $s \neq 1$, neither of them lies on the unit circle. Therefore

$$b_2(\hat{\lambda}_\varphi) \neq 0 \qquad \text{for all } \varphi \in (0, \pi).$$

Consequently, the quantity

$$\frac{d}{d\varphi} q_2(r, \varphi) = -2i \, \frac{b_2(\hat{\lambda}_\varphi)}{\hat{\nu}_\varphi}$$

is continuous and never vanishes on the connected interval $(0, \pi)$. Consequently, it has a fixed sign throughout $(0, \pi)$.

Equivalently, once one chooses one of the two continuous lifts

$$\varphi \longmapsto (\hat{\lambda}_\varphi, \hat{\nu}_\varphi)$$

of the unit circle to the spectral curve, the sign of $\frac{d}{d\varphi} q_2(r, \varphi)$ is fixed along the entire interval. We choose the lift compatible with the normalization of the distinguished real period direction.

By the discussion preceding the lemma, the endpoints of the fixed–r curve satisfy

$$\lim_{\varphi \to 0^+} q_2(r, \varphi) = -1, \qquad \lim_{\varphi \to \pi^-} q_2(r, \varphi) = 1.$$

Hence $q_2(r, \varphi)$ increases from -1 to 1 as φ runs from 0 to π. Since its derivative has a constant sign on $(0, \pi)$, that sign must be positive. Therefore

$$\frac{d}{d\varphi} q_2(r, \varphi) > 0 \qquad \text{for all } \varphi \in (0, \pi).$$

Thus $\varphi \mapsto q_2(r, \varphi)$ is strictly increasing. $\square$

As a consequence, for fixed r the image of $\varphi \mapsto S(r, \varphi)$ is a simple curve which runs from the left boundary $\Re(\tau) = -1$ to the right boundary $\Re(\tau) = 1$. In particular, q_2 may be used as a parameter along this curve.

Remark 6.2. *In the paper Knopf et al. [2025], elliptic function theory was used to derive an explicit formula for the real and imaginary parts of the conformal classes for $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$. At the end of Section 4 of this paper, the curves for fixed r were plotted:*

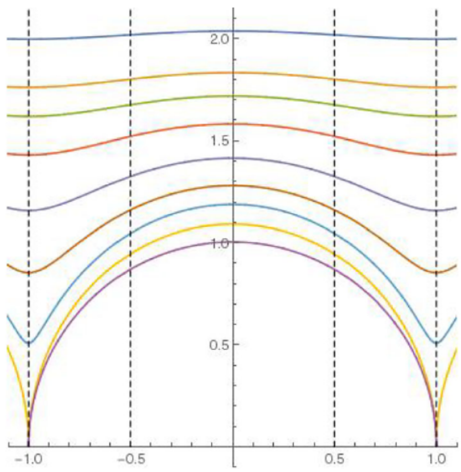

Figure 6.1: Reproduced from Knopf, Knopf et al. [2025], τ_a for different values of r

This image serves as a visualization. All qualitative properties shown in the figure will be proven rigorously below

We now turn to the imaginary part. Since the reduced polynomial $\hat{a}$ depends only on r, the corresponding coefficients β and γ are functions of r alone. In particular, for fixed r all φ–dependence enters only through the evaluation point

$$\hat{\lambda} = -e^{-2i\varphi}.$$

Lemma 6.3. *For fixed $r \in (0,1)$, the function $q_1(r,\varphi)$ is given by*

$$q_1(r,\varphi) = 2\gamma(r)\sqrt{\alpha(r) - 2\cos(2\varphi)}.$$

Proof. Recall that

$$dq_1 = \Theta_{b_1} = \frac{i\gamma(\hat{\lambda}^2 - 1)}{\hat{\nu}}\,\frac{d\hat{\lambda}}{\hat{\lambda}}. \tag{6.1}$$

We claim that

$$q_1(\hat{\lambda},\hat{\nu}) = -2i\gamma\,\frac{\hat{\nu}}{\hat{\lambda}}$$

is a primitive of (6.1). Indeed, differentiating $\hat{\nu}^2 = -\hat{\lambda}\hat{a}(\hat{\lambda})$ with respect to $\hat{\lambda}$ gives

$$2\hat{\nu}\,d\hat{\nu} = -\big(\hat{a}(\hat{\lambda}) + \hat{\lambda}\hat{a}'(\hat{\lambda})\big)\,d\hat{\lambda},$$

hence

$$d\hat{\nu} = -\frac{\hat{a}(\hat{\lambda}) + \hat{\lambda}\hat{a}'(\hat{\lambda})}{2\hat{\nu}}\,d\hat{\lambda}.$$

Therefore

$$d\left(\frac{\hat{\nu}}{\hat{\lambda}}\right) = \frac{1}{\hat{\lambda}}\,d\hat{\nu} - \frac{\hat{\nu}}{\hat{\lambda}^2}\,d\hat{\lambda} = -\frac{\hat{a} + \hat{\lambda}\hat{a}'}{2\hat{\lambda}\hat{\nu}}\,d\hat{\lambda} - \frac{\hat{\nu}}{\hat{\lambda}^2}\,d\hat{\lambda}.$$

Using again $\hat{\nu}^2 = -\hat{\lambda}\hat{a}(\hat{\lambda})$, the second term becomes

$$-\frac{\hat{\nu}}{\hat{\lambda}^2}\,d\hat{\lambda} = \frac{\hat{a}(\hat{\lambda})}{\hat{\lambda}\hat{\nu}}\,d\hat{\lambda}.$$

It follows directly that

$$d\left(\frac{\hat{\nu}}{\hat{\lambda}}\right) = \frac{\hat{a}(\hat{\lambda}) - \hat{\lambda}\hat{a}'(\hat{\lambda})}{2\hat{\lambda}\hat{\nu}}\,d\hat{\lambda}.$$

Since

$$\hat{a}(\hat{\lambda}) = \hat{\lambda}^2 + \alpha\hat{\lambda} + 1, \qquad \hat{a}'(\hat{\lambda}) = 2\hat{\lambda} + \alpha,$$

we obtain

$$\hat{a}(\hat{\lambda}) - \hat{\lambda}\hat{a}'(\hat{\lambda}) = 1 - \hat{\lambda}^2.$$

Hence

$$d\left(\frac{\hat{\nu}}{\hat{\lambda}}\right) = \frac{1 - \hat{\lambda}^2}{2\hat{\lambda}\hat{\nu}}\,d\hat{\lambda}.$$

Multiplying by $-2i\gamma$ yields

$$d\left(-2i\gamma\,\frac{\hat{\nu}}{\hat{\lambda}}\right) = i\gamma\frac{\hat{\lambda}^2 - 1}{\hat{\nu}}\,\frac{d\hat{\lambda}}{\hat{\lambda}} = \Theta_{b_1},$$

which proves the claim.

It remains to evaluate this primitive at the point

$$\hat{\lambda}_\varphi = -e^{-2i\varphi}.$$

Since $|\hat{\lambda}_\varphi| = 1$, we compute

$$\left(\frac{\hat{\nu}(\hat{\lambda}_\varphi)}{\hat{\lambda}_\varphi}\right)^2 = \frac{\hat{\nu}(\hat{\lambda}_\varphi)^2}{\hat{\lambda}_\varphi^2} = -\frac{\hat{\lambda}_\varphi^2 + \alpha\hat{\lambda}_\varphi + 1}{\hat{\lambda}_\varphi} = -\big(\hat{\lambda}_\varphi + \alpha + \hat{\lambda}_\varphi^{-1}\big).$$

Now

$$\hat{\lambda}_\varphi + \hat{\lambda}_\varphi^{-1} = -e^{-2i\varphi} - e^{2i\varphi} = -2\cos(2\varphi),$$

and therefore

$$\left(\frac{\hat{\nu}(\hat{\lambda}_\varphi)}{\hat{\lambda}_\varphi}\right)^2 = -\big(\alpha - 2\cos(2\varphi)\big).$$

Since $\alpha > 2$, the quantity $\alpha - 2\cos(2\varphi)$ is strictly positive. Choosing the branch compatible with $q_1 > 0$, we obtain

$$\frac{\hat{\nu}(\hat{\lambda}_\varphi)}{\hat{\lambda}_\varphi} = i\sqrt{\alpha - 2\cos(2\varphi)}.$$

Substituting this into the primitive gives

$$q_1(r,\varphi) = -2i\gamma(r)\,\frac{\hat{\nu}(\hat{\lambda}_\varphi)}{\hat{\lambda}_\varphi} = 2\gamma(r)\sqrt{\alpha(r) - 2\cos(2\varphi)}.$$

□

The explicit formula immediately identifies the lowest point of each fixed r curve.

Lemma 6.4. *For fixed $r \in (0,1)$, the function $\varphi \mapsto q_1(r,\varphi)$ attains its minimum at the boundary points corresponding to $\Re(\tau) = \pm 1$.*

Proof. By Lemma 6.3,

$$q_1(r,\varphi) = 2\gamma(r)\sqrt{\alpha(r) - 2\cos(2\varphi)}.$$

For fixed r, the factor $2\gamma(r)$ is constant and strictly positive. Thus it suffices to minimize the square root. Since $\cos(2\varphi) \le 1$ for all $\varphi \in [0,\pi]$, the expression

$$\alpha(r) - 2\cos(2\varphi)$$

is minimal exactly when $\cos(2\varphi) = 1$, that is, for $\varphi = 0$ or $\varphi = \pi$. These are precisely the two endpoints of the fixed r curve, and they correspond to the two boundary values $\Re(\tau) = -1$ and $\Re(\tau) = 1$. □

We now study these boundary values more closely. Along the two boundary branches one has $\varphi = 0$ or $\varphi = \pi$, and therefore

$$q_1(r) = 2\gamma(r)\sqrt{\alpha(r) - 2}.$$

We shall show that this quantity is strictly increasing along the Whitham flow and tends to infinity as $r \to 0$.

Lemma 6.5. *Along the two boundary branches $\Re(\tau) = \pm 1$, the quantity q_1 satisfies*

$$\dot{q}_1 > 0.$$

Moreover,

$$q_1(r) \to \infty \qquad \text{as } r \to 0.$$

Proof. Differentiating along the boundary branches with respect to the Whitham parameter t gives

$$\dot{q}_1 = 2\dot{\gamma}\sqrt{\alpha - 2} + \frac{\gamma\dot{\alpha}}{\sqrt{\alpha - 2}}.$$

Using the Whitham system from Chapter 5, this becomes

$$\dot{q}_1 = \frac{\beta - \alpha}{\gamma(\beta^2 - 4)}\sqrt{\alpha - 2} + \frac{\alpha^2 - 4}{\gamma(\beta^2 - 4)\sqrt{\alpha - 2}}.$$

After simplification one obtains

$$\dot{q}_1 = \frac{\sqrt{\alpha - 2}}{\gamma(\beta - 2)}. \tag{6.2}$$

By the description of the zeros of b_2, one has $\beta > 2$. Since also $\gamma > 0$ and $\alpha > 2$, it follows immediately from (6.2) that

$$\dot{q}_1 > 0.$$

It remains to analyze the behaviour as $r \to 0$. Since

$$\alpha = r + r^{-1} \sim r^{-1}, \qquad \sqrt{\alpha} \sim r^{-1/2},$$

we only need the corresponding asymptotic scale for γ. The rescaling argument from the proof in Chapter 5 shows that the relevant period of Θ_{b_2} on the collapsing cycle stays non–zero, and therefore $\gamma\beta$ has the same asymptotic order as $\sqrt{\alpha}$. In particular, there are constants $C_1, C_2 > 0$ such that for all sufficiently small r,

$$C_1\sqrt{\alpha} \leq \gamma\beta \leq C_2\sqrt{\alpha}.$$

Since $\beta > 2$, the factor $\beta/(\beta - 2)$ is bounded from below by 1, and by (6.2) we may rewrite

$$\dot{q}_1 = \frac{\beta}{\beta - 2} \cdot \frac{\sqrt{\alpha - 2}}{\gamma\beta}.$$

For small r, the second factor stays bounded away from zero because $\gamma\beta \asymp \sqrt{\alpha}$ and $\sqrt{\alpha - 2} \asymp \sqrt{\alpha}$. Thus there exists $c_1 > 0$ such that

$$\dot{q}_1 \geq c_1 > 0$$

for all sufficiently small r.

Finally, the Whitham time needed to reach $r = 0$ is infinite. Indeed,

$$\dot{\alpha} = \frac{\alpha^2 - 4}{\gamma^2(\beta^2 - 4)} < \frac{\alpha^2}{(\gamma\beta)^2} \leq \frac{1}{C_1^2}\alpha$$

for small r, hence

$$\frac{dt}{d\alpha} \geq C_1^2 \frac{1}{\alpha}.$$

Integrating from α_0 to A yields

$$t(A) - t(\alpha_0) \geq C_1^2 \int_{\alpha_0}^{A} \frac{d\alpha}{\alpha} = C_1^2(\log A - \log \alpha_0),$$

and therefore the time needed to reach $\alpha = \infty$, equivalently $r = 0$, is infinite. Since $\dot{q}_1$ stays bounded away from zero for large time, it follows that

$$q_1(r) \to \infty \qquad \text{as } r \to 0.$$

□

We can now compare the curves corresponding to different values of r. For fixed r, the map $\varphi \mapsto S(r, \varphi)$ is a simple curve from $\Re(\tau) = -1$ to $\Re(\tau) = 1$, and by Lemma 6.4 its lowest point lies at the two endpoints. By Lemma 6.5, these endpoint heights move strictly upward as $r \to 0$. At the other end, when $r \to 1$, the curves converge to the image of the limiting stratum $\mathcal{M}_2^3$, which is the upper semicircle $|\tau| = 1$, $\Im(\tau) > 0$. Thus the family of curves starts at the circular boundary of $\mathcal{F}_3$ and moves upward as r decreases.

Next we show, that for different r values, the curves don't intersect.

Lemma 6.6. *The determinant of the Jacobian Matrix of S is strictly negative.*

Proof. We compute the Jacobian determinant of the map $S(\varphi, r) = (q_2(\varphi, r), q_1(\varphi, r))$ with respect to the coordinates (φ, r). By definition, we have:

$$\det J_S = \frac{\partial q_2}{\partial \varphi}\frac{\partial q_1}{\partial r} - \frac{\partial q_1}{\partial \varphi}\frac{\partial q_2}{\partial r}.$$

Since the radial parameter r depends on the Whitham flow parameter t, we apply the chain rule $\partial_r = \frac{1}{\dot{r}}\partial_t$ to rewrite the determinant as:

$$\det J_S = \frac{1}{\dot{r}}\left(\frac{\partial q_2}{\partial \varphi}\dot{q}_1 - \frac{\partial q_1}{\partial \varphi}\dot{q}_2\right).$$

From Lemma 5.10, the variations of the primitives with respect to the Whitham parameter are given by $\dot{q}_l = i\frac{c_l(\hat{\lambda})}{\hat{\nu}}$ for $l = 1, 2$. as in the proof of Lemma 6.1, we obtain:

$$\frac{\partial q_l}{\partial \varphi} = -2i\frac{b_l(\hat{\lambda})}{\hat{\nu}}.$$

Substituting these expressions into the determinant yields:

$$\det J_S = \frac{1}{\dot{r}}\left[\left(-2i\frac{b_2}{\hat{\nu}}\right)\left(i\frac{c_1}{\hat{\nu}}\right) - \left(-2i\frac{b_1}{\hat{\nu}}\right)\left(i\frac{c_2}{\hat{\nu}}\right)\right] = \frac{2}{\dot{r}\hat{\nu}^2}(b_2c_1 - b_1c_2).$$

By the compatibility equation (5.16), normalized by $Q(\hat{\lambda}) = \hat{\lambda}$, we know that $b_2c_1 - b_1c_2 = \hat{\lambda}\hat{a}(\hat{\lambda})$. Thus, the Jacobian determinant takes the form:

$$\det J_S = \frac{2}{\dot{r}\hat{\nu}^2}\hat{\lambda}\hat{a}(\hat{\lambda}).$$

By construction, the roots of the reduced determinant polynomial $\hat{a}$ are $-r$ and $-r^{-1}$. Since $r \in (0, 1)$, neither root lies on the unit circle $|\hat{\lambda}| = 1$. Consequently, $\hat{a}(\hat{\lambda}) \neq 0$ for all $\varphi \in [0, \pi]$. Since the determinant is a continuous function and is never zero, it must maintain a constant sign throughout its domain.

Because the sign is constant, it is sufficient to evaluate $\det J_S$ at a single boundary configuration. We choose $\varphi = 0$. By Lemma 6.3, the function $\varphi \mapsto q_1(r, \varphi)$ attains its minimum at this boundary, which implies:

$$\left.\frac{\partial q_1}{\partial \varphi}\right|_{\varphi=0} = 0.$$

The Jacobian determinant therefore simplifies to:

$$\det J_S(\varphi = 0) = \frac{\partial q_2}{\partial \varphi}\frac{\partial q_1}{\partial r}.$$

We determine the signs of these two factors individually:

- By Lemma 6.1, the map $\varphi \mapsto q_2(r, \varphi)$ is strictly increasing, hence $\frac{\partial q_2}{\partial \varphi} > 0$.
- To find the sign of $\frac{\partial q_1}{\partial r}$, we expand it as $\frac{\dot{q}_1}{\dot{r}}$. From Lemma 6.5, it is established that $\dot{q}_1 > 0$ along the boundary branch. Furthermore, in the proof of the same Lemma we have seen that $\dot{r} < 0$. It immediately follows that $\frac{\partial q_1}{\partial r} < 0$.

Taking the product of a strictly positive and a strictly negative term, we obtain:

$$\det J_S(\varphi = 0) < 0.$$

Therefore, the determinant of the Jacobian matrix of S is strictly negative everywhere. □

For every fixed $r \in (0,1)$, Lemma 6.1 has already shown that $\varphi \mapsto q_2(r,\varphi)$ is strictly increasing from -1 to 1. Hence, for every $x \in (-1,1)$ and every $r \in (0,1)$, there exists a unique $\varphi_x(r) \in (0,\pi)$ such that

$$q_2(r, \varphi_x(r)) = x. \tag{6.3}$$

Since $\partial_\varphi q_2(r,\varphi) > 0$, the Implicit Function Theorem A.2 implies that $r \mapsto \varphi_x(r)$ is continuous. We may therefore define

$$h_x(r) := q_1(r, \varphi_x(r)).$$

We claim that h_x is strictly decreasing on $(0,1)$. Differentiating (6.3) with respect to r, we obtain

$$\partial_r q_2(r,\varphi_x(r)) + \partial_\varphi q_2(r,\varphi_x(r))\, \varphi_x'(r) = 0,$$

and therefore

$$\varphi_x'(r) = -\frac{\partial_r q_2(r,\varphi_x(r))}{\partial_\varphi q_2(r,\varphi_x(r))}.$$

Using this in the derivative of h_x, we get

$$\begin{aligned} h_x'(r) &= \partial_r q_1(r,\varphi_x(r)) + \partial_\varphi q_1(r,\varphi_x(r))\, \varphi_x'(r) \\ &= \partial_r q_1(r,\varphi_x(r)) - \partial_\varphi q_1(r,\varphi_x(r)) \frac{\partial_r q_2(r,\varphi_x(r))}{\partial_\varphi q_2(r,\varphi_x(r))} \\ &= \frac{\partial_\varphi q_2(r,\varphi_x(r))\, \partial_r q_1(r,\varphi_x(r)) - \partial_\varphi q_1(r,\varphi_x(r))\, \partial_r q_2(r,\varphi_x(r))}{\partial_\varphi q_2(r,\varphi_x(r))}. \end{aligned}$$

The numerator is equal to $\det J_S(r,\varphi_x(r))$, which is by Lemma 6.6 strictly negative. Since $\partial_\varphi q_2(r,\varphi_x(r)) > 0$ by Lemma 6.1, it follows that

$$h_x'(r) < 0.$$

Thus, for every fixed $x \in (-1,1)$, the imaginary part q_1 increases strictly as r decreases.

We now determine the image of S. Fix $x \in (-1,1)$. By section 5.4, we already know that, as $r \to 1$, the image curve $\varphi \mapsto S(r,\varphi)$ converges to the upper semicircle, i.e. so that

$$h_x(r) \longrightarrow \sqrt{1-x^2} \qquad \text{as } r \to 1.$$

On the other hand, by Lemma 6.5, the endpoints of the curve tend to infinity in the positive imaginary direction as $r \to 0$. Since for fixed r the minimum of $q_1(r,\varphi)$ is attained at the endpoints, it follows that

$$h_x(r) \longrightarrow \infty \qquad \text{as } r \to 0,$$

for any fixed $x \in (-1,1)$. Because h_x is continuous and strictly decreasing, its image is precisely the interval

$$(\sqrt{1-x^2}, \infty).$$

Equivalently, on the vertical line $\Re(\tau) = x$, the image of S consists exactly of those points strictly above the unit semicircle. Since this holds for every $x \in (-1,1)$, and since the limiting stratum $\mathcal{M}_2^3$ maps exactly to the upper semicircle, we conclude that

$$\operatorname{im}(S) = \mathcal{F}_3.$$

We have thereby shown

Corollary 6.7. *The image of S is equal to $\mathcal{F}_3$.*

We finally show that every point of $\mathcal{F}_3$ has a unique preimage. This is the statement announced at the beginning of the chapter.

Theorem 6.8. *The map*

$$S : \mathcal{M}_2^2 \cup \mathcal{M}_2^3 \to \mathcal{F}_3, \qquad a \mapsto q_2(a) + iq_1(a)$$

is a bijection.

Proof. By Corollary 6.7, the map S is surjective.

To prove injectivity, let $\tau = x + iy \in \mathcal{F}_3$ be given. If τ lies on the unit semicircle $|\tau| = 1$, $\Im(\tau) > 0$, then by the description of the limiting stratum there is a unique point of $\mathcal{M}_2^3$ mapping to τ.

Now assume that $|\tau| > 1$. Then $x \in (-1, 1)$ and $y > \sqrt{1 - x^2}$.

Suppose that τ has two preimages in $\mathcal{M}_2^2$, corresponding to parameters (r_1, φ_1) and (r_2, φ_2). Since both points map to τ, we have

$$q_2(r_1, \varphi_1) = x = q_2(r_2, \varphi_2).$$

For each fixed r, the map $\varphi \mapsto q_2(r, \varphi)$ is strictly increasing, hence

$$\varphi_1 = \varphi_x(r_1), \qquad \varphi_2 = \varphi_x(r_2).$$

Moreover,

$$y = q_1(r_1, \varphi_1) = h_x(r_1), \qquad y = q_1(r_2, \varphi_2) = h_x(r_2).$$

Since h_x is strictly decreasing, it follows that $r_1 = r_2$. Then also

$$\varphi_1 = \varphi_x(r_1) = \varphi_x(r_2) = \varphi_2.$$

Thus the two preimages coincide. Hence every point of the interior of $\mathcal{F}_3$ has a unique preimage in $\mathcal{M}_2^2$.

Therefore S is injective. Since it is also surjective, it is bijective. □

The interpretation is therefore simple: on the boundary, the reduced genus–one conformal class contains exactly the information needed to recover the spectral data. For fixed real part $\Re(\tau) = x$, the imaginary part determines the radial parameter r through the strict monotonicity of h_x, and once r is fixed, the real part determines the angular parameter φ via the strictly increasing map $\varphi \mapsto q_2(r, \varphi)$. Hence the boundary strata $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$ are completely described by their conformal classes.

Appendices

Appendix A

Important Theorems

Theorem A.1 (Picard-Lindelöf)**.** *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^n$ be continuous and locally Lipschitz continuous in the second variable. Then for every $(x_0, y_0) \in U$ there exist an open interval $I \subset \mathbb{R}$ with $x_0 \in I$ and a unique differentiable map $y : I \to \mathbb{R}^n$ such that*

$$y'(x) = f(x, y(x)) \quad \text{for all } x \in I, \qquad y(x_0) = y_0.$$

Proof. For the proof, see Teschl [2012], Theorem 2.2. □

Theorem A.2 (Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^m$ be of class C^1. Assume that for some $p \in U$ the differential $DF(p)$ has rank m. Then there exists a neighborhood V of p such that*

$$F^{-1}(F(p)) \cap V$$

is a C^1-submanifold of $\mathbb{R}^n$ of dimension $n - m$.

Proof. For the proof, see Lee [2012], Corollary 5.14. □

Theorem A.3 (Liouville's Theorem)**.** *Every meromorphic function on the Riemann sphere without poles is constant.*

Proof. For the proof, see Forster [2012], Theorem 2.10. □

Theorem A.4 (Spectral Theorem)**.** *Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e. $AA^* = A^*A$. Then there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ such that*

$$A = UDU^*.$$

The diagonal entries of D are exactly the eigenvalues of A.

Proof. For the proof, see Axler [2024], Section 7B. □

Theorem A.5 (Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $u : \Omega \to \mathbb{R}$ be harmonic. If u attains a maximum or a minimum at an interior point of Ω, then u is constant.*

Proof. For the proof, see Axler et al. [2013], Theorem 1.8. □

Theorem A.6 (Removable singularities theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, let $a \in \Omega$, and let $u : \Omega \setminus \{a\} \to \mathbb{R}$ be harmonic. If u is bounded in a neighborhood of a, then there exists a unique harmonic function $\tilde{u} : \Omega \to \mathbb{R}$ with*

$$\tilde{u} = u \quad \text{on } \Omega \setminus \{a\}.$$

Proof. For the proof, see Axler et al. [2013], Theorem 2.3. □

Theorem A.7 (Orbit–stabilizer theorem for transitive Lie group actions)**.** *Let G be a Lie group acting smoothly and transitively on a smooth manifold M, and let $p \in M$. Denote by*

$$G_p := \{g \in G \mid g \cdot p = p\}$$

the stabilizer of p. Then G_p is a closed Lie subgroup of G, and the orbit map

$$G \to M, \qquad g \mapsto g \cdot p$$

induces a diffeomorphism

$$G/G_p \longrightarrow M.$$

In particular, if $G = \mathbb{R}^n$ and $\dim M = n$, then G_p is discrete. Moreover, if M is compact, then $\mathbb{R}^n/G_p$ is compact.

Proof. For the proof, see Lee [2012], Theorem 21.18. □

Hiermit versichere ich, dass diese Arbeit von mir persönlich verfasst wurde und dass ich keinerlei fremde Hilfe in Anspruch genommen habe. Ebenso versichere ich, dass diese Arbeit oder Teile daraus weder von mir selbst noch von anderen als Leistungsnachweise andernorts eingereicht wurden. Wörtliche oder sinngemäße Übernahmen aus anderen Schriften und Veröffentlichungen in gedruckter oder elektronischer Form sind gekennzeichnet. Sämtliche Sekundärliteratur und sonstige Quellen sind nachgewiesen und in der Bibliographie aufgeführt. Das Gleiche gilt für graphische Darstellungen und Bilder sowie für alle Internet-Quellen. Ich bin ferner damit einverstanden, dass meine Arbeit zum Zwecke eines Plagiatsabgleichs in elektronischer Form anonymisiert versendet und gespeichert werden kann. Mir ist bekannt, dass von der Korrektur der Arbeit ab gesehen werden kann, wenn diese Erklärung nicht erteilt wird.

Datum, Ort

Unterschrift

Bibliography

Sheldon Axler. *Linear algebra done right.* Springer, 2024.

Sheldon Axler, Paul Bourdon, and Ramey Wade. *Harmonic function theory.* Springer Science & Business Media, 2013.

Emma Carberry and Martin Ulrich Schmidt. The closure of spectral data for constant mean curvature tori in S^3. *arXiv preprint arXiv:1202.4517*, 2012.

Emma Carberry, Martin Kilian, Sebastian Klein, and Martin Ulrich Schmidt. The space of genus two spectral curves of constant mean curvature tori in $\mathbb{R}^3$. *arXiv preprint arXiv:2110.00436*, 2021.

Josef Dorfmeister, Franz Pedit, and Hongyong Wu. Weierstrass type representation of harmonic maps into symmetric spaces. *Communications in analysis and geometry*, 6(4):633–668, 1998.

Otto Forster. *Lectures on Riemann surfaces.* Springer Science & Business Media, 2012.

Eberhard Freitag and Rolf Busam. *Complex analysis.* Springer, 2005.

Brian C Hall. An elementary introduction to groups and representations. *arXiv preprint math-ph/0005032*, 2000.

Robin Hartshorne. *Algebraic geometry.* Springer Science & Business Media, 2013.

Ricardo Pena Hoepner. Solutions of the sinh-gordon equation of spectral genus two. Master's thesis, University of Mannheim, 2015.

Martin Kilian, Martin Ulrich Schmidt, and Nicholas Schmitt. Flows of constant mean curvature tori in the 3-sphere: the equivariant case. *arXiv preprint arXiv:1011.2875*, 2010.

Markus Knopf, Ricardo Peña Hoepner, and Martin U Schmidt. Solutions of the sinh-gordon equation of spectral genus two and constrained willmore tori i. *The Journal of Geometric Analysis*, 35(1): 29, 2025.

John M Lee. Submanifolds. In *Introduction to smooth manifolds*, pages 98–124. Springer, 2012.

Jan R Magnus and Heinz Neudecker. *Matrix differential calculus with applications in statistics and econometrics.* John Wiley & Sons, 2019.

Andrew Pressley. Loop groups, grassmannians and kdv equations. In *Infinite Dimensional Groups with Applications*, pages 285–306. Springer, 1985.

Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140. American Mathematical Soc., 2012.

Valerii V Trofimov. *Introduction to geometry of manifolds with symmetry*, volume 270. Springer Science & Business Media, 2013.