



UNIVERSITY OF MANNHEIM

Algebraic-geometric Solutions of the KdV Equation

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Declaration of Authorship

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Remark: AI-based tools were used in the preparation of this thesis to assist with linguistic revisions and, on occasion, to discuss mathematical formulations and lines of argument.

A handwritten signature in black ink, appearing to read 'Eda TL'.

Mannheim, 22.09.2026

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Chapter 1

Introduction

1.1 Motivation: Integrable systems and the KdV equation

The study of nonlinear wave phenomena has been a central theme in mathematical physics for well over a century. Among the many nonlinear partial differential equations that arise in this context, the Korteweg-de-Vries (KdV) equation occupies a special place. Originally derived to model the propagation of shallow water waves, it has since become a great example of an *integrable system*: a nonlinear equation that, despite its complexity, admits exact solutions and possesses a rich mathematical structure.

A central feature of this structure is the existence of *solitons*, stable solitary wave solutions that propagate without changing their shape or speed. When two solitons collide, they pass through each other and emerge unchanged, as if the interaction had never taken place. This particle-like behavior is a direct consequence of the integrability of the KdV equation.

Underlying this integrability is an infinite family of conserved quantities: functionals of the solution that remain constant under the KdV flow. The existence of infinitely many conserved quantities severely constrains the dynamics of the equation, and is ultimately responsible for the orderly behavior of solitons. It also gives rise to a hierarchy of commuting flows, the *KdV hierarchy*, each of which shares the same conserved quantities and the same class of special solutions.

Among these special solutions, the *algebraic-geometric* or *finite-gap* solutions play a distinguished role. They are the solutions whose associated spectral data has a simple structure: only finitely many spectral gaps. These solutions are the central object of study in this thesis.

1.2 Historical background and context

The Korteweg-de Vries equation was first derived in 1895 to model the propagation of shallow water waves. Its mathematical significance was not fully appreciated until the 1960s, when numerical experiments revealed the soliton phenomenon [21]. The key math-

ematical breakthrough came with the discovery that the KdV equation could be written as the compatibility condition of two linear operators, a structure that Lax formalized through the concept of a Lax pair.

The theory of finite-gap potentials emerged in the 1970s from the work of several authors [18, 17, 11, 5]. It was shown that the stationary solutions of the KdV hierarchy are precisely the finite-gap potentials, and that these can be expressed explicitly in terms of Riemann theta functions on hyperelliptic curves. Particular finite-gap potentials had in fact been studied much earlier, in the classical theory of Lamé's equation, a special case of Hill's equation (see e.g. Erdélyi et al. [6]). This algebro-geometric description connects the spectral theory of the Hill operator with the geometry of Riemann surfaces, and remains an active area of research.

The approach taken in this thesis is inspired by the work of Pinkall and Sterling [19], who developed an analogous theory for the sinh-Gordon equation using the framework of polynomial Killing fields, see also Hitchin, Segal and Ward [10] for the general theory of integrable systems from the loop group and twistor perspective. We follow this approach for the KdV equation, using the formal Laurent series solution of the Lax equation as our central tool, in place of the Riemann surface methods used in the classical theory.

1.3 Notational remark

Throughout this thesis, we write f' for the derivative of a function f of a single variable, and use the operator notation $\frac{d}{dx}$ when emphasizing the differential operator itself or when the variable of differentiation needs to be made explicit (e.g. when several variables are present).

1.4 Aim and scope of this thesis

The aim of this thesis is to present, in a self-contained and rigorous manner, the theory of algebraic-geometric solutions of the KdV equation, in analogy with the treatment of finite-type solutions of the sinh-Gordon equation by Pinkall and Sterling [19].

The central result is Theorem 4.8, which characterizes finite-type potentials in four equivalent ways: through the existence of a special solution of the Lax equation with finitely many non-vanishing coefficients, through a linear dependence relation among the coefficients of the formal Laurent series solution, through a formal power series scaling of the canonical solution, and through the existence of a commuting differential operator of odd order. The equivalence of these four characterizations is proved in full detail, with all auxiliary results established from first principles.

For periodic potentials, Theorem 5.14 establishes the equivalence of finite type with the finiteness of the number of spectral gaps of the associated Hill operator. This connects the algebraic theory of Chapter 4 with the spectral theory of periodic Schrödinger operators developed in Chapter 5.

The scope of this thesis is deliberately focused: we develop the theory of finite-type potentials in sufficient depth to establish the two main theorems. Chapter 6 extends the spectral theory of Chapter 5 to the larger class of L^1 potentials, motivated by the long-term goal of showing that periodic finite-type potentials are dense in $L^1([0, 1], \mathbb{R})$. This density question, which connects the algebraic-geometric theory of Chapters 3-4 with functional analysis, is discussed as an outlook in Chapter 7.

1.5 Outline of this thesis

The thesis is organized as follows.

Chapter 2 introduces the KdV equation and its Lax pair formulation. We present the Schrödinger operator as the Lax operator for the KdV equation, discuss the associated eigenvalue problem and its spectral invariance, and motivate the role of the Lax formulation in finite-gap theory.

Chapter 3 develops the formal solution of the Lax equation. We introduce the matrix formulation of the Lax equation and the ansatz for the formal Laurent series solution, derive the recursion relations for the coefficients, and establish the connection with the Gelfand-Dikii structure and the KdV hierarchy.

Chapter 4 presents the central theory of the thesis. We introduce the notion of a potential of finite type and prove Theorem 4.8, which establishes four equivalent characterizations of finite-type potentials. The proof relies on several preparatory lemmas concerning commutator identities and the structure of the differential operator P_{2N+1} .

Chapter 5 develops the spectral theory of periodic Schrödinger operators. We introduce the Hill operator, the monodromy matrix and the discriminant, and discuss the band-gap structure of the spectrum. The main result is Theorem 5.14, which characterizes periodic finite-type potentials as those with finitely many spectral gaps.

Chapter 6 extends the spectral theory of periodic Schrödinger operators to the class of L^1 potentials. Following the approach of Pöschel and Trubowitz [20], we show that the fundamental solutions y_1 and y_2 , the basic estimates, and the properties of the monodromy matrix remain valid in this more general setting. This extension is motivated by the density question discussed in Chapter 7.

Chapter 7 concludes the thesis with an outlook on the density of periodic finite-type potentials in $L^1([0, 1], \mathbb{R})$, and on further connections to algebraic geometry and Riemann surfaces.

Chapter 2

The KdV Equation and Its Lax Pair

This chapter introduces the Korteweg-de-Vries (KdV) equation and its Lax pair formulation, which forms the foundation for the theory developed in the subsequent chapters. We begin by presenting the KdV equation and its physical motivation, then introduce the concept of integrability and the Lax pair formalism. We show how the Schrödinger operator arises naturally as the Lax operator for the KdV equation, and discuss the associated eigenvalue problem and its spectral invariance. The chapter concludes with a discussion of the role of the Lax formulation in finite-gap theory, motivating the approach taken in Chapters 3 and 4.

2.1 The Korteweg-de-Vries equation

The Korteweg-de-Vries (KdV) equation is a nonlinear partial differential equation for a function $q = q(x, t)$ of space $x \in \mathbb{R}$ and time $t \in \mathbb{R}$:

$$q_t + \frac{1}{4}q_{xxx} - \frac{3}{2}qq_x = 0. \quad (2.1)$$

It was first derived by Korteweg and de Vries in 1895 [12] to model the propagation of shallow water waves in a canal. The term q_{xxx} accounts for dispersion, whereas the term qq_x is an advection term, which, in general, describes the transport of a substance or quantity by bulk motion of a fluid. The stable solitary wave solutions are called *solitons*. A soliton is a localized wave packet that propagates without changing its shape or speed. The existence of solitons is a hallmark of integrable systems: the nonlinear and dispersive effects balance each other exactly, preventing the wave from breaking or dispersing. The simplest soliton solution of (2.1) is:

$$q(x, t) = -\frac{c}{2} \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2}(x - ct) \right), \quad c > 0, \quad (2.2)$$

with *sech* standing for the hyperbolic secant, which is a wave of amplitude $-c/2$ traveling to the right with speed c .

The KdV equation has a remarkable property: it admits infinitely many conserved quantities, which are functionals of q that remain constant under the KdV flow. The first few are:

$$I_1 = \int_{-\infty}^{\infty} q \, dx, \quad (2.3)$$

$$I_2 = \int_{-\infty}^{\infty} q^2 \, dx, \quad (2.4)$$

$$I_3 = \int_{-\infty}^{\infty} \left(q^3 - \frac{1}{2} q_x^2 \right) dx. \quad (2.5)$$

The existence of infinitely many conserved quantities is tightly connected to the integrability of the KdV equation, as we will discuss in the next section.

2.2 Integrability and Lax pairs

A nonlinear partial differential equation is called *integrable* if it possesses a sufficiently rich structure to allow for exact solutions. One of the most powerful frameworks for understanding integrability is the *Lax pair* formalism, introduced by Lax in 1968 [13].

Definition 2.1. A *Lax pair* for a nonlinear evolution equation is a pair of linear operators (L, P) , depending on a solution q of the equation, such that the equation is equivalent to the *Lax equation*:

$$\frac{\partial L}{\partial t} = [P, L] := PL - LP. \quad (2.6)$$

The key insight of the Lax formalism is that the nonlinear evolution equation can be encoded in the compatibility condition of two linear problems. This allows one to use the powerful machinery of linear algebra and spectral theory to study the nonlinear equation. The Lax equation (2.6) has an important consequence: the spectrum of L is invariant under the KdV flow. To see this, suppose $Ly(t) = \lambda(t)y(t)$ for all t , where $y(t)$ evolves according to $\frac{\partial y}{\partial t} = Py$. Differentiating $Ly = \lambda y$ with respect to t :

$$\frac{\partial L}{\partial t} y + L \frac{\partial y}{\partial t} = \lambda' y + \lambda \frac{\partial y}{\partial t}. \quad (2.7)$$

Substituting $\frac{\partial L}{\partial t} = [P, L]$ and $\frac{\partial y}{\partial t} = Py$:

$$[P, L]y + LPy = \lambda' y + \lambda Py. \quad (2.8)$$

Since $[P, L]y = PLy - LPy = \lambda Py - LPy$, the left-hand side becomes:

$$\lambda Py - LPy + LPy = \lambda Py. \quad (2.9)$$

Therefore:

$$\lambda Py = \lambda' y + \lambda Py, \quad (2.10)$$

which gives $\lambda'y = 0$. Since $y \not\equiv 0$, we conclude $\lambda' = 0$, i.e., λ is independent of t . Therefore the eigenvalues of L are conserved quantities of the KdV flow, and the spectrum of L is invariant under the time evolution.

This spectral invariance is the source of the infinitely many conserved quantities of integrable systems.

2.3 The Schrödinger operator as Lax operator

For the KdV equation (2.1), the appropriate Lax pair was discovered by Gardner, Greene, Kruskal and Miura in 1967 [8]. The Lax operator is the one-dimensional Schrödinger operator:

$$L = -\frac{d^2}{dx^2} + q(x, t), \quad (2.11)$$

and the operator P is the third-order differential operator:

$$P = -4\frac{d^3}{dx^3} + 6q\frac{d}{dx} + 3q_x. \quad (2.12)$$

A direct computation of the commutator $[P, L]$, using the operator identities of Lemma 4.1, yields:

$$[P, L] = -q_{xxx} + 6qq_x. \quad (2.13)$$

Hence the Lax equation $L_t = [P, L]$ is equivalent to $q_t = -q_{xxx} + 6qq_x$, which is four times the KdV equation (2.1).

The choice of the Schrödinger operator (2.11) as the Lax operator is not accidental. The Schrödinger operator is self-adjoint on $L^2(\mathbb{R})$ for real potentials q , which ensures that its spectrum is real. Moreover, the spectral theory of the Schrödinger operator is very well developed, providing powerful tools for the analysis of the KdV equation.

2.4 The associated eigenvalue problem

The Lax operator $L = -\frac{d^2}{dx^2} + q$ gives rise to the eigenvalue problem:

$$Ly = \lambda y, \quad \text{i.e.,} \quad -y''(x) + q(x, t)y(x) = \lambda y(x), \quad (2.14)$$

for $\lambda \in \mathbb{C}$ and $y \in C^\infty(\mathbb{R})$. This is the time-independent Schrödinger equation with potential q and energy λ , where t enters only as a parameter through the potential $q = q(x, t)$.

As a first-order system, (2.14) is equivalent to:

$$\frac{d}{dx} \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q(x, t) - \lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}. \quad (2.15)$$

This matrix formulation is the starting point for the construction of the formal Laurent series in Chapter 3.

The time evolution of y is governed by the equation $y_t = Py$, which as a first-order system reads:

$$\frac{d}{dt} \begin{pmatrix} y \\ y' \end{pmatrix} = V(\lambda) \begin{pmatrix} y \\ y' \end{pmatrix}, \quad (2.16)$$

for a matrix $V(\lambda)$ whose explicit form depends on the normalization of P . The compatibility condition of these two systems is precisely the Lax equation (2.6), and hence the KdV equation (2.1).

2.5 Spectral invariance and conserved quantities

As discussed in Section 2.2, the spectrum of the Lax operator L is invariant under the KdV flow. This means that if $q(x, t)$ is a solution of the KdV equation, then the spectrum of $L = -\frac{d^2}{dx^2} + q(\cdot, t)$ is independent of t .

This spectral invariance has profound consequences. Each eigenvalue λ_n of L is a conserved quantity of the KdV flow, and the associated eigenfunctions evolve in a simple, linear fashion. The conserved quantities $I_1, I_2, I_3, \dots$ introduced in Section 2.1 can be expressed in terms of the spectral data of L .

More precisely, the KdV equation is part of an infinite hierarchy of commuting flows, the *KdV hierarchy*. Each flow is generated by a higher-order differential operator P_{2n+1} commuting with L , and takes the form of a nonlinear evolution equation for q . The first few flows are:

$$q_{t_1} = q_x, \quad (2.17)$$

$$q_{t_2} = -\frac{1}{4}q_{xxx} + \frac{3}{2}qq_x, \quad (2.18)$$

$$q_{t_3} = \frac{1}{16}q_{xxxxx} - \frac{5}{8}qq_{xxx} - \frac{5}{4}q_xq_{xx} + \frac{15}{8}q^2q_x. \quad (2.19)$$

The second flow q_{t_2} is exactly the KdV equation (2.1). The precise structure of these flows, and the differential polynomials f_{n+1} in terms of which they can be written as $q_{t_n} = \frac{d}{dx}f_{n+1}(q)$, will be introduced in Section 3.5 via the Gelfand-Dikii recursion. The flows are compatible in the sense that they commute: if q satisfies $q_{t_n} = 0$ for some n , then it also satisfies $q_{t_m} = 0$ for all $m \geq n$, after a suitable choice of integration constants (cf. Remark 1.5 in [9]).

The *stationary KdV hierarchy* is now obtained by setting $q_{t_n} = 0$.

A potential $q \in C^\infty(I, \mathbb{K})$ satisfying one of these stationary equations is called an *algebro-geometric KdV solution*, or simply a *KdV potential*, following Gesztesy and Holden [9]. As we will show in Chapter 4, these are precisely the finite-type potentials characterized by Theorem 4.8: a potential is of finite type if and only if it satisfies one of the stationary KdV equations $\frac{d}{dx}f_{n+1}(q) = 0$ for some $n \in \mathbb{N}_0$.

We note that from this point on, we consider q as a function of x alone, so $q = q(x) \in C^\infty(I, \mathbb{K})$. This is justified by the fact that the stationary KdV equations $\frac{d}{dx}f_{n+1}(q) = 0$

are ordinary differential equations in x that contain no time derivatives of q . Therefore, setting $q_{t_n} = 0$ reduces the time-dependent problem to a purely spatial one, and t no longer plays a role.

2.6 Role of the Lax formulation in finite-gap theory

The Lax pair formulation provides the natural framework for the theory of finite-gap potentials developed in this thesis. The key idea is to look for solutions of the Lax equation (2.6) that have a special algebraic structure, namely that they can be expressed as formal Laurent series in the spectral parameter λ with only finitely many non-vanishing coefficients.

As we will show in Chapter 3, the Lax equation (2.6) admits a unique solution $\xi(x, \lambda)$ in the form of a formal Laurent series in λ^{-1} , whose coefficients are differential polynomials in q . A potential q is of finite type if and only if this series truncates, i.e., only finitely many coefficients are non-zero. This is one of the four equivalent characterizations of finite-type potentials given in Theorem 4.8.

The connection to spectral theory, developed in Chapter 5, arises from the fact that for periodic potentials, finite type is equivalent to the associated Hill operator having only finitely many spectral gaps. This connects the algebraic notion of finite type with the geometric structure of the spectrum, and ultimately leads to the description of finite-type potentials in terms of algebraic curves, as discussed in Chapter 7.

Chapter 3

Formal Solution of the Lax Equation

In Chapters 3 and 4, we work over a general field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and write $q \in C^\infty(I, \mathbb{K})$ for the potential. The restriction to $\mathbb{K} = \mathbb{R}$ will be made in Chapter 5, where the spectral theory requires self-adjointness of the Hill operator.

3.1 Matrix formulation of the Lax equation

In the classical Lax formulation of the KdV equation one considers the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q(x, t), \quad (3.1)$$

together with a differential operator P of odd order such that the Lax equation

$$\frac{d}{dt}L = [P, L] \quad (3.2)$$

is equivalent to the KdV equation. In order to obtain a first-order system suitable for a matrix formulation, we rewrite the spectral problem

$$Ly = \lambda y \quad (3.3)$$

as a 2×2 system. Introducing

$$y_1 = y, \quad y_2 = y',$$

and noting that

$$y_1' = y_2, \quad y_2' = y'' = (q - \lambda)y_1,$$

where the last equality follows directly from the spectral equation (3.3), the spectral problem (3.3) is equivalent to the matrix equation

$$\frac{d}{dx}\Psi = U(\lambda)\Psi, \quad \Psi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad (3.4)$$

where

$$U(\lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix}. \quad (3.5)$$

Indeed, the first row of $U(\lambda)\Psi$ gives $y'_1 = y_2$, and the second row gives $y'_2 = (q(x) - \lambda)y_1$, which together are equivalent to $-y'' + qy = \lambda y$.

To describe the time evolution we rewrite the equation $y_t = Py$ in the same first-order form. This yields an evolution equation

$$\frac{d}{dt}\Psi = V(\lambda)\Psi, \quad (3.6)$$

with a traceless matrix $V(\lambda)$. The pair of equations (3.4)-(3.6) constitutes a *matrix Lax pair* for the KdV equation, and the compatibility condition

$$U_t - V_x + [U, V] = 0 \quad (3.7)$$

is equivalent to the KdV equation

$$q_t = -\frac{1}{4}q''' + \frac{3}{2}qq',$$

up to a rescaling of the time variable.

The compatibility condition arises from requiring that the two equations (3.4) and (3.6) are simultaneously solvable, i.e., that mixed partial derivatives of Ψ commute. Indeed, differentiating $\frac{d}{dx}\Psi = U\Psi$ with respect to t and $\frac{d}{dt}\Psi = V\Psi$ with respect to x and subtracting yields the desired condition.

In what follows, we will work exclusively with the x -equation (3.4). This matrix formulation will serve as the starting point for the formal Laurent expansion developed in the following sections.

The matrix Lax pair (3.4)-(3.6) provides a first-order formulation of the KdV spectral problem and its time evolution. In order to analyze the structure of solutions and to prepare the construction of finite-type potentials, we follow the approach of Pinkall and Sterling and seek a formal solution of the x -equation in the form of a Laurent expansion in the spectral parameter λ . This formal series will encode the algebraic structure underlying the KdV hierarchy and will lead to a system of recursion relations for its coefficients. We now introduce this ansatz.

3.2 Ansatz for the formal Laurent series

Following the method of Pinkall and Sterling, we now construct a formal solution of the x -equation

$$\frac{d}{dx}\Psi = U(\lambda)\Psi, \quad U(\lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix},$$

in the form of a Laurent series in the spectral parameter λ . We look for a matrix-valued function

$$\xi(x, \lambda) = \xi_{-1}(x)\lambda + \sum_{n=0}^{\infty} \xi_n(x) \lambda^{-n}, \quad \xi_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & -\alpha_n \end{pmatrix} \text{ for } n \geq 0, \quad (3.8)$$

where the coefficients α_n , β_n , and γ_n are smooth functions of x . The tracelessness of ξ_n reflects the fact that the Lax matrices $U(\lambda)$ and $V(\lambda)$ lie in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The formal series $\xi(\lambda)$ is required to satisfy the auxiliary linear equation

$$\frac{d}{dx} \xi = [U(\lambda), \xi], \quad (3.9)$$

which is compatible with the Lax pair and preserves the algebraic structure of the coefficients.

Remark 3.1. The Lax equation (3.9) is linear in ξ : if ξ is a solution and $c \in \mathbb{K}$ is a constant, then $c\xi$ is also a solution. More generally, any $\mathbb{K}$ -linear combination of solutions is again a solution.

Substituting the Laurent expansion (3.8) into (3.9) and comparing coefficients of equal powers of λ yields an infinite system of coupled differential equations for the functions α_n , β_n , and γ_n . These relations will be derived explicitly and will lead to a recursive determination of all coefficients in Section 3.3. The formal Laurent series (3.8) plays the same role for the KdV equation as the polynomial Killing fields in the work of Pinkall and Sterling: it encodes the integrable structure of the equation and provides a natural framework for the notion of finite-type potentials.

Remark 3.2. The formal series $\xi(x, \lambda)$ is understood as a formal Laurent series in λ^{-1} with a pole of order at most 1, i.e. as an element of $\lambda \cdot \mathbb{K}[[\lambda^{-1}]]$:

$$\xi(x, \lambda) = \xi_{-1}\lambda + \sum_{n=0}^{\infty} \xi_n(x)\lambda^{-n}.$$

The specific form of ξ_{-1} is determined by substituting the Laurent series into the Lax equation (3.9) and comparing coefficients of λ^1 . This gives:

$$\frac{d}{dx} \xi_{-1} = \left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \xi_{-1} \right].$$

Writing $\xi_{-1} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ for a general traceless matrix and computing the commutator explicitly:

$$\left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & -2a \\ 0 & b \end{pmatrix}.$$

For $\frac{d}{dx}\xi_{-1} = \begin{pmatrix} -b & -2a \\ 0 & b \end{pmatrix}$ to have a solution with constant ξ_{-1} (as required by the formal series ansatz), we need $b = 0$ and $a = 0$. Therefore the only traceless solution is:

$$\xi_{-1} = \begin{pmatrix} 0 & 0 \\ \gamma_{-1} & 0 \end{pmatrix}$$

for a constant $\gamma_{-1} \in \mathbb{K}$. The value $\gamma_{-1} = -1$ is then fixed by the normalization $\beta_0 = 1$, as shown in Section 3.4.

Remark 3.3 (Relation to products of eigenfunctions). Fix $\lambda \in \mathbb{K}$ and let y_1, y_2 be two linearly independent solutions of $Ly = \lambda y$ with constant Wronskian $W(y_1, y_2) := y_1 y_2' - y_1' y_2 = 1$. Then the traceless matrix

$$\eta(x, \lambda) = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \quad \beta := y_1 y_2, \quad \alpha := -\frac{1}{2}(y_1 y_2)' = -\frac{1}{2}(y_1' y_2 + y_1 y_2'), \quad \gamma := -y_1' y_2',$$

is a solution of the Lax equation (3.9) at this fixed λ . Indeed, $\beta' = y_1' y_2 + y_1 y_2' = -2\alpha$ by definition of α , which is the second component equation of (3.10)-(3.12). Differentiating α once more and using $y_1'' = (q - \lambda)y_1$, $y_2'' = (q - \lambda)y_2$ gives

$$\alpha' = -\frac{1}{2}(y_1'' y_2 + 2y_1' y_2' + y_1 y_2'') = -(q - \lambda)y_1 y_2 - y_1' y_2' = \gamma - (q - \lambda)\beta,$$

which is the first component equation. Finally

$$\gamma' = -y_1'' y_2' - y_1' y_2'' = -(q - \lambda)(y_1 y_2' + y_1' y_2) = -(q - \lambda)\beta' = 2\alpha(q - \lambda),$$

which is the third component equation. By linearity of (3.9), every $\mathbb{K}$ -linear combination of such matrices, built from any pairs of solutions of $Ly = \lambda y$, again solves (3.9) at that λ .

This connects the abstract construction of $\xi(x, \lambda)$ to the classical objects of finite-gap theory: solutions of the Lax equation (3.9) at a fixed λ arise from quadratic combinations $y_1^2, y_1 y_2, y_2^2$ of solutions of the spectral problem $Ly = \lambda y$. Every solution of (3.9) at fixed λ arises this way is the classical converse statement (see [9]), which we do not reprove here. The normalized solution ξ constructed in Section 3.4, and in particular the explicit formula

$$\xi(x, \lambda) = \frac{1}{\sqrt{f(\lambda)}} F(x, \lambda) (2M(\lambda) - \Delta(\lambda)I_2) F^{-1}(x, \lambda)$$

obtained in the proof of Theorem 5.14, is a specific instance of this correspondence, built from the fundamental solutions y_1, y_2 of Definition 5.4, the entire factor $1/\sqrt{f(\lambda)}$ there renormalizes it so that $\det \xi = \lambda$ globally in λ , as required by Section 3.4.2.

3.3 The recursion formulas for α_n , β_n and γ_n

We will now start with the explicit derivation of recursion formulas for the coefficients α_n , β_n and γ_n .

3.3.1 Derivation of the coefficient equations

We start from equation (3.9), which requires for the right-hand side:

$$\begin{aligned}
 & \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix} \cdot \xi(x, \lambda) - \xi(x, \lambda) \cdot \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \gamma & -\alpha \\ \alpha \cdot (q(x) - \lambda) & \beta \cdot (q(x) - \lambda) \end{pmatrix} - \begin{pmatrix} \beta \cdot (q(x) - \lambda) & \alpha \\ -\alpha \cdot (q(x) - \lambda) & \gamma \end{pmatrix} \\
 &= \begin{pmatrix} \gamma - \beta \cdot (q(x) - \lambda) & -2\alpha \\ 2\alpha \cdot (q(x) - \lambda) & -(\gamma - \beta \cdot (q(x) - \lambda)) \end{pmatrix}.
 \end{aligned}$$

Since the derivative of $\xi(x, \lambda)$ with respect to x is a component-wise derivative, we conclude that

$$\begin{aligned}
 \gamma - \beta \cdot (q(x) - \lambda) &= \alpha'(x, \lambda), \\
 -2\alpha &= \beta'(x, \lambda), \\
 2\alpha \cdot (q(x) - \lambda) &= \gamma'(x, \lambda) \text{ and} \\
 -(\gamma - \beta \cdot (q(x) - \lambda)) &= -\alpha'(x, \lambda)
 \end{aligned}$$

Note that the last equation coincides with the negative of the first, and is therefore redundant.

Rewriting these equations yields:

$$\begin{aligned}
 \beta' &= -2\alpha, \\
 \alpha' &= \gamma - \beta \cdot (q(x) - \lambda) \text{ and} \\
 \gamma' &= 2\alpha \cdot (q(x) - \lambda)
 \end{aligned}$$

On the other hand, we can look at the Laurent expansion (3.8). The formula for ξ can be translated into formulas for the components, e.g.

$$\begin{aligned}
 \alpha(x, \lambda) &= \sum_{n=0}^{\infty} \alpha_n(x) \lambda^{-n}, \\
 \beta(x, \lambda) &= \sum_{n=0}^{\infty} \beta_n(x) \lambda^{-n} \text{ and} \\
 \gamma(x, \lambda) &= \gamma_{-1} \lambda + \sum_{n=0}^{\infty} \gamma_n(x) \lambda^{-n}
 \end{aligned}$$

In view of Theorem 4.8, we will subsequently restrict attention to solutions ξ for which the Laurent series terminates, i.e., for which only finitely many coefficients β_n are non-zero. In this case, the sums above are finite and run from $n = 0$ to N for some $N \in \mathbb{N}_0$.

3.3.2 Recursion formulas by Coefficient comparison

We begin with the first equation. Substituting the Laurent expansions of α , β and γ into $\alpha' = \gamma - (q - \lambda)\beta$ yields:

$$\begin{aligned}
\alpha' &= \sum_{n=0}^{\infty} \alpha'_n(x) \lambda^{-n} \\
&\stackrel{!}{=} \gamma - (q - \lambda)\beta \\
&= \left(\gamma_{-1}\lambda + \sum_{n=0}^{\infty} \gamma_n(x) \lambda^{-n} \right) - (q - \lambda) \left(\sum_{n=0}^{\infty} \beta_n(x) \lambda^{-n} \right) \\
&= \gamma_{-1}\lambda + \sum_{n=0}^{\infty} \gamma_n(x) \lambda^{-n} - q \sum_{n=0}^{\infty} \beta_n(x) \lambda^{-n} + \sum_{n=0}^{\infty} \beta_n(x) \lambda^{1-n} \\
&= \gamma_{-1}\lambda + \sum_{n=0}^{\infty} \gamma_n(x) \lambda^{-n} - \sum_{n=0}^{\infty} (q\beta_n)(x) \lambda^{-n} + \sum_{n=0}^{\infty} \beta_n(x) \lambda^{1-n}.
\end{aligned}$$

Grouping by powers of λ (each power of λ^{-i} gets its own coefficient) gives on the right-hand side:

$$\begin{aligned}
\lambda^1 : & \quad \gamma_{-1} + \beta_0, \\
\lambda^0 : & \quad \gamma_0 - q\beta_0 + \beta_1, \\
\lambda^{-1} : & \quad \gamma_1 - q\beta_1 + \beta_2, \\
& \quad \vdots \\
\lambda^{-i} : & \quad \gamma_i - q\beta_i + \beta_{i+1}.
\end{aligned}$$

The left-hand side is already grouped:

$$\alpha' = \sum_{n=0}^{\infty} \alpha'_n(x) \lambda^{-n}.$$

Comparing coefficients yields:

$$\begin{aligned}
\lambda^1 : & \quad 0 = \gamma_{-1} + \beta_0, \\
\lambda^0 : & \quad \alpha'_0 = \gamma_0 - q\beta_0 + \beta_1, \\
\lambda^{-1} : & \quad \alpha'_1 = \gamma_1 - q\beta_1 + \beta_2, \\
& \quad \vdots \\
\lambda^{-i} : & \quad \alpha'_i = \gamma_i - q\beta_i + \beta_{i+1}.
\end{aligned}$$

The last line gives the inductive recursion:

$$\alpha'_i = \gamma_i - q\beta_i + \beta_{i+1}. \tag{3.10}$$

For the second equation $\beta' = -2\alpha$, substituting the Laurent expansions yields:

$$\beta' = \sum_{n=0}^{\infty} \beta'_n(x) \lambda^{-n} \stackrel{!}{=} -2\alpha = -2 \sum_{n=0}^{\infty} \alpha_n(x) \lambda^{-n}.$$

Comparing coefficients immediately gives:

$$\beta'_n = -2\alpha_n, \quad \text{and hence} \quad \alpha_n = -\frac{1}{2}\beta'_n. \quad (3.11)$$

For the third equation $\gamma' = 2\alpha(q - \lambda)$, substituting the Laurent expansions yields:

$$\begin{aligned} \gamma' &= \gamma'_{-1}\lambda + \sum_{n=0}^{\infty} \gamma'_n(x) \lambda^{-n} \\ &\stackrel{!}{=} 2\alpha(q - \lambda) \\ &= 2 \sum_{n=0}^{\infty} (\alpha_n(x) \cdot q) \lambda^{-n} - 2 \sum_{n=0}^{\infty} \alpha_n(x) \lambda^{1-n}. \end{aligned}$$

Grouping by powers of λ :

$$\begin{aligned} \lambda^1 : & \quad -2\alpha_0, \\ \lambda^0 : & \quad 2q\alpha_0 - 2\alpha_1, \\ & \quad \vdots \\ \lambda^{-n} : & \quad 2q\alpha_n - 2\alpha_{n+1}. \end{aligned}$$

Comparing coefficients yields the recursion:

$$\gamma'_n = 2q\alpha_n - 2\alpha_{n+1}. \quad (3.12)$$

We now express α_n and γ_n in terms of β_n . From (3.11) we have $\alpha_n = -\frac{1}{2}\beta'_n$. Substituting into (3.10):

$$\alpha'_n = -\frac{1}{2}\beta''_n \stackrel{!}{=} \gamma_n - q\beta_n + \beta_{n+1},$$

which gives:

$$\gamma_n = -\frac{1}{2}\beta''_n + q\beta_n - \beta_{n+1}. \quad (3.13)$$

It remains to derive the recursion for β_{n+1} . Differentiating (3.13) with respect to x and using (3.12):

$$\begin{aligned} \gamma'_n &= -\frac{1}{2}\beta'''_n + q' \cdot \beta_n + q\beta'_n - \beta'_{n+1} \\ &\stackrel{!}{=} 2q\alpha_n - 2\alpha_{n+1} \\ &= 2q \left(-\frac{1}{2}\beta'_n \right) - 2 \left(-\frac{1}{2}\beta'_{n+1} \right) \\ &= -q\beta'_n + \beta'_{n+1}. \end{aligned}$$

Rearranging gives:

$$2\beta'_{n+1} = -\frac{1}{2}\beta_n''' + q' \cdot \beta_n + 2q\beta'_n,$$

and hence:

$$\beta'_{n+1} = -\frac{1}{4}\beta_n''' + q\beta'_n + \frac{1}{2}q' \cdot \beta_n. \quad (3.14)$$

Note that (3.14) determines β'_{n+1} uniquely from β_n , and hence β_{n+1} only up to an integration constant. Consequently, γ_n is also determined only up to an integration constant via (3.13), while α_n is uniquely determined by β_n via (3.11). The precise choice of integration constants, and how it leads to the unique solution ξ with $\det \xi = \lambda$, will be discussed in detail in Section 3.4.

3.4 Normalization and Differential Polynomials

Having derived the recursion relations for the coefficients α_n , β_n and γ_n in Section 3.3, we now turn to the question of uniqueness. We show that a suitable normalization of the solution ξ leads to a unique sequence of coefficients $(\beta_n)_{n \in \mathbb{N}_0}$, and that these coefficients are differential polynomials in q .

3.4.1 Normalization of the solution

Since the Lax equation $\frac{d}{dx}\xi = [U(\lambda), \xi]$ is linear, if $\xi(x, \lambda)$ is a solution, then so is $\lambda^k \xi(x, \lambda)$ for any $k \in \mathbb{Z}$. Indeed, since the derivative does not act on λ and the commutator is bilinear:

$$\frac{d}{dx}(\lambda^k \xi) = \lambda^k \frac{d}{dx}\xi = \lambda^k [U(\lambda), \xi] = [U(\lambda), \lambda^k \xi].$$

By multiplying ξ with a suitable power of λ , we may therefore assume that $\beta_n = 0$ for $n < 0$ and $\beta_0 \neq 0$, where $n = 0$ is the smallest index such that $\beta_n \neq 0$. Setting $n = -1$ in the recursion relation (3.14):

$$\beta'_0 = -\frac{1}{4}\beta_{-1}''' + q\beta'_{-1} + \frac{1}{2}q' \cdot \beta_{-1} = 0,$$

since $\beta_{-1} = 0$. Therefore β_0 is constant. Without loss of generality we set $\beta_0 = 1$.

Furthermore, setting $n = -1$ in (3.13):

$$\gamma_{-1} = -\frac{1}{2}\beta_{-1}'' + q\beta_{-1} - \beta_0 = -\beta_0 = -1.$$

Thus $\gamma_{-1} = -1$ is also a constant. All other solutions of the recursion are then obtained from this normalized solution by choosing constants $c_0, c_1, c_2, \dots \in \mathbb{K}$ and setting $\tilde{\beta}_n := \beta_n + \sum_{l=0}^{n-1} c_l \beta_l$ for each $n \in \mathbb{N}_0$, as shown in Lemma 4.7.

3.4.2 The determinant of ξ

We show that $\det \xi(x, \lambda)$ is independent of x by direct computation, without assuming invertibility. Writing $\det \xi = -\alpha^2 - \beta\gamma$ and differentiating with respect to x using the component equations (3.10)-(3.12):

$$\begin{aligned} \frac{d}{dx} \det \xi &= -2\alpha\alpha' - \beta' \cdot \gamma - \beta \cdot \gamma' \\ &= -2\alpha(\gamma - (q - \lambda)\beta) - (-2\alpha)\gamma - \beta \cdot 2\alpha(q - \lambda) \\ &= -2\alpha\gamma + 2\alpha(q - \lambda)\beta + 2\alpha\gamma - 2\alpha\beta(q - \lambda) \\ &= 0. \end{aligned}$$

Therefore $\det \xi(x, \lambda)$ is constant in x . We now determine the explicit form of $\det \xi$. Computing the determinant:

$$\det(\xi(x, \lambda)) = - \left(\sum_{n=0}^{\infty} \alpha_n \lambda^{-n} \right)^2 - \left(\gamma_{-1} \lambda + \sum_{n=0}^{\infty} \gamma_n \lambda^{-n} \right) \left(\sum_{n=0}^{\infty} \beta_n \lambda^{-n} \right).$$

The term of highest order in λ comes from $-\gamma_{-1}\lambda \cdot \beta_0 = -\gamma_{-1}\beta_0\lambda$. Since $\gamma_{-1} = -1$ and $\beta_0 = 1$, this gives λ . All remaining terms have non-positive powers of λ . Therefore:

$$\det \xi(x, \lambda) = \lambda + p(\lambda^{-1}),$$

where $p(\lambda^{-1}) = \det \xi(x, \lambda) - \lambda \in \mathbb{K}[[\lambda^{-1}]]$ is independent of x . Since $\det \xi(x, \lambda) = \lambda + p(\lambda^{-1})$ is non-zero for $\lambda \neq 0$, ξ is invertible for $\lambda \neq 0$. We now construct a normalized solution $\tilde{\xi}$ with $\det \tilde{\xi} = \lambda$ as follows.

Since $(1+x)^{-\frac{1}{2}}$ is analytic at $x=0$, there exists a formal power series:

$$r(t) = 1 + \sum_{\ell=1}^{\infty} \binom{-\frac{1}{2}}{\ell} t^{\ell} p^{\ell}(t) = (1 + tp(t))^{-\frac{1}{2}} \in 1 + t\mathbb{K}[[t]],$$

satisfying $r(t)^2(1 + tp(t)) = 1$, i.e., $r(\lambda^{-1})^2(1 + \lambda^{-1}p(\lambda^{-1})) = 1$. By linearity of the Lax equation, $\tilde{\xi}(x, \lambda) := r(\lambda^{-1})\xi(x, \lambda)$ is again a solution. Its determinant satisfies:

$$\begin{aligned} \det \tilde{\xi}(x, \lambda) &= r(\lambda^{-1})^2 \det \xi(x, \lambda) \\ &= r(\lambda^{-1})^2 (\lambda + p(\lambda^{-1})) \\ &= r(\lambda^{-1})^2 \lambda (1 + \lambda^{-1}p(\lambda^{-1})) \\ &= \lambda \cdot r(\lambda^{-1})^2 (1 + \lambda^{-1}p(\lambda^{-1})) \\ &= \lambda \cdot 1 = \lambda. \end{aligned}$$

Thus $\tilde{\xi}$ is the unique solution of the Lax equation with $\det \tilde{\xi}(x, \lambda) = \lambda$ and $\beta_0 = 1$, which we again denote by ξ .

3.4.3 The β_n as differential polynomials

The normalized solution $\tilde{\xi}$ with $\det \tilde{\xi} = \lambda$, which we again denote by ξ , satisfies:

$$\det \xi = -\alpha^2 - \beta\gamma \stackrel{!}{=} \lambda.$$

We first express γ as a function of β and λ . From Section 3.3, we know that $\gamma_n = -\frac{1}{2}\beta_n'' + q\beta_n - \beta_{n+1}$. Summing over all n with $\gamma_{-1} = -1$:

$$\begin{aligned} \gamma &= \gamma_{-1}\lambda + \sum_{n=0}^{\infty} \gamma_n \lambda^{-n} \\ &= -\lambda + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\beta_n'' + q\beta_n - \beta_{n+1} \right) \lambda^{-n} \\ &= -\lambda + q\beta - \frac{1}{2}\beta'' - \lambda(\beta - \beta_0) \\ &= -\lambda\beta + q\beta - \frac{1}{2}\beta'' \\ &= (q - \lambda)\beta - \frac{1}{2}\beta'', \end{aligned}$$

where we used $\beta_0 = 1$ and the index shift $\sum_{n=0}^{\infty} \beta_{n+1} \lambda^{-n} = \lambda(\beta - \beta_0) = \lambda(\beta - 1)$. Therefore:

$$\begin{aligned} -\alpha^2 - \beta\gamma &= -\left(\frac{1}{2}\beta'\right)^2 - \beta \left((q - \lambda)\beta - \frac{1}{2}\beta'' \right) \\ &= -\frac{1}{4}(\beta')^2 + \frac{1}{2}\beta\beta'' - (q - \lambda)\beta^2 \\ &\stackrel{!}{=} \lambda. \end{aligned}$$

Term $-\frac{1}{4}(\beta')^2$:

$$-\frac{1}{4}(\beta')^2 = -\frac{1}{4} \left(\sum_{n=0}^{\infty} \beta_n' \lambda^{-n} \right)^2 = -\frac{1}{4} \sum_{m=0}^{\infty} \left(\sum_{i=0}^m \beta_i' \beta_{m-i}' \right) \lambda^{-m}.$$

Term $\frac{1}{2}\beta\beta''$:

$$\frac{1}{2}\beta\beta'' = \frac{1}{2} \left(\sum_{n=0}^{\infty} \beta_n \lambda^{-n} \right) \left(\sum_{n=0}^{\infty} \beta_n'' \lambda^{-n} \right) = \frac{1}{2} \sum_{m=0}^{\infty} \left(\sum_{i=0}^m \beta_i \beta_{m-i}'' \right) \lambda^{-m}.$$

Term $-(q - \lambda)\beta^2$:

$$-(q - \lambda)\beta^2 = \lambda\beta^2 - q\beta^2.$$

For the first part:

$$\lambda\beta^2 = \lambda \sum_{m=0}^{\infty} \left(\sum_{i=0}^m \beta_i \beta_{m-i} \right) \lambda^{-m} = \sum_{m=0}^{\infty} \left(\sum_{i=0}^m \beta_i \beta_{m-i} \right) \lambda^{1-m}.$$

Applying the index shift $k = m - 1$:

$$\lambda\beta^2 = \sum_{k=-1}^{\infty} \left(\sum_{i=0}^{k+1} \beta_i \beta_{k+1-i} \right) \lambda^{-k}.$$

For $k = -1$ this gives $\beta_0^2 \lambda = \lambda$, which matches the right-hand side. For the second part:

$$-q\beta^2 = -q \sum_{m=0}^{\infty} \left(\sum_{i=0}^m \beta_i \beta_{m-i} \right) \lambda^{-m}.$$

Coefficient comparison for λ^{-n} , $n \geq 0$:

Since the right-hand side λ has only a term at λ^1 , all coefficients of λ^{-n} for $n \geq 0$ must vanish on the left-hand side:

$$-\frac{1}{4} \sum_{i=0}^n \beta'_i \beta'_{n-i} + \frac{1}{2} \sum_{i=0}^n \beta_i \beta''_{n-i} + \sum_{i=0}^{n+1} \beta_i \beta_{n+1-i} - q \sum_{i=0}^n \beta_i \beta_{n-i} \stackrel{!}{=} 0.$$

We isolate the terms containing β_{n+1} in the third sum. Since $\beta_0 = 1$, the terms with $i = 0$ and $i = n + 1$ give $2\beta_0\beta_{n+1} = 2\beta_{n+1}$. Note that β_{n+1} does not appear in any of the other three sums, since their indices run only up to n . Therefore the coefficient of β_{n+1} on the left-hand side is exactly $2 \neq 0$, and we can solve uniquely for β_{n+1} :

$$\beta_{n+1} = \sum_{i=0}^n \left(\frac{1}{8} \beta'_i \beta'_{n-i} - \frac{1}{4} \beta_i \beta''_{n-i} + \frac{1}{2} q \beta_i \beta_{n-i} \right) - \sum_{i=1}^n \frac{1}{2} \beta_i \beta_{n+1-i}. \quad (3.15)$$

This formula determines β_{n+1} *algebraically* from $\beta_0, \dots, \beta_n$, without integration. Since $\beta_0 = 1$ is a differential polynomial in q (trivially), it follows by induction that all β_n are differential polynomials in q , i.e., polynomials in q and finitely many of its derivatives. The first few are:

$$\begin{aligned} \beta_1 &= \frac{1}{2}q, \\ \beta_2 &= -\frac{1}{8}q'' + \frac{3}{8}q^2, \\ \beta_3 &= \frac{1}{32}q^{(4)} - \frac{5}{16}qq'' - \frac{5}{32}(q')^2 + \frac{5}{16}q^3. \end{aligned}$$

3.4.4 Connection to the Gelfand-Dikii polynomials

Having established that the β_n are differential polynomials in q , we now show that they coincide with the homogeneous Gelfand-Dikii polynomials $\hat{f}_n$, which will be introduced in Section 3.5.

Both sequences satisfy the same recursion relation (3.14) with the same initial condition $\beta_0 = f_0 = 1$ and with all integration constants set to zero.

For the β_n , this is enforced by the determinant condition (3.15), which determines β_{n+1} algebraically from $\beta_0, \dots, \beta_n$ without integration. For the $\hat{f}_n$, it holds by definition.

We verify that both sequences coincide by induction. The base case $\beta_0 = f_0 = 1$ holds by normalization. For the inductive step, assume $\beta_l = \hat{f}_l$ for all $l \leq n$. By Section 3.4, β_{n+1} is determined algebraically from $\beta_0, \dots, \beta_n$ via (3.15) without integration constant. By the derivation in Section 3.3, β_{n+1} also satisfies the linear recursion (3.14), so by the inductive hypothesis:

$$\beta'_{n+1} = -\frac{1}{4}\hat{f}_n''' + q\hat{f}_n' + \frac{1}{2}q' \cdot \hat{f}_n = \hat{f}'_{n+1}.$$

Therefore $\beta_{n+1} - \hat{f}_{n+1}$ is a constant. Since β_{n+1} is determined algebraically by (3.15) without integration constant, and $\hat{f}_{n+1}$ is the antiderivative of $\hat{f}'_{n+1}$ with $c_{n+1} = 0$, the algebraic formula (3.15) produces a differential polynomial with no free constant term (since all terms involve derivatives or products of $\beta_0, \dots, \beta_n$, which are differential polynomials with no free constant term by induction). Therefore the constant is zero, and $\beta_{n+1} = \hat{f}_{n+1}$. By induction, both sequences coincide:

$$\beta_n = \hat{f}_n = f_n|_{c_k=0, k=1, \dots, n}, \quad n \in \mathbb{N}_0. \quad (3.16)$$

In view of Theorem 4.8, we will subsequently restrict attention to solutions ξ for which the Laurent series terminates, i.e., for which only finitely many coefficients β_n are non-zero. In this case, the sums in the Laurent expansions of α , β and γ are finite and run from $n = 0$ to N for some $N \in \mathbb{N}_0$:

$$\begin{aligned} \alpha(x, \lambda) &= \sum_{n=0}^N \alpha_n(x) \lambda^{-n}, \\ \beta(x, \lambda) &= \sum_{n=0}^N \beta_n(x) \lambda^{-n}, \\ \gamma(x, \lambda) &= \gamma_{-1}\lambda + \sum_{n=0}^N \gamma_n(x) \lambda^{-n}. \end{aligned}$$

This finite-type condition is precisely the subject of Chapter 4.

3.5 Differential Polynomials and the Gelfand-Dikii Structure

In this section we connect the formal Laurent expansion of the matrix Lax pair developed in Sections 3.1-3.4 with the classical scalar Lax formulation of the KdV hierarchy. The key objects are the differential polynomials introduced by Gelfand and Dikii, which encode the structure of the hierarchy and appear naturally when studying the commutator $[P, L]$ for differential operators P commuting with the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q(x).$$

As shown in Section 3.4, the coefficients β_n of the formal Laurent series solution ξ are differential polynomials in q , determined by the nonlinear recursion (3.15), and coincide with the homogeneous Gelfand-Dikii polynomials $\hat{f}_n$, as established in Section 3.4. The Gelfand-Dikii polynomials f_ℓ introduced below satisfy the same recursion relation as the β_n , namely (3.14).

3.5.1 The Gelfand-Dikii Recursion

We define a sequence of differential polynomials $\{f_\ell\}_{\ell \geq 0}$ by

$$f_0 = 1, \quad f'_\ell = -\frac{1}{4}f'''_{\ell-1} + qf'_{\ell-1} + \frac{1}{2}q' \cdot f_{\ell-1}, \quad \ell \geq 1. \quad (3.17)$$

This recursion arises naturally when the commutator $[P_{2n+1}, L]$ does not contain negative powers of $\frac{d}{dx}$. The first few polynomials are

$$\begin{aligned} f_1 &= \frac{1}{2}q, \\ f_2 &= -\frac{1}{8}q'' + \frac{3}{8}q^2, \\ f_3 &= \frac{1}{32}q^{(4)} - \frac{5}{16}qq'' - \frac{5}{32}(q')^2 + \frac{5}{16}q^3. \end{aligned}$$

Comparing with the coefficients $\beta_1, \beta_2, \beta_3$ computed in Section 3.4.3, one observes that $\beta_n = \hat{f}_n$ for $n = 1, 2, 3$, where $\hat{f}_n = f_n|_{c_k=0, k=1, \dots, n}$ denotes the homogeneous Gelfand-Dikii polynomial. This has been established in general in Section 3.4.

3.5.2 Construction of the Operators P_{2n+1}

Using the polynomials f_ℓ we define differential operators of odd order

$$P_{2n+1} = \sum_{\ell=0}^n \left(f_{n-\ell} \frac{d}{dx} - \frac{1}{2} f'_{n-\ell} \right) L^\ell, \quad n \in \mathbb{N}_0. \quad (3.18)$$

The specific combination

$$f \frac{d}{dx} - \frac{1}{2} f'$$

is chosen so that the commutator $[P_{2n+1}, L]$ contains no negative powers of $\frac{d}{dx}$. This ensures that the Lax equation

$$L_{t_n} = [P_{2n+1}, L]$$

produces a local evolution equation for q .

3.5.3 Explicit Computation of P_1 , P_3 , and P_5

The operator P_1 For $n = 0$ we obtain

$$P_1 = \frac{d}{dx}.$$

The operator P_3 For $n = 1$ we have

$$P_3 = \left(f_1 \frac{d}{dx} - \frac{1}{2} f_1' \right) + \left(f_0 \frac{d}{dx} \right) L.$$

Using $f_1 = \frac{1}{2}q$, $f_1' = \frac{1}{2}q'$, $f_0 = 1$, and

$$\frac{d}{dx} L = -\frac{d^3}{dx^3} + q' + q \frac{d}{dx},$$

we obtain

$$P_3 = -\frac{d^3}{dx^3} + \frac{3}{2}q \frac{d}{dx} + \frac{3}{4}q'. \quad (3.19)$$

Note that $P = 4P_3$, where P is the operator introduced in (2.12); the factor 4 accounts precisely for the rescaling of the time variable t mentioned in Section 3.5.4 below, and reconciles the fact that $[P_3, L] = q_t$ reproduces (2.1) exactly (Section 3.5.4), whereas $[P, L] = 4[P_3, L]$ reproduces it only up to this rescaling by the factor 4.

The operator P_5 For $n = 2$ we write

$$P_5 = A_0 + A_1 + A_2,$$

where

$$A_0 = f_2 \frac{d}{dx} - \frac{1}{2} f_2', \quad A_1 = \left(f_1 \frac{d}{dx} - \frac{1}{2} f_1' \right) L, \quad A_2 = \frac{d}{dx} L^2.$$

A direct computation yields

$$\begin{aligned} P_5 = & \frac{d^5}{dx^5} - \frac{5}{2}q \frac{d^3}{dx^3} - \frac{15}{4}q' \cdot \frac{d^2}{dx^2} \\ & + \left(\frac{15}{8}q^2 - \frac{25}{8}q'' \right) \frac{d}{dx} + \left(-\frac{15}{16}q''' + \frac{15}{8}qq' \right). \end{aligned}$$

3.5.4 The Commutator $[P_3, L]$ and the KdV Equation

Using the operator identities

$$\begin{aligned}\left[\frac{d}{dx}, q\right] &= q', \\ \left[\frac{d^2}{dx^2}, q\right] &= 2q' \cdot \frac{d}{dx} + q'', \\ \left[\frac{d^3}{dx^3}, q\right] &= 3q' \cdot \frac{d^2}{dx^2} + 3q'' \cdot \frac{d}{dx} + q''',\end{aligned}$$

one finds that all terms of order $\left(\frac{d}{dx}\right)^k$ with $k \geq 1$ cancel in the commutator $[P_3, L]$, leaving

$$[P_3, L] = -\frac{1}{4}q''' + \frac{3}{2}qq'.$$

Since $L_t = q_t$, the Lax equation $L_t = [P_3, L]$ yields

$$q_t = -\frac{1}{4}q''' + \frac{3}{2}qq',$$

which is the KdV equation up to a rescaling of the time variable.

3.5.5 The KdV Hierarchy

The general evolution equation generated by P_{2n+1} is

$$q_{t_n} = \frac{d}{dx} f_{n+1}(q).$$

The first few flows are

$$\begin{aligned}q_{t_1} &= q', \\ q_{t_2} &= -\frac{1}{4}q''' + \frac{3}{2}qq', \\ q_{t_3} &= \frac{1}{16}q^{(5)} - \frac{5}{8}qq''' - \frac{5}{4}q'q'' + \frac{15}{8}q^2q'.$$

This infinite family of compatible flows is called the *KdV hierarchy*. It will play a central role in the characterization of finite-gap potentials in Chapter 4.

The stationary KdV equations $\frac{d}{dx} f_{n+1}(q) = 0$ are therefore equivalent to $\beta'_{n+1} = 0$, i.e., β_{n+1} is constant. As discussed in Section 3.4.4, this is precisely the finite-type condition of Theorem 4.8.

Chapter 4

Potentials of Finite Type

In this chapter we develop the central theory of this thesis. Building on the formal Laurent series and recursion relations established in Chapter 3, we introduce the notion of a potential of finite type and characterize it in four equivalent ways. The key result is Theorem 4.8, which establishes the equivalence of these characterisations. The proof relies on several preparatory lemmas concerning commutator identities and the structure of the differential operator P_{2N+1} , which we develop in the subsequent sections. Recall from Chapter 3 that the unique solution $\xi(x, \lambda)$ of the Lax equation (3.9) with $\det \xi = \lambda$ and $\beta_0 = 1$ gives rise to a sequence of differential polynomials $(\beta_n)_{n \in \mathbb{N}_0}$ in q , determined recursively by (3.14)-(3.15). We now use this to single out a special class of potentials.

4.1 Commutator identities

The proof of Theorem 4.8 requires a number of commutator identities for differential operators. We collect these in the following lemma, which will be applied repeatedly in the subsequent sections. We will not prove the identities here.

Lemma 4.1 (Commutator identities). *Let $q \in C^\infty(I, \mathbb{K})$ and let $f \in C^\infty(I, \mathbb{K})$. Then the following commutator identities hold as identities of differential operators:*

$$\left[\frac{d}{dx}, f \right] = f', \quad (4.1)$$

$$\left[\frac{d^2}{dx^2}, f \right] = f'' + 2f' \frac{d}{dx}, \quad (4.2)$$

$$\left[f \frac{d}{dx}, -\frac{d^2}{dx^2} \right] = f'' \frac{d}{dx} + 2f' \frac{d^2}{dx^2}, \quad (4.3)$$

$$\left[f \frac{d}{dx}, q \right] = f q', \quad (4.4)$$

$$\left[\frac{1}{2} f', L \right] = \frac{1}{2} f''' + f'' \frac{d}{dx}, \quad (4.5)$$

where $L = -\frac{d^2}{dx^2} + q$ and $f' = \frac{d}{dx}f$ denotes the derivative of f with respect to x .

4.2 The commutator $[P_{2n+1}, L]$

Having established the necessary commutator identities, we now turn to the computation of the commutator of the differential operator P_{2N+1} with L . This result is the key ingredient in the proof of the equivalence of conditions (ii) and (iv) in Theorem 4.8.

Lemma 4.2. *Let $q \in C^\infty(I, \mathbb{K})$ and let P_{2N+1} be the differential operator defined by*

$$P_{2N+1} = \sum_{\ell=0}^N \left(\beta_{N-\ell} \frac{d}{dx} - \frac{1}{2} \beta'_{N-\ell} \right) L^\ell, \quad (4.6)$$

where $L = -\frac{d^2}{dx^2} + q$ and $(\beta_n)_{n \in \mathbb{N}_0}$ is the unique solution of (3.14)-(3.15) with $\beta_0 = 1$. Then:

$$[P_{2N+1}, L] = 2\beta'_{N+1}. \quad (4.7)$$

Proof. We compute the commutator $[P_{2N+1}, L]$ directly. Using the general identity $[AB, C] = A[B, C] + [A, C]B$, we first observe that for any differential operator A :

$$[A \cdot L^\ell, L] = A[L^\ell, L] + [A, L]L^\ell = [A, L]L^\ell, \quad (4.8)$$

since L^ℓ and L commute. Therefore:

$$[P_{2N+1}, L] = \sum_{\ell=0}^N \left[\beta_{N-\ell} \frac{d}{dx} - \frac{1}{2} \beta'_{N-\ell}, L \right] L^\ell. \quad (4.9)$$

It remains to compute the commutator of a single term with L . For a smooth function $f \in C^\infty(I, \mathbb{K})$ we compute:

$$\left[f \frac{d}{dx} - \frac{1}{2} f', L \right] = \left[f \frac{d}{dx}, L \right] - \left[\frac{1}{2} f', L \right]. \quad (4.10)$$

For the first term, we split $L = -\frac{d^2}{dx^2} + q$ and apply Lemma 4.1:

$$\begin{aligned} \left[f \frac{d}{dx}, L \right] &= \left[f \frac{d}{dx}, -\frac{d^2}{dx^2} \right] + \left[f \frac{d}{dx}, q \right] \\ &= f'' \frac{d}{dx} + 2f' \frac{d^2}{dx^2} + f q', \end{aligned} \quad (4.11)$$

where we used identities (4.3) and (4.4). For the second term, we apply identity (4.5):

$$\left[\frac{1}{2} f', L \right] = \frac{1}{2} f''' + f'' \frac{d}{dx}. \quad (4.12)$$

Subtracting yields:

$$\begin{aligned} \left[f \frac{d}{dx} - \frac{1}{2} f', L \right] &= f'' \frac{d}{dx} + 2f' \frac{d^2}{dx^2} + f q' - \frac{1}{2} f''' - f'' \frac{d}{dx} \\ &= 2f' \frac{d^2}{dx^2} + f q' - \frac{1}{2} f'''. \end{aligned} \quad (4.13)$$

We now replace $\frac{d^2}{dx^2} = q - L$ to obtain:

$$\begin{aligned} \left[f \frac{d}{dx} - \frac{1}{2} f', L \right] &= 2f'(q - L) + f q' - \frac{1}{2} f''' \\ &= -2f' L + 2f' q + f q' - \frac{1}{2} f'''. \end{aligned} \quad (4.14)$$

Substituting $f = \beta_{N-\ell}$ and summing over ℓ :

$$\begin{aligned} [P_{2N+1}, L] &= \sum_{\ell=0}^N \left(-2\beta'_{N-\ell} \cdot L + 2\beta'_{N-\ell} \cdot q + \beta_{N-\ell} q' - \frac{1}{2} \beta'''_{N-\ell} \right) L^\ell \\ &= -2 \sum_{\ell=0}^N \beta'_{N-\ell} \cdot L^{\ell+1} + \sum_{\ell=0}^N \left(2\beta'_{N-\ell} \cdot q + \beta_{N-\ell} q' - \frac{1}{2} \beta'''_{N-\ell} \right) L^\ell. \end{aligned} \quad (4.15)$$

We now apply the recursion formula (3.14) to the second sum. Indeed, the recursion formula states:

$$\beta'_{n+1} = -\frac{1}{4} \beta'''_n + \frac{1}{2} q' \cdot \beta_n + q \cdot \beta'_n, \quad (4.16)$$

which can be rewritten as:

$$2\beta'_{N-\ell} \cdot q + \beta_{N-\ell} q' - \frac{1}{2} \beta'''_{N-\ell} = 2\beta'_{N+1-\ell}. \quad (4.17)$$

Therefore the second sum becomes:

$$\sum_{\ell=0}^N 2\beta'_{N+1-\ell} \cdot L^\ell. \quad (4.18)$$

Substituting the index shift $k = \ell + 1$ in the first sum:

$$-2 \sum_{\ell=0}^N \beta'_{N-\ell} \cdot L^{\ell+1} = -2 \sum_{k=1}^{N+1} \beta'_{N+1-k} \cdot L^k. \quad (4.19)$$

Adding both sums:

$$\begin{aligned} [P_{2N+1}, L] &= -2 \sum_{k=1}^{N+1} \beta'_{N+1-k} \cdot L^k + 2 \sum_{\ell=0}^N \beta'_{N+1-\ell} \cdot L^\ell \\ &= 2\beta'_{N+1} \cdot L^0 + 2 \sum_{\ell=1}^N \beta'_{N+1-\ell} \cdot L^\ell - 2 \sum_{k=1}^N \beta'_{N+1-k} \cdot L^k - 2\beta'_0 \cdot L^{N+1} \\ &= 2\beta'_{N+1} - 2\beta'_0 \cdot L^{N+1}. \end{aligned} \quad (4.20)$$

Since $\beta_0 = 1$ is constant, we have $\beta'_0 = 0$, and therefore:

$$[P_{2N+1}, L] = 2\beta'_{N+1}. \quad (4.21)$$

□

4.3 Structure of P_{2n+1}

In order to connect the operator P_{2N+1} to the commuting operator Q required in condition (iv) of Theorem 4.8, we need to analyse the structure of P_{2N+1} as a differential operator. The following lemma shows that P_{2N+1} has order $2N + 1$ with smooth coefficients, and that a simple rescaling yields an operator of exactly the form required in (iv).

Lemma 4.3. *Let $q \in C^\infty(I, \mathbb{K})$ and let P_{2N+1} be defined as in Lemma 4.2. Then P_{2N+1} is a differential operator of order $2N + 1$ with leading term $(-1)^N \left(\frac{d}{dx}\right)^{2N+1}$ and smooth coefficients. In particular, $Q := (-1)^N P_{2N+1}$ has the form*

$$Q = \left(\frac{d}{dx}\right)^{2N+1} + \sum_{l=0}^{2N} \left(\frac{d}{dx}\right)^l q_l(x) \quad (4.22)$$

with $q_l \in C^\infty(I, \mathbb{K})$, and satisfies $[Q, L] = 0$ if and only if $[P_{2N+1}, L] = 0$.

Proof. We first determine the structure of L^ℓ for $\ell \in \mathbb{N}_0$.

Claim: For each $\ell \in \mathbb{N}_0$, L^ℓ is a differential operator of the form

$$L^\ell = (-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} + \sum_{k=0}^{2\ell-1} a_k^{(\ell)}(x) \frac{d^k}{dx^k} \quad (4.23)$$

with smooth coefficients $a_k^{(\ell)} \in C^\infty(I, \mathbb{K})$.

We prove this by induction on ℓ .

Base case: For $\ell = 1$ we have $L = -\frac{d^2}{dx^2} + q$, which has leading term $(-1)^1 \frac{d^2}{dx^2}$ and smooth coefficient $a_0^{(1)} = q$.

Induction hypothesis: Suppose that for some $\ell \in \mathbb{N}_0$:

$$L^\ell = (-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} + \sum_{k=0}^{2\ell-1} a_k^{(\ell)}(x) \frac{d^k}{dx^k}. \quad (4.24)$$

Inductive step: We compute:

$$\begin{aligned} L^{\ell+1} &= L \cdot L^\ell \\ &= \left(-\frac{d^2}{dx^2} + q\right) \left((-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} + \sum_{k=0}^{2\ell-1} a_k^{(\ell)} \frac{d^k}{dx^k}\right). \end{aligned} \quad (4.25)$$

Term 1:

$$-\frac{d^2}{dx^2} \cdot (-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} = (-1)^{\ell+1} \frac{d^{2\ell+2}}{dx^{2\ell+2}}. \quad (4.26)$$

Term 2:

$$-\frac{d^2}{dx^2} \cdot \sum_{k=0}^{2\ell-1} a_k^{(\ell)} \frac{d^k}{dx^k}. \quad (4.27)$$

For each individual summand, we apply the product rule twice. Writing $\frac{d^k}{dx^k} \psi = \psi^{(k)}$ for a test function $\psi \in C^\infty(I, \mathbb{K})$, the first application of $\frac{d}{dx}$ gives:

$$\frac{d}{dx} \left(a_k^{(\ell)} \psi^{(k)} \right) = a_k^{(\ell)'} \psi^{(k)} + a_k^{(\ell)} \psi^{(k+1)}. \quad (4.28)$$

Applying $\frac{d}{dx}$ once more and using the product rule on each summand:

$$\frac{d}{dx} \left(a_k^{(\ell)'} \psi^{(k)} \right) = a_k^{(\ell)''} \psi^{(k)} + a_k^{(\ell)'} \psi^{(k+1)}, \quad (4.29)$$

$$\frac{d}{dx} \left(a_k^{(\ell)} \psi^{(k+1)} \right) = a_k^{(\ell)'} \psi^{(k+1)} + a_k^{(\ell)} \psi^{(k+2)}. \quad (4.30)$$

Adding these yields:

$$\frac{d^2}{dx^2} \left(a_k^{(\ell)} \psi^{(k)} \right) = a_k^{(\ell)''} \psi^{(k)} + 2a_k^{(\ell)'} \psi^{(k+1)} + a_k^{(\ell)} \psi^{(k+2)}. \quad (4.31)$$

Thus, as a differential operator:

$$-\frac{d^2}{dx^2} \circ \left(a_k^{(\ell)} \frac{d^k}{dx^k} \right) = -a_k^{(\ell)} \frac{d^{k+2}}{dx^{k+2}} - 2a_k^{(\ell)'} \frac{d^{k+1}}{dx^{k+1}} - a_k^{(\ell)''} \frac{d^k}{dx^k}. \quad (4.32)$$

This produces terms of orders $k+2$, $k+1$, and k , with smooth coefficients $-a_k^{(\ell)}$, $-2a_k^{(\ell)'}$, and $-a_k^{(\ell)''}$, respectively, since $a_k^{(\ell)} \in C^\infty(I, \mathbb{K})$ implies that all its derivatives are smooth. Since $k \leq 2\ell - 1$, the highest order is $k+2 \leq 2\ell + 1$, which is strictly less than $2\ell + 2$.

Term 3:

$$q \cdot (-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}}. \quad (4.33)$$

This is a differential operator of order 2ℓ with smooth coefficient $(-1)^\ell q \in C^\infty(I, \mathbb{K})$.

Term 4:

$$q \cdot \sum_{k=0}^{2\ell-1} a_k^{(\ell)} \frac{d^k}{dx^k}. \quad (4.34)$$

This is a differential operator of order at most $2\ell - 1$ with smooth coefficients $q \cdot a_k^{(\ell)} \in C^\infty(I, \mathbb{K})$, since the product of smooth functions is again smooth.

Combining all four terms, $L^{\ell+1}$ has the form:

$$L^{\ell+1} = (-1)^{\ell+1} \frac{d^{2\ell+2}}{dx^{2\ell+2}} + \sum_{k=0}^{2\ell+1} a_k^{(\ell+1)} \frac{d^k}{dx^k} \quad (4.35)$$

with smooth coefficients $a_k^{(\ell+1)} \in C^\infty(I, \mathbb{K})$, which completes the inductive step.

We now analyze the structure of P_{2N+1} . A single term in the sum is:

$$\left(\beta_{N-\ell} \frac{d}{dx} - \frac{1}{2} \beta'_{N-\ell} \right) L^\ell. \quad (4.36)$$

Using the structure of L^ℓ established above, we compute each part separately. For the first part:

$$\begin{aligned} \beta_{N-\ell} \frac{d}{dx} \cdot L^\ell &= \beta_{N-\ell} \frac{d}{dx} \left((-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} + \sum_{k=0}^{2\ell-1} a_k^{(\ell)} \frac{d^k}{dx^k} \right) \\ &= (-1)^\ell \beta_{N-\ell} \frac{d^{2\ell+1}}{dx^{2\ell+1}} + \sum_{k=0}^{2\ell-1} \beta_{N-\ell} a_k^{(\ell)} \frac{d^{k+1}}{dx^{k+1}}, \end{aligned} \quad (4.37)$$

where all coefficients are smooth since $\beta_{N-\ell}$ is a differential polynomial in q and hence smooth, and products of smooth functions are smooth. For the second part:

$$\begin{aligned} -\frac{1}{2} \beta'_{N-\ell} \cdot L^\ell &= -\frac{1}{2} \beta'_{N-\ell} \left((-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} + \sum_{k=0}^{2\ell-1} a_k^{(\ell)} \frac{d^k}{dx^k} \right) \\ &= (-1)^{\ell+1} \frac{1}{2} \beta'_{N-\ell} \frac{d^{2\ell}}{dx^{2\ell}} + \sum_{k=0}^{2\ell-1} \left(-\frac{1}{2} \beta'_{N-\ell} a_k^{(\ell)} \right) \frac{d^k}{dx^k}, \end{aligned} \quad (4.38)$$

again with smooth coefficients. Therefore the ℓ -th term of P_{2N+1} has the form:

$$(-1)^\ell \beta_{N-\ell} \frac{d^{2\ell+1}}{dx^{2\ell+1}} + \text{terms of order} \leq 2\ell \text{ with smooth coefficients.} \quad (4.39)$$

The leading term of P_{2N+1} comes from $\ell = N$:

$$(-1)^N \beta_0 \frac{d^{2N+1}}{dx^{2N+1}} = (-1)^N \frac{d^{2N+1}}{dx^{2N+1}}, \quad (4.40)$$

since $\beta_0 = 1$. All remaining terms have order at most $2N$ with smooth coefficients, since:

- the $\beta_{N-\ell}$ are differential polynomials in q and hence smooth,
- the $a_k^{(\ell)}$ are smooth by the inductive argument above,
- products and derivatives of smooth functions are again smooth.

Therefore P_{2N+1} is a differential operator of order $2N+1$ of the form:

$$P_{2N+1} = (-1)^N \frac{d^{2N+1}}{dx^{2N+1}} + \sum_{l=0}^{2N} \left(\frac{d}{dx} \right)^l r_l(x) \quad (4.41)$$

with smooth coefficients $r_l \in C^\infty(I, \mathbb{K})$.

Setting $Q := (-1)^N P_{2N+1}$ yields:

$$Q = \left(\frac{d}{dx}\right)^{2N+1} + \sum_{l=0}^{2N} \left(\frac{d}{dx}\right)^l q_l(x) \quad (4.42)$$

with $q_l := (-1)^N r_l \in C^\infty(I, \mathbb{K})$, which is exactly the form required in (iv). Finally, since $(-1)^N \neq 0$:

$$[Q, L] = (-1)^N [P_{2N+1}, L], \quad (4.43)$$

and therefore $[Q, L] = 0$ if and only if $[P_{2N+1}, L] = 0$. $\square$

Lemma 4.3 shows that P_{2N+1} has smooth coefficients, provided the potential q itself is already assumed to be smooth, which is an assumption we have made throughout since the beginning of Chapter 3. The following lemma justifies this assumption retroactively: we show that a potential q for which a solution ξ of the Lax equation (3.9) of finite type exists is automatically smooth, even if the coefficients $\alpha_n, \beta_n, \gamma_n$ are a priori only assumed to satisfy the much weaker regularity condition $\alpha_n, \beta_n, \gamma_n \in L^1_{loc}(\mathbb{R})$.

Since the proof of the following lemma relies on basic properties of Sobolev spaces on subintervals of $\mathbb{R}$, we briefly recall the relevant definitions and the one-dimensional embedding theorem before stating and proving the result.

Definition 4.4 (see e.g. [2], chapter 8, or [7], section 5.2). For an open interval $I \subset \mathbb{R}$, let $L^1_{loc}(I)$ denote the space of (equivalence classes of) measurable functions $f : I \rightarrow \mathbb{R}$ that are Lebesgue integrable on every compact subset of I . For $k \in \mathbb{N}_0$, the Sobolev space $W^{k,1}_{loc}(I)$ consists of all $f \in L^1_{loc}(I)$ that admit weak derivatives $f', f'', \dots, f^{(k)}$ up to order k , all lying in $L^1_{loc}(I)$; here $f^{(l)} \in L^1_{loc}(I)$ is called a weak derivative of $f^{(l-1)}$ if

$$\int_I f^{(l-1)}(x) \varphi'(x) dx = - \int_I f^{(l)}(x) \varphi(x) dx \quad \text{for all } \varphi \in C_c^\infty(I).$$

We will use the standard one-dimensional Sobolev embedding $W^{1,1}_{loc}(I) \subset C(I)$: every $f \in W^{1,1}_{loc}(I)$ has a (unique) continuous representative, given by $f(x) = f(x_0) + \int_{x_0}^x f'(t) dt$ for any fixed $x_0 \in I$. In particular, if $f \in W^{k,1}_{loc}(I)$ then $f, f', \dots, f^{(k-1)} \in C(I)$, and $\bigcap_{k \in \mathbb{N}} W^{k,1}_{loc}(I) = C^\infty(I)$.

Lemma 4.5 (Sobolev embedding in one dimension, see [2], Theorem 8.2). *Let $I \subset \mathbb{R}$ be an open interval and $f \in W^{1,1}_{loc}(I)$. Then f has a (unique) representative that is continuous on I , and for this representative,*

$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt \quad \text{for all } x, x_0 \in I,$$

where f' denotes the weak derivative of f . In particular, $W^{1,1}_{loc}(I) \subset C(I)$.

Proof. Fix $x_0 \in I$ and define $g : I \rightarrow \mathbb{R}$ by $g(x) := \int_{x_0}^x f'(t) dt$. Since $f' \in L^1_{loc}(I)$, this integral is well-defined for every $x \in I$, and g is continuous on I (absolute continuity of the integral: for $x_n \rightarrow x$, $|g(x_n) - g(x)| \leq \int_{[x, x_n]} |f'(t)| dt \rightarrow 0$ by dominated convergence, since $f' \in L^1$ on any compact set containing x and all x_n).

It remains to show $f = g + f(x_0)$ almost everywhere, i.e. that g is itself a weak derivative-antiderivative of f' matching f . For any $\varphi \in C_c^\infty(I)$, Fubini's theorem gives

$$\int_I g(x) \varphi'(x) dx = \int_I \left(\int_{x_0}^x f'(t) dt \right) \varphi'(x) dx = - \int_I f'(t) \varphi(t) dt,$$

(the sign and limits of integration work out by swapping the order of integration over the region between x_0 and x , using that φ has compact support). But by definition of the weak derivative of f , $\int_I f'(t) \varphi(t) dt = - \int_I f(x) \varphi'(x) dx$. Combining the two displays,

$$\int_I (g(x) - f(x)) \varphi'(x) dx = 0 \quad \text{for all } \varphi \in C_c^\infty(I),$$

which means $g - f$ has weak derivative 0 on I . A standard fact (a function with vanishing weak derivative on an interval is a.e. equal to a constant) gives $g - f = c$ a.e. for some constant $c \in \mathbb{R}$. Evaluating at x_0 (where $g(x_0) = 0$) gives $c = -f(x_0)$, hence $f(x) = g(x) + f(x_0)$ for a.e. $x \in I$. Since the right-hand side is continuous, f agrees a.e. with a continuous function, which we take as its continuous representative. $\square$

We are now ready to state and prove the following lemma:

Lemma 4.6. *Let equation (3.9) hold for some $N \in \mathbb{N}$ with $\beta_n = 0$ for $n < 0$ and $\beta_0 = 1$ for all powers λ^{-n} , $n \in \mathbb{Z} \ni n \leq N$, and let all coefficients $\alpha_n, \beta_n, \gamma_n$ lie in $L^1_{loc}(\mathbb{R})$, then $q \in W^{2N-1,1}_{loc}(\mathbb{R})$. In particular, q is smooth if a solution of (3.9) exists such that the set of indices $n \in \mathbb{Z}$ with β_n not identically vanishing has a finite minimum.*

Proof. Step 1: Structure of β_n . We first show, by induction on n , that β_n (for the normalization $\beta_0 = 1$) is homogeneous of weight $2n$, if we assign weight $l + 2$ to each derivative $q^{(l)}$ (so that a product of factors has weight equal to the sum of the individual weights, and differentiating a monomial raises its weight by 1, by the product rule). The base case $\beta_0 = 1$ has weight 0. For the inductive step, assume β_j is homogeneous of weight $2j$ for $j = 0, \dots, n$. By the recursion (3.14),

$$\beta'_{n+1} = -\frac{1}{4}\beta'''_n + \frac{1}{2}q'\beta_n + q\beta'_n,$$

every term on the right has weight $2n + 3$: β'''_n has weight $2n + 3$ (three derivatives applied to a weight- $2n$ object), $q'\beta_n$ has weight $3 + 2n = 2n + 3$, and $q\beta'_n$ has weight $2 + (2n + 1) = 2n + 3$. Hence β'_{n+1} is homogeneous of weight $2n + 3$ and therefore β_{n+1} is homogeneous of weight $2n + 2$, completing the induction. A monomial of weight $2n$ that is linear in q (i.e. consists of a single factor $q^{(l)}$) must satisfy $l + 2 = 2n$, so $l = 2n - 2$; hence there is at most one such term, say $c_n q^{(2n-2)}$. Every other monomial is a product

of $k \geq 2$ factors $q^{(l_1)} \cdots q^{(l_k)}$ with $\sum_i (l_i + 2) = 2n$, so $\sum_i l_i = 2n - 2k \leq 2n - 4$, and in particular every individual $l_i \leq 2n - 4$. We may therefore write

$$\beta_n = c_n q^{(2n-2)} + R_n, \quad R_n = (\text{finite sum of products of } \geq 2 \text{ factors } q^{(l_i)}, l_i \leq 2n - 4).$$

Tracking only the linear term through the recursion (3.14) (the terms $q \cdot \beta'_n$ and $\frac{1}{2} q' \beta_n$ only ever contribute products of ≥ 2 factors to β'_{n+1} , since they multiply q or q' onto β_n), we get $c_{n+1} = -\frac{1}{4} c_n$. Since $\beta_1 = \frac{1}{2} q$ gives $c_1 = \frac{1}{2}$, this yields $c_n = \frac{1}{2} \left(-\frac{1}{4}\right)^{n-1} = -\frac{(-1)^n}{2^{2n-1}} \neq 0$ for all $n \geq 1$.

Step 2: Base case ($n = 1$). Since $R_1 = 0$ (a product of ≥ 2 factors would need $l_i \leq 2 \cdot 1 - 4 = -2$, impossible), we have $\beta_1 = \frac{1}{2} q$, so $\beta_1 \in L^1_{loc}(\mathbb{R})$ gives $q = 2\beta_1 \in L^1_{loc}(\mathbb{R})$. By (3.11), $\alpha_1 = -\frac{1}{2} \beta'_1$, so $\alpha_1 \in L^1_{loc}(\mathbb{R})$ gives $\beta'_1 \in L^1_{loc}(\mathbb{R})$, i.e. $q' = 2\beta'_1 \in L^1_{loc}(\mathbb{R})$. Hence $q \in W^{1,1}_{loc}(\mathbb{R})$.

Step 3: Inductive step. Suppose $q \in W^{2n-1,1}_{loc}(\mathbb{R})$, and suppose additionally $\alpha_{n+1}, \beta_{n+1} \in L^1_{loc}(\mathbb{R})$. By the one-dimensional Sobolev embedding $W^{1,1}_{loc}(\mathbb{R}) \subset C(\mathbb{R})$, applied successively, $q^{(l)} \in C(\mathbb{R})$ for all $l \leq 2n - 2$. Since by the induction hypothesis, $q \in W^{2n-1,1}_{loc}(\mathbb{R})$, $q^{(l)} \in W^{1,1}_{loc}(\mathbb{R})$ holds for all $l \leq 2n - 2$, Lemma 4.5 gives $q^{(l)} \in C(\mathbb{R})$. Writing $\beta_{n+1} = c_{n+1} q^{(2n)} + R_{n+1}$ as in Step 1, every factor occurring in R_{n+1} has order $\leq 2n - 2$, hence is continuous, so $R_{n+1} \in C(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$. Since $c_{n+1} \neq 0$, we obtain

$$q^{(2n)} = c_{n+1}^{-1} (\beta_{n+1} - R_{n+1}) \in L^1_{loc}(\mathbb{R}),$$

so $q \in W^{2n,1}_{loc}(\mathbb{R})$. Differentiating, $\beta'_{n+1} = c_{n+1} q^{(2n+1)} + R'_{n+1}$, where R'_{n+1} is (by the product rule) a finite sum of products in which every factor except one has order $\leq 2n - 2$ (hence continuous, in particular bounded on compact sets) and the remaining factor has order $\leq 2n - 1$ (hence in $L^1_{loc}(\mathbb{R})$ by the regularity just established). A product of a locally bounded and a locally integrable function is again locally integrable, so $R'_{n+1} \in L^1_{loc}(\mathbb{R})$. By (3.11), $\beta'_{n+1} = -2\alpha_{n+1} \in L^1_{loc}(\mathbb{R})$, so

$$q^{(2n+1)} = c_{n+1}^{-1} (\beta'_{n+1} - R'_{n+1}) \in L^1_{loc}(\mathbb{R}),$$

giving $q \in W^{2n+1,1}_{loc}(\mathbb{R}) = W^{2(n+1)-1,1}_{loc}(\mathbb{R})$. This completes the induction.

Step 4: Finite minimum. Suppose a solution of (3.9) exists such that the set of indices $n \in \mathbb{Z}$ with β_n not identically vanishing has a finite minimum $M \in \mathbb{Z}$, i.e. $\beta_n \equiv 0$ for $n < M$ and $\beta_M \not\equiv 0$. By the linearity of (3.9), multiplying the solution ξ by λ^M produces another solution $\tilde{\xi} = \lambda^M \xi$, whose coefficients $\tilde{\alpha}_n := \alpha_{n+M}$, $\tilde{\beta}_n := \beta_{n+M}$, $\tilde{\gamma}_n := \gamma_{n+M}$ satisfy $\tilde{\beta}_n = 0$ for $n < 0$ and $\tilde{\beta}_0 = \beta_M \neq 0$ (after the usual normalization $\tilde{\beta}_0 = 1$). This rescaling does not change q , and since ξ is by hypothesis a genuine solution of (3.9), its recursion continues indefinitely, so $\alpha_n, \beta_n, \gamma_n \in L^1_{loc}(\mathbb{R})$ for every $n \geq M$, hence $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n \in L^1_{loc}(\mathbb{R})$ for every $n \geq 0$. Thus the hypothesis of the first part of the lemma is satisfied for every $N \in \mathbb{N}$, giving $q \in W^{2N-1,1}_{loc}(\mathbb{R})$ for every N , i.e. $q \in \bigcap_{N \in \mathbb{N}} W^{2N-1,1}_{loc}(\mathbb{R})$. By the Sobolev embedding applied at every order, this intersection is contained in $\bigcap_{k \in \mathbb{N}_0} C^k(\mathbb{R}) = C^\infty(\mathbb{R})$, so $q \in C^\infty(\mathbb{R})$. $\square$

4.4 Characterization of finite-type potentials

We now have all the tools necessary to prove the main theorem of this chapter. The following lemma, which characterizes all solutions of the recursion formula (3.14) in terms of the unique reference solution $(\beta_n)_{n \in \mathbb{N}_0}$, will be used directly in the proof of implication (i) $\Rightarrow$ (ii).

Lemma 4.7. *Let $q \in C^\infty(I, \mathbb{K})$ and let $(\beta_n)_{n \in \mathbb{N}_0}$ be the uniquely determined sequence of differential polynomials with $\beta_0 = 1$ given by (3.14)-(3.15). Then every solution $(\tilde{\beta}_n)_{n \in \mathbb{N}_0}$ of the recursion formula (3.14) with $\tilde{\beta}_0 = 1$ can be written as*

$$\tilde{\beta}_n = \beta_n + \sum_{l=0}^{n-1} d_l \beta_l \quad (4.44)$$

for uniquely determined constants $d_0, \dots, d_{n-1} \in \mathbb{K}$.

Proof. We prove the claim by induction on n .

Base case: For $n = 0$ we have $\tilde{\beta}_0 = 1 = \beta_0$, which corresponds to the claim with an empty sum.

Induction hypothesis: Suppose that

$$\tilde{\beta}_n = \beta_n + \sum_{l=0}^{n-1} d_l \beta_l \quad (4.45)$$

holds for some $n \in \mathbb{N}_0$ and suitable constants $d_0, \dots, d_{n-1} \in \mathbb{K}$.

Inductive step: The recursion formula (3.14) gives:

$$\tilde{\beta}'_{n+1} = -\frac{1}{4}\tilde{\beta}'''_n + \frac{1}{2}q' \cdot \tilde{\beta}_n + q \cdot \tilde{\beta}'_n. \quad (4.46)$$

Since the right-hand side is linear in $\tilde{\beta}_n$, we can substitute the induction hypothesis $\tilde{\beta}_n = \beta_n + \sum_{l=0}^{n-1} d_l \beta_l$ and treat each of the three summands separately:

$$-\frac{1}{4}\tilde{\beta}'''_n = -\frac{1}{4}\beta'''_n + \sum_{l=0}^{n-1} d_l \left(-\frac{1}{4}\beta'''_l \right), \quad (4.47)$$

$$\frac{1}{2}q' \cdot \tilde{\beta}_n = \frac{1}{2}q' \cdot \beta_n + \sum_{l=0}^{n-1} d_l \left(\frac{1}{2}q' \cdot \beta_l \right), \quad (4.48)$$

$$q \cdot \tilde{\beta}'_n = q \cdot \beta'_n + \sum_{l=0}^{n-1} d_l (q \cdot \beta'_l). \quad (4.49)$$

Adding all three summands and regrouping yields:

$$\begin{aligned}
\tilde{\beta}'_{n+1} &= \underbrace{\left(-\frac{1}{4}\beta_n''' + \frac{1}{2}q' \cdot \beta_n + q \cdot \beta_n'\right)}_{=\beta'_{n+1}} \\
&\quad + \sum_{l=0}^{n-1} d_l \underbrace{\left(-\frac{1}{4}\beta_l''' + \frac{1}{2}q' \cdot \beta_l + q \cdot \beta_l'\right)}_{=\beta'_{l+1}} \\
&= \beta'_{n+1} + \sum_{l=0}^{n-1} d_l \beta'_{l+1},
\end{aligned} \tag{4.50}$$

where we used the fact that β_{n+1} and β_{l+1} each satisfy the recursion formula (3.14). Integrating both sides yields:

$$\tilde{\beta}_{n+1} = \beta_{n+1} + \sum_{l=0}^{n-1} d_l \beta_{l+1} + c_n \tag{4.51}$$

for an integration constant $c_n \in \mathbb{K}$. Since $\beta_0 = 1$, we write $c_n = c_n \cdot \beta_0$ and obtain:

$$\tilde{\beta}_{n+1} = \beta_{n+1} + \sum_{l=0}^{n-1} d_l \beta_{l+1} + c_n \beta_0 = \beta_{n+1} + \sum_{l=0}^n d'_l \beta_l, \tag{4.52}$$

where we set $d'_0 := c_n$ and $d'_l := d_{l-1}$ for $l = 1, \dots, n$. This is exactly the claim for $n+1$. $\square$

We are now in a position to state and prove the main theorem of this chapter.

Theorem 4.8. *Let $q \in C^\infty(I, \mathbb{K})$ on an open interval I and let $\xi(x, \lambda)$ be the unique solution (3.14)-(3.15) of (3.9) with $\det \xi(x, \lambda) = \lambda$. Then the following statements are equivalent:*

- (i) *There exists a solution $\tilde{\xi}$ of the Lax equation (3.9) such that only finitely many, but at least one, of the smooth functions $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n \in C^\infty(I, \mathbb{K})$ do not vanish identically. The smoothness $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n \in C^\infty(I, \mathbb{K})$ follows from the smoothness of q and standard ODE theory applied to (3.9).*
- (ii) *For the above solution (3.14)-(3.15) with $\beta_n = 0$ for $n < 0$ and $\beta_0 = 1$, there exist $N \in \mathbb{N}_0$ and finitely many constants $c_0, \dots, c_{N-1} \in \mathbb{K}$ such that $\beta_N + \sum_{l=0}^{N-1} c_l \beta_l$ vanishes identically on I . In particular, $N \geq 1$ ensures that at least one non-trivial relation exists.*
- (iii) *There exists $p(t) \in \mathbb{K}[[t]]$ such that $\tilde{\xi} = (1 + \lambda^{-1}p(\lambda^{-1})) \xi$ satisfies condition (i).*
- (iv) *There exists $N \in \mathbb{N}_0$ and an operator $Q = \left(\frac{d}{dx}\right)^{2N+1} + \sum_{l=0}^{2N} \left(\frac{d}{dx}\right)^l q_l(x)$ with $q_l \in C^\infty(I, \mathbb{K})$, such that $QLy = LQy$ for $L = -\left(\frac{d}{dx}\right)^2 + q(x)$ and all $y \in C^\infty(I, \mathbb{K})$.*

Proof of Theorem 4.8. (i) $\Rightarrow$ (ii):

Let $\tilde{\xi}$ be a solution of the Lax equation (3.9) as in (i), that is, there exists $N \in \mathbb{N}_0$ such that only finitely many of the functions $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n$ do not vanish identically, but at least one does not. By linearity of the Lax equation (3.9), we may multiply $\tilde{\xi}$ by a suitable power of λ so that $\tilde{\beta}_n = 0$ for $n < 0$ and $\tilde{\beta}_0 \neq 0$. From the recursion formula (3.14) for $n = -1$ it follows that $\tilde{\beta}'_0 = 0$, so $\tilde{\beta}_0$ is constant. Without loss of generality we set $\tilde{\beta}_0 = 1$. Since $\tilde{\xi}$ satisfies the same Lax equation (3.9), its coefficients $\tilde{\beta}_n$ satisfy the same recursion formula (3.14) as β_n . By Lemma 4.7 we therefore have:

$$\tilde{\beta}_n = \beta_n + \sum_{l=0}^{n-1} d_l \beta_l \quad (4.53)$$

for suitable constants $d_0, \dots, d_{n-1} \in \mathbb{K}$. Since by assumption only finitely many $\tilde{\beta}_n$ do not vanish identically, we define:

$$N := \min\{n \in \mathbb{N} \mid \tilde{\beta}_m \equiv 0 \text{ for all } m \geq n\}. \quad (4.54)$$

Then $\tilde{\beta}_N \equiv 0$, and by Lemma 4.7:

$$0 \equiv \tilde{\beta}_N = \beta_N + \sum_{l=0}^{N-1} d_l \beta_l. \quad (4.55)$$

Setting $c_l := d_l$ for $l = 0, \dots, N-1$, this is exactly condition (ii).

(ii) $\Rightarrow$ (iii):

Assume that (ii) holds, that is, there exist $N \in \mathbb{N}_0$ and constants $c_0, \dots, c_{N-1} \in \mathbb{K}$ such that:

$$\beta_N + \sum_{l=0}^{N-1} c_l \beta_l \equiv 0 \quad \text{on } I. \quad (4.56)$$

We define:

$$p(t) := \sum_{j=0}^{N-1} c_{N-1-j} t^j \in \mathbb{K}[t] \subset \mathbb{K}[[t]], \quad (4.57)$$

where $\mathbb{K}[t]$ denotes the ring of polynomials in t and $\mathbb{K}[[t]]$ the ring of formal power series in t . Note that $p(t)$ is a polynomial of degree $N-1$, which is in particular an element of $\mathbb{K}[[t]]$ as required by (iii). We set $\tilde{\xi} := (1 + \lambda^{-1} p(\lambda^{-1})) \xi$. Since the Lax equation (3.9) is linear, $\tilde{\xi}$ is again a solution.

We compute the scalar factor explicitly. Since $p(t) = \sum_{j=0}^{N-1} c_{N-1-j} t^j$, we have:

$$\lambda^{-1} p(\lambda^{-1}) = \lambda^{-1} \sum_{j=0}^{N-1} c_{N-1-j} \lambda^{-j} = \sum_{j=0}^{N-1} c_{N-1-j} \lambda^{-(j+1)} = \sum_{k=1}^N c_{N-k} \lambda^{-k}, \quad (4.58)$$

where we set $k = j + 1$. Thus:

$$1 + \lambda^{-1}p(\lambda^{-1}) = 1 + \sum_{k=1}^N c_{N-k}\lambda^{-k}. \quad (4.59)$$

Since $\tilde{\xi} = (1 + \lambda^{-1}p(\lambda^{-1}))\xi$ is a scalar multiple of the matrix ξ , the multiplication acts entrywise. For the β -entry we compute the Cauchy product of the scalar factor with the formal Laurent series $\sum_{m=0}^{\infty} \beta_m \lambda^{-m}$:

$$\tilde{\beta} = \left(1 + \sum_{k=1}^N c_{N-k}\lambda^{-k}\right) \cdot \sum_{m=0}^{\infty} \beta_m \lambda^{-m} = \sum_{m=0}^{\infty} \beta_m \lambda^{-m} + \sum_{k=1}^N c_{N-k}\lambda^{-k} \cdot \sum_{m=0}^{\infty} \beta_m \lambda^{-m}. \quad (4.60)$$

In the second term, for fixed k , the coefficient of λ^{-n} is $c_{N-k}\beta_{n-k}$, since $\lambda^{-k} \cdot \lambda^{-m} = \lambda^{-(k+m)}$ and we set $n = k + m$. Thus the total coefficient of λ^{-n} is:

$$\tilde{\beta}_n = \beta_n + \sum_{k=1}^N c_{N-k}\beta_{n-k}, \quad (4.61)$$

where $\beta_{n-k} = 0$ for $n - k < 0$. For $n = N$, substituting $l = N - k$ yields:

$$\tilde{\beta}_N = \beta_N + \sum_{k=1}^N c_{N-k}\beta_{N-k} = \beta_N + \sum_{l=0}^{N-1} c_l \beta_l \equiv 0 \quad (4.62)$$

by condition (ii). Since $\tilde{\beta}_N \equiv 0$, so are all its derivatives, and the recursion formula (3.14) gives:

$$\tilde{\beta}'_{N+1} = -\frac{1}{4}\tilde{\beta}'''_N + \frac{1}{2}q' \cdot \tilde{\beta}_N + q \cdot \tilde{\beta}'_N = 0. \quad (4.63)$$

Thus $\tilde{\beta}_{N+1}$ is constant, and we choose the integration constant to be zero, so $\tilde{\beta}_{N+1} = 0$.

The choice of integration constant $c_N = 0$ is justified as follows. Since $\tilde{\beta}'_{N+1} = 0$, the function $\tilde{\beta}_{N+1}$ is a constant. By linearity of the Lax equation, replacing $\tilde{\xi}$ by $\tilde{\xi} - \tilde{\beta}_{N+1}\lambda^{-(N+1)}\xi$ yields a new solution of the Lax equation, whose $(N+1)$ -th β -coefficient vanishes identically. Indeed, the β -coefficient of $\lambda^{-(N+1)}\xi$ at index $N+1$ is $\beta_0 = 1$, so subtracting $\tilde{\beta}_{N+1}\lambda^{-(N+1)}\xi$ cancels $\tilde{\beta}_{N+1}$. We therefore replace $\tilde{\xi}$ by this new solution and set $\tilde{\beta}_{N+1} = 0$ without loss of generality.

Proceeding inductively, we can set all subsequent coefficients $\tilde{\beta}_{N+2}, \tilde{\beta}_{N+3}, \dots$ to zero in the same way, since each satisfies $\tilde{\beta}'_{n+1} = 0$ once $\tilde{\beta}_n = 0$. The resulting solution $\tilde{\xi}$ is a polynomial in λ^{-1} of degree N . Inductively, $\tilde{\beta}_n \equiv 0$ for all $n > N$: if $\tilde{\beta}_n \equiv 0$, then:

$$\tilde{\beta}'_{n+1} = -\frac{1}{4}\tilde{\beta}'''_n + \frac{1}{2}q' \cdot \tilde{\beta}_n + q \cdot \tilde{\beta}'_n = 0, \quad (4.64)$$

and we again choose the integration constant to be zero. Correspondingly, $\tilde{\alpha}_n$ and $\tilde{\gamma}_n$ vanish for all $n > N$ by:

$$\tilde{\alpha}_n = -\frac{1}{2}\tilde{\beta}'_n = 0, \quad \tilde{\gamma}_n = -\frac{1}{2}\tilde{\beta}''_n + q\tilde{\beta}_n - \tilde{\beta}_{n+1} = 0. \quad (4.65)$$

Hence $\tilde{\xi}$ has only finitely many non-vanishing coefficients, and thus satisfies condition (i). Together with the explicit construction of $p(t)$, this establishes condition (iii).

(iii) $\Rightarrow$ (i):

Let $p(t) \in \mathbb{K}[[t]]$ be as in (iii), so that $\tilde{\xi} = (1 + \lambda^{-1}p(\lambda^{-1}))\xi$ satisfies condition (i). Since the Lax equation (3.9) is linear, $\tilde{\xi}$ is again a solution of the Lax equation as a scalar multiple of ξ . By assumption, $\tilde{\xi}$ has only finitely many non-vanishing coefficients $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n$, and at least one of these functions does not vanish identically. This is exactly condition (i).

(ii) $\Leftrightarrow$ (iv):

We show that (ii) and (iv) are equivalent by establishing a chain of equivalences.

We first clarify the relationship between condition (ii) and the vanishing of $\tilde{\beta}_{N+1}$. By Lemma 4.7, every solution of the linear recursion (3.14) can be written as:

$$\tilde{\beta}_n = \beta_n + \sum_{l=0}^{n-1} d_l \beta_l$$

for constants $d_l \in \mathbb{K}$, where (β_n) is the canonical sequence determined by (3.14)-(3.15). Note that the modified sequence $(\tilde{\beta}_n)$ satisfies only the linear recursion (3.14), not the algebraic condition (3.15); the latter uniquely fixes the canonical sequence (β_n) without integration constants, whereas $(\tilde{\beta}_n)$ has a free integration constant at each step.

Condition (ii) states that there exist $N \in \mathbb{N}$ and constants $c_0, \dots, c_{N-1} \in \mathbb{K}$ such that:

$$\beta_N + \sum_{l=0}^{N-1} c_l \beta_l \equiv 0 \quad \text{on } I. \quad (4.66)$$

Setting $d_l := c_l$ for $l = 0, \dots, N-1$, this is equivalent to saying that the modified sequence $(\tilde{\beta}_n)$ with these constants satisfies $\tilde{\beta}_N \equiv 0$. Since $(\tilde{\beta}_n)$ satisfies the linear recursion (3.14), and $\tilde{\beta}_N \equiv 0$, we obtain:

$$\tilde{\beta}'_{N+1} = -\frac{1}{4}\tilde{\beta}'''_N + q\tilde{\beta}'_N + \frac{1}{2}q' \cdot \tilde{\beta}_N = 0.$$

Therefore $\tilde{\beta}_{N+1}$ is constant. Since $(\tilde{\beta}_n)$ satisfies only the linear recursion (3.14) (not the algebraic condition (3.15)), this constant is a free integration constant, which we may choose to be zero. With this choice, $\tilde{\beta}_{N+1} \equiv 0$.

Conversely, if $\tilde{\beta}_{N+1} \equiv 0$ for some modified sequence $(\tilde{\beta}_n)$ with constants $d_0, \dots, d_N$, then by Lemma 4.7:

$$0 = \tilde{\beta}_{N+1} = \beta_{N+1} + \sum_{l=0}^N d_l \beta_l,$$

which is a linear relation of the form (ii) with N replaced by $N + 1$ and $c_l := d_l$. Therefore condition (ii) is equivalent to the existence of a modified sequence $(\tilde{\beta}_n)$ with $\tilde{\beta}_{N+1} \equiv 0$. By Lemma 4.2, this is equivalent to:

$$[P_{2N+1}, L] = 2\tilde{\beta}'_{N+1} = 0. \quad (4.67)$$

Setting $Q := (-1)^N P_{2N+1}$, and noting that $(-1)^N \neq 0$, this is in turn equivalent to:

$$[Q, L] = (-1)^N [P_{2N+1}, L] = 0. \quad (4.68)$$

By Lemma 4.3, Q is a differential operator of order $2N + 1$ of the form:

$$Q = \left(\frac{d}{dx}\right)^{2N+1} + \sum_{l=0}^{2N} \left(\frac{d}{dx}\right)^l q_l(x) \quad (4.69)$$

with $q_l \in C^\infty(I, \mathbb{K})$, satisfying $QLy = LQy$ for all $y \in C^\infty(I, \mathbb{K})$. This is exactly condition (iv). Thus all four statements (i)-(iv) are mutually equivalent. $\square$

Now that Theorem 4.8 has been established, we are ready to define the central object of this thesis:

Definition 4.9. A potential $q \in C^\infty(I, \mathbb{K})$ is called of *finite type*, or *algebraic-geometric*, if it satisfies one of the equivalent conditions (i)-(iv) of Theorem 4.8.

Remark 4.10. The assumption $q \in C^\infty(I, \mathbb{K})$ made here is in fact not just additional: Lemma 4.6 above shows that if a potential q admits a solution ξ of the Lax equation (3.9) of finite type, then q is automatically smooth, even if the coefficients $\alpha_n, \beta_n, \gamma_n$ are a priori only assumed to lie in the much weaker space $L^1_{loc}(\mathbb{R})$.

We note that condition (ii) provides the most explicit characterization: a potential is of finite type if and only if the sequence $(\beta_n)_{n \in \mathbb{N}_0}$ satisfies a linear dependence relation with constant coefficients. Condition (iv) characterizes finite-type potentials as those for which the Schrödinger operator L admits a commuting differential operator of odd order, which is the classical Burchnell-Chaundy characterization. Conditions (i) and (iii) provide equivalent formulations in terms of the Lax equation and formal power series respectively.

Chapter 5

Periodic Potentials and Spectral Theory

In this chapter we turn to the special case of periodic potentials and study their spectral properties. The main result is Theorem 5.14, which characterizes periodic potentials of finite type in terms of the spectral theory of the associated Schrödinger operator. Specifically, we show that a periodic potential is of finite type if and only if its stable spectrum consists of only finitely many intervals, or equivalently, if the associated Hill operator has only finitely many spectral gaps.

The spectral theory of periodic Schrödinger operators is a rich subject in its own right. For a detailed treatment of the eigenvalues of Hill's equation and their properties, we refer to the book of Pöschel and Trubowitz [20], whose results we will use freely in this chapter. An independent, self-contained development of this spectral theory is given by Crutoi [4] for potentials $q \in L^2([0, 1], \mathbb{R})$, extended 1-periodically to $\mathbb{R}$. Every smooth 1-periodic potential $q \in C^\infty(\mathbb{R}, \mathbb{R})$ considered in this thesis arises this way: its restriction $q|_{[0, 1]}$ lies in $L^2([0, 1], \mathbb{R})$, and extending this restriction periodically recovers q itself. Her results, stated for the extended potential together with the associated Hill operator, solutions, and monodromy matrix (all defined, as in our setting, on all of $\mathbb{R}$) therefore apply directly to the potentials of this chapter. We point to the relevant results of [4] throughout this chapter in place of reproducing their proofs. The results of Chapters 3 and 4 were developed for a general field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. From this point on, we specialize to $\mathbb{K} = \mathbb{R}$, i.e., we consider only real-valued potentials $q \in C^\infty(\mathbb{R}, \mathbb{R})$. This restriction is necessary for the Hill operator $L = -\frac{d^2}{dx^2} + q$ to be self-adjoint on $L^2(\mathbb{R})$, which ensures that its spectrum is real.

5.1 The Hill operator and periodic Schrödinger operators

We consider a potential $q \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying the periodicity condition

$$q(x + 1) = q(x) \quad \text{for all } x \in \mathbb{R}. \quad (5.1)$$

Remark 5.1. The choice of period 1 is a normalization convention and not a restriction

of generality. Indeed, if q has period $T > 0$, the substitution $\tilde{x} = x/T$ yields a potential $\tilde{q}(\tilde{x}) := q(T\tilde{x})$ of period 1. The choice of period 1 simplifies the notation considerably, in particular the evaluation of the fundamental solution at $x = 1$ and the definition of the monodromy matrix below.

The associated one-dimensional Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q \quad (5.2)$$

is called the *Hill operator*. It acts on the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions:

$$L^2(\mathbb{R}) = \left\{ f: \mathbb{R} \rightarrow \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}. \quad (5.3)$$

The spectral theory of L is governed by the behavior of solutions to the eigenvalue equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}. \quad (5.4)$$

For each $\lambda \in \mathbb{C}$, the solution space of (5.4) is two-dimensional. Depending on λ , the solutions exhibit qualitatively different behavior, which leads to the band-gap structure of the spectrum described below.

5.2 Eigenvalues of Hill's equation

Following Pöschel and Trubowitz [20], we distinguish four types of eigenvalues of Hill's equation (5.4).

Definition 5.2. Let $q \in C^\infty(\mathbb{R}, \mathbb{R})$ with $q(x+1) = q(x)$. We call $\lambda \in \mathbb{C}$:

- a *periodic eigenvalue* if there exists a nontrivial solution y of (5.4) such that $y(x+n) = y(x)$ for all $n \in \mathbb{Z}$,
- an *anti-periodic eigenvalue* if there exists a nontrivial solution y of (5.4) such that $y(x+n) = (-1)^n y(x)$ for all $n \in \mathbb{Z}$,
- a *Dirichlet eigenvalue* if there exists a nontrivial solution y of (5.4) such that $y(0) = 0 = y(1)$,
- a *Neumann eigenvalue* if there exists a nontrivial solution y of (5.4) such that $y'(0) = 0 = y'(1)$.

These four eigenvalue types, together with a proof of their basic properties, are treated for the more general class $q \in L^2([0, 1], \mathbb{R})$ by Crutoi [4, Definitions 1.7-1.10, 3.1].

The following theorem, proved in Pöschel and Trubowitz [20], describes the ordering and interlacing properties of these eigenvalues. It is the key spectral result that we will use in the proof of Theorem 5.14.

Theorem 5.3 (Pöschel-Trubowitz [20], see also Crutoi [4], Satz 2.3). *For every $q \in L^2([0, 1], \mathbb{R})$, the periodic eigenvalues $(\lambda_n)_{n \in \mathbb{N}_0}$, anti-periodic eigenvalues $(\lambda'_n)_{n \in \mathbb{N}}$, Dirichlet eigenvalues $(\mu_n)_{n \in \mathbb{N}}$, and Neumann eigenvalues $(\nu_n)_{n \in \mathbb{N}_0}$ are real sequences satisfying:*

$$\lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 < \lambda_3 \leq \lambda_4 < \cdots, \quad (5.5)$$

as well as $\nu_0 \leq \lambda_0$ and

$$\mu_{2n-1}, \nu_{2n-1} \in [\lambda'_{2n-1}, \lambda'_{2n}], \quad \mu_{2n}, \nu_{2n} \in [\lambda_{2n-1}, \lambda_{2n}] \quad \text{for all } n \in \mathbb{N}. \quad (5.6)$$

The stable spectrum of L consists of the disjoint union of intervals:

$$\sigma(L) = [\lambda_0, \lambda'_1] \cup [\lambda'_2, \lambda_1] \cup [\lambda_2, \lambda'_3] \cup [\lambda'_4, \lambda_3] \cup \cdots \quad (5.7)$$

Proof. See Pöschel and Trubowitz [20, Chapter 2]. A complete, independently derived proof for $q \in L^2([0, 1], \mathbb{R})$, comprising the Floquet/oscillation argument (Crutoi's Oszillationstheorem) together with the placement of the Dirichlet and Neumann eigenvalues within the spectral gaps, is given by Crutoi [4, Satz 2.3, Korollar 3.8, Korollar 3.13]. $\square$

5.3 Fundamental solutions and monodromy matrix

To analyze the spectral properties of L , we introduce the fundamental solution matrix and the monodromy matrix, which are the central tools in the proof of Theorem 5.14.

Definition 5.4. The *fundamental solution matrix* of (5.4) is the 2×2 matrix

$$F(x, \lambda) = \begin{pmatrix} y_1(x, \lambda) & y_2(x, \lambda) \\ y'_1(x, \lambda) & y'_2(x, \lambda) \end{pmatrix}, \quad (5.8)$$

where y_1 and y_2 are the two linearly independent solutions of (5.4) determined by the initial conditions

$$y_1(0, \lambda) = 1, \quad y'_1(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y'_2(0, \lambda) = 1, \quad (5.9)$$

so that $F(0, \lambda) = I_2$, the 2×2 identity matrix.

Since q is periodic with period 1, if y is a solution of (5.4), then so is $x \mapsto y(x + 1)$. The transition from a solution to its translate by one period is described by the monodromy matrix.

Definition 5.5. The *monodromy matrix* of L is defined by

$$M(\lambda) := F(1, \lambda) = \begin{pmatrix} y_1(1, \lambda) & y_2(1, \lambda) \\ y'_1(1, \lambda) & y'_2(1, \lambda) \end{pmatrix}. \quad (5.10)$$

Its trace

$$\Delta(\lambda) := \text{tr}(M(\lambda)) = y_1(1, \lambda) + y'_2(1, \lambda) \quad (5.11)$$

is called the *discriminant* of L .

The monodromy matrix describes how solutions evolve over one period. Concretely, for any solution y of (5.4), the values of y and y' after n periods are given by:

$$\begin{pmatrix} y(x+n) \\ y'(x+n) \end{pmatrix} = M(\lambda)^n \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}. \quad (5.12)$$

The monodromy matrix has the following key properties.

Lemma 5.6. *The monodromy matrix $M(\lambda)$ satisfies:*

- (i) $\det M(\lambda) = 1$ for all $\lambda \in \mathbb{C}$.
- (ii) The eigenvalues of $M(\lambda)$ are $\rho_{\pm}(\lambda) = \frac{\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4}}{2}$, and satisfy $\rho_+(\lambda) \cdot \rho_-(\lambda) = 1$.
- (iii) A solution y of (5.4) is bounded on all of $\mathbb{R}$ if and only if either $|\Delta(\lambda)| < 2$, or $|\Delta(\lambda)| = 2$ and $M(\lambda)$ is diagonalizable.

Proof. Parts (i) and (ii) follow from the constancy of the Wronski determinant of y_1, y_2 , see Crutoi [4, Lemma 1.4, Definition 1.6], who proves this for the general class $q \in L^2([0, 1], \mathbb{R})$, and whose argument applies directly to our smooth periodic potentials. Part (iii) is the standard Floquet boundedness criterion for the monodromy matrix; see Pöschel and Trubowitz [20, Chapter 1]. $\square$

Definition 5.7. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function. A point $\lambda_0 \in \mathbb{C}$ is called a *simple zero* of f if $f(\lambda_0) = 0$ and $f'(\lambda_0) \neq 0$, and a *double zero* if $f(\lambda_0) = 0$, $f'(\lambda_0) = 0$, and $f''(\lambda_0) \neq 0$.

In the context of the discriminant, a simple zero λ_0 of $\Delta(\lambda)^2 - 4$ satisfies:

$$\Delta(\lambda_0)^2 - 4 = 0 \quad \text{and} \quad \left. \frac{d}{d\lambda}(\Delta(\lambda)^2 - 4) \right|_{\lambda_0} = 2\Delta(\lambda_0)\Delta'(\lambda_0) \neq 0.$$

Since $\Delta(\lambda_0) = \pm 2 \neq 0$, this is equivalent to $\Delta'(\lambda_0) \neq 0$. A double zero λ_0 of $\Delta(\lambda)^2 - 4$ satisfies:

$$\Delta(\lambda_0)^2 - 4 = 0 \quad \text{and} \quad \Delta'(\lambda_0) = 0.$$

The following lemma, due to Magnus and Winkler [15], describes the behavior of the monodromy matrix at zeros of $\Delta(\lambda)^2 - 4$. It will be used crucially in the proof of Theorem 5.14.

Lemma 5.8 (Magnus-Winkler [15], Lemma 2.5, see also Crutoi [4], Lemma 2.16). *Let $q \in C^\infty(\mathbb{R}, \mathbb{R})$ with $q(x+1) = q(x)$. Then:*

- (i) *At a simple zero λ_0 of $\Delta(\lambda)^2 - 4$, the monodromy matrix satisfies $M(\lambda_0) \neq \pm I_2$.*
- (ii) *At a double zero λ_0 of $\Delta(\lambda)^2 - 4$, the monodromy matrix satisfies $M(\lambda_0) = \pm I_2$.*

Proof. This is proved by Magnus and Winkler [15, Lemma 2.5] under the assumption that q is smooth (or continuous). Crutoi [4, Lemma 2.16, Korollar 2.17] gives an independent proof for $q \in L^2([0, 1], \mathbb{R})$, extended 1-periodically to $\mathbb{R}$, based on the multiplicity analysis of the zeros of $\Delta(\lambda)^2 - 4$. In particular, both statements follow from the equivalence established in Korollar 2.17 of [4]: the Hill equation possesses two linearly independent (anti-)periodic solutions at λ if and only if λ is a double zero of $\Delta(\lambda)^2 - 4$, which is in turn equivalent to $M(\lambda) = \pm I_2$. Indeed, a simple zero λ_0 of $\Delta(\lambda)^2 - 4$ satisfies $\Delta'(\lambda_0) \neq 0$, so by the sign formula discussed in [4], the geometric and algebraic multiplicities of the corresponding eigenvalue of $M(\lambda_0)$ differ, giving $M(\lambda_0) \neq \pm I_2$. Conversely, a double zero λ_0 satisfies $\Delta'(\lambda_0) = 0$, which is equivalent to $M(\lambda_0) = \pm I_2$ by the same result. Since our potential $q \in C^\infty(\mathbb{R}, \mathbb{R})$ is itself 1-periodic, its restriction $q|_{[0,1]}$ lies in $L^2([0, 1], \mathbb{R})$, and periodically extending this restriction recovers q itself. Her result, stated for the extended potential and the associated globally defined monodromy matrix, therefore applies directly to our setting. $\square$

5.4 The discriminant $\Delta(\lambda)$

The discriminant $\Delta(\lambda)$ encodes all spectral information about L . By Lemma 5.6 (iii), the stable spectrum of L is exactly:

$$\sigma(L) = \{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \leq 2\}. \quad (5.13)$$

The complement $\{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| > 2\}$ consists of open intervals called *spectral gaps*. The boundary points of the gaps are exactly the zeros of $\Delta(\lambda)^2 - 4$, which are related to the periodic and anti-periodic eigenvalues as follows:

- The periodic eigenvalues $(\lambda_n)_{n \in \mathbb{N}_0}$ are exactly the zeros of $\Delta(\lambda) - 2$.
- The anti-periodic eigenvalues $(\lambda'_n)_{n \in \mathbb{N}}$ are exactly the zeros of $\Delta(\lambda) + 2$.

For a detailed proof of this correspondence, see Crutoi [4, Proposition 1.9]. A spectral gap $(\lambda'_{2n-1}, \lambda'_{2n})$ or $(\lambda_{2n-1}, \lambda_{2n})$ is called *open* if its endpoints are distinct, and *closed* if they coincide. A closed gap corresponds to a double zero of $\Delta(\lambda)^2 - 4$.

5.5 Band-gap structure of the spectrum

By Theorem 5.3, the stable spectrum of L consists of the bands:

$$\sigma(L) = [\lambda_0, \lambda'_1] \cup [\lambda'_2, \lambda_1] \cup [\lambda_2, \lambda'_3] \cup \dots, \quad (5.14)$$

and the spectral gaps are the open intervals between consecutive bands. The n -th gap has length $|\lambda'_{2n-1} - \lambda'_{2n}|$ or $|\lambda_{2n-1} - \lambda_{2n}|$ respectively, and is open if and only if this length is positive. This band-gap decomposition is worked out in detail, for $q \in L^2([0, 1], \mathbb{R})$, in Crutoi [4, Abschnitt 2.2.4].

5.6 Finite number of spectral gaps

We say that L has *finitely many spectral gaps* if only finitely many of the gaps are open, i.e., there exists $N \in \mathbb{N}$ such that $\lambda'_{2n-1} = \lambda'_{2n}$ and $\lambda_{2n-1} = \lambda_{2n}$ for all $n > N$. This is a consequence of the structure of the eigenvalue sequences: by Theorem 5.3, the periodic and anti-periodic eigenvalues form a strictly increasing sequence (up to the inequalities $\leq$), so the spectral gaps appear exactly once in a fixed order and cannot repeat. More precisely, the n -th gap is the interval $(\lambda'_{2n-1}, \lambda'_{2n})$ or $(\lambda_{2n-1}, \lambda_{2n})$, and since the eigenvalue sequences are monotonically increasing, each gap corresponds to a unique interval in this sequence. Therefore, saying that only finitely many gaps are open is the same as saying that from some index N onwards, all gaps are closed, i.e., the corresponding eigenvalues coincide. Equivalently, $\Delta(\lambda)^2 - 4$ has only finitely many simple zeros; this equivalence rests on the fact that every zero of $\Delta(\lambda)^2 - 4$ has multiplicity at most two, with a double zero corresponding to a closed gap, which is proved for $q \in L^2([0, 1], \mathbb{R})$ by Crutoi [4, Lemma 2.16, Korollar 2.17]. This is a very special property: a generic periodic potential has infinitely many open gaps. As we will show in Theorem 5.14, potentials with finitely many open gaps are precisely the finite-type potentials.

5.7 Characterization of periodic finite-type potentials

Remark 5.9. Recall that a $\mathbb{C}$ -valued function is called *analytic* at a point if it can be represented as a convergent power series in a neighborhood of that point, and *entire* if it is analytic on all of $\mathbb{C}$. A matrix-valued function is called analytic (resp. entire) if each of its entries is analytic (resp. entire).

The proof of the main theorem below will require finer control on the zero structure of certain entire functions of finite order, in particular the construction of an entire square root of $\Delta(\lambda)^2 - 4$. We therefore recall the relevant notions of order and genus of an entire function, and the Hadamard factorization theorem.

Definition 5.10 (Order of an entire function). Let g be an entire function. The *order* of g is

$$\rho(g) := \limsup_{r \rightarrow \infty} \frac{\ln \ln M(r)}{\ln r}, \quad M(r) := \max_{|\lambda|=r} |g(\lambda)|.$$

Definition 5.11 (Genus of an entire function). Let g be an entire function of finite order, with a zero of order $m \geq 0$ at $\lambda = 0$ and remaining zeros $(\lambda_n)_n$ (listed with multiplicity). Let s be the smallest non-negative integer such that

$$\sum_n \frac{1}{|\lambda_n|^{s+1}} < \infty,$$

called the *rank* of the sequence (λ_n) , and for $s \geq 0$ define the *Weierstrass elementary factor of degree s* by

$$E_s(w) := (1 - w) \exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^s}{s}\right), \quad E_0(w) := 1 - w.$$

By the Weierstrass factorization theorem, g can be written as

$$g(\lambda) = \lambda^m e^{Q(\lambda)} \prod_n E_s\left(\frac{\lambda}{\lambda_n}\right)$$

for some entire function Q , and since g has finite order, Q may be taken to be a polynomial. The *genus* of g is

$$\mu(g) := \max(s, \deg Q).$$

Theorem 5.12 (Hadamard Factorization Theorem, see [3]). *Let g be an entire function of finite order $\rho(g)$. Then g has finite genus $\mu(g)$, related to the order by Borel's inequality:*

$$\mu(g) \leq \rho(g) \leq \mu(g) + 1.$$

Moreover, g admits the factorization

$$g(\lambda) = \lambda^m e^{Q(\lambda)} \prod_n E_{\mu(g)}\left(\frac{\lambda}{\lambda_n}\right),$$

where Q is a polynomial of degree at most $\mu(g)$, m is the order of g at $\lambda = 0$, and $(\lambda_n)_n$ are the remaining zeros of g , listed with multiplicity.

Corollary 5.13 (Entire square roots). *Let g be an entire function of finite order all of whose zeros have even multiplicity. Then g admits an entire square root, i.e. there exists an entire function $\sqrt{g}$ with $(\sqrt{g})^2 = g$, unique up to sign.*

We now have all the tools necessary to state and prove the main result of this chapter, which connects the algebraic notion of finite type from Chapter 4 with the spectral theory developed above.

Theorem 5.14. *A potential $q \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying $q(x+1) = q(x)$ for all $x \in \mathbb{R}$ is of finite type if and only if the stable spectrum of the associated Hill operator $L = -\frac{d^2}{dx^2} + q$ consists of only finitely many intervals, or equivalently, L has only finitely many open spectral gaps.*

Proof. ($\Rightarrow$): **Finite type implies finitely many spectral gaps.**

Suppose q is of finite type. By Theorem 4.8 (i), there exists a solution $\tilde{\xi}$ of the Lax equation (3.9) with only finitely many non-vanishing coefficients $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n$.

Step 1: ξ is periodic.

By Lemma 4.7 and the proof of (i) $\Rightarrow$ (ii) in Theorem 4.8, the coefficients $\tilde{\beta}_n$ can be written as:

$$\tilde{\beta}_n = \beta_n + \sum_{l=0}^{n-1} d_l \beta_l \tag{5.15}$$

for constants $d_l \in \mathbb{K}$. Since the β_n are differential polynomials in q by Chapter 3, and since q is periodic with $q(x+1) = q(x)$, all derivatives of q are also periodic:

$$q^{(k)}(x+1) = q^{(k)}(x) \quad \text{for all } k \in \mathbb{N}_0. \tag{5.16}$$

Therefore each $\beta_n(x+1) = \beta_n(x)$, and since the d_l are constants:

$$\tilde{\beta}_n(x+1) = \beta_n(x+1) + \sum_{l=0}^{n-1} d_l \beta_l(x+1) = \beta_n(x) + \sum_{l=0}^{n-1} d_l \beta_l(x) = \tilde{\beta}_n(x). \quad (5.17)$$

The same argument applies to $\tilde{\alpha}_n$ and $\tilde{\gamma}_n$, so $\tilde{\xi}(x+1, \lambda) = \tilde{\xi}(x, \lambda)$ for all $x \in \mathbb{R}$.

Step 2: $M(\lambda)$ commutes with $\tilde{\xi}(0, \lambda)$.

We claim that:

$$\tilde{\xi}(x, \lambda) = F(x, \lambda) \tilde{\xi}(0, \lambda) F^{-1}(x, \lambda). \quad (5.18)$$

To see this, let $\eta(x) := F(x, \lambda) \tilde{\xi}(0, \lambda) F^{-1}(x, \lambda)$. At $x = 0$:

$$\eta(0) = F(0, \lambda) \tilde{\xi}(0, \lambda) F^{-1}(0, \lambda) = I_2 \cdot \tilde{\xi}(0, \lambda) \cdot I_2 = \tilde{\xi}(0, \lambda).$$

We now show that η satisfies the same Lax equation as $\tilde{\xi}$. Using $\frac{d}{dx} F = AF$ (since F is the fundamental solution) and, by differentiating $F \cdot F^{-1} = I_2$ with the product rule:

$$\frac{d}{dx} F \cdot F^{-1} + F \cdot \frac{d}{dx} F^{-1} = 0 \implies \frac{d}{dx} F^{-1} = -F^{-1} \frac{d}{dx} F \cdot F^{-1} = -F^{-1} A,$$

where $A(x, \lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix}$, we compute:

$$\begin{aligned} \frac{d}{dx} \eta &= \frac{d}{dx} F \cdot \tilde{\xi}(0, \lambda) \cdot F^{-1} + F \cdot \tilde{\xi}(0, \lambda) \cdot \frac{d}{dx} F^{-1} \\ &= AF \tilde{\xi}(0, \lambda) F^{-1} - F \tilde{\xi}(0, \lambda) F^{-1} A \\ &= [A, \eta]. \end{aligned} \quad (5.19)$$

Thus η satisfies the Lax equation $\frac{d}{dx} \eta = [A, \eta]$, which is the same as (3.9) since $A = U(\lambda)$, and agrees with $\tilde{\xi}$ at $x = 0$. By uniqueness of solutions of the Lax equation, we conclude:

$$\tilde{\xi}(x, \lambda) = F(x, \lambda) \tilde{\xi}(0, \lambda) F^{-1}(x, \lambda).$$

Setting $x = 1$ and using $\tilde{\xi}(1, \lambda) = \tilde{\xi}(0, \lambda)$ from Step 1, and recalling that $M(\lambda) = F(1, \lambda)$:

$$\tilde{\xi}(0, \lambda) = F(1, \lambda) \tilde{\xi}(0, \lambda) F^{-1}(1, \lambda) = M(\lambda) \tilde{\xi}(0, \lambda) M^{-1}(\lambda), \quad (5.20)$$

which gives:

$$M(\lambda) \tilde{\xi}(0, \lambda) = \tilde{\xi}(0, \lambda) M(\lambda). \quad (5.21)$$

Step 3: $\Delta(\lambda)^2 - 4$ has only finitely many simple zeros.

Let λ_0 be a simple zero of $\Delta(\lambda)^2 - 4$. By Lemma 5.8 (i), $M(\lambda_0) \neq \pm I_2$.

Since λ_0 is a zero of $\Delta(\lambda)^2 - 4$, we have $\Delta(\lambda_0) = \pm 2$, so the characteristic polynomial of $M(\lambda_0)$ is $\rho^2 \mp 2\rho + 1 = (\rho \mp 1)^2$, i.e., $\rho = \pm 1$ is a repeated eigenvalue of $M(\lambda_0)$. A 2×2 matrix with a repeated eigenvalue ρ is diagonalizable if and only if it equals ρI_2 : if $\ker(M(\lambda_0) \mp I_2) = \mathbb{K}^2$, then $M(\lambda_0) \mp I_2$ vanishes on every vector, so $M(\lambda_0) = \pm I_2$;

otherwise $\ker(M(\lambda_0) \mp I_2)$ is only one-dimensional and $M(\lambda_0)$ has a nontrivial Jordan block, hence is not diagonalizable. Since $M(\lambda_0) \neq \pm I_2$, we conclude that $M(\lambda_0)$ is not diagonalizable.

We now write $M(\lambda_0) = \pm I_2 + N$, where $N := M(\lambda_0) \mp I_2$. By the Cayley-Hamilton theorem, $M(\lambda_0)$ satisfies its characteristic equation $M(\lambda_0)^2 \mp 2M(\lambda_0) + I_2 = 0$, which gives:

$$N^2 = (M(\lambda_0) \mp I_2)^2 = M(\lambda_0)^2 \mp 2M(\lambda_0) + I_2 = 0,$$

so N is nilpotent. Moreover, $N \neq 0$ since $M(\lambda_0) \neq \pm I_2$.

Since $\tilde{\xi}(0, \lambda_0)$ commutes with $M(\lambda_0)$ by Step 2, it also commutes with $N = M(\lambda_0) \mp I_2$. We now show that a traceless 2×2 matrix commuting with a nonzero nilpotent 2×2 matrix must have determinant zero. Since $N^2 = 0$, every eigenvalue ρ of N satisfies $\rho^2 = 0$ (as $N^2 v = \rho^2 v = 0$ for any eigenvector v), hence $\rho = 0$. The eigenspace $\ker(N)$ is one-dimensional: it cannot be trivial since $\det(N) = 0$ (as all eigenvalues are zero), and it cannot be two-dimensional since that would imply $N = 0$, contradicting $N \neq 0$. Since N is a 2×2 matrix with the single eigenvalue $\rho = 0$ and one-dimensional eigenspace $\ker(N)$, we can find a basis $\{v_1, v_2\}$ of $\mathbb{K}^2$ with $Nv_1 = 0$ and $Nv_2 = v_1$: indeed, choose any $v_2 \notin \ker(N)$; then $Nv_2 \neq 0$ and $N^2 v_2 = 0$, so $Nv_2 \in \ker(N)$, and we can set $v_1 := Nv_2$. In this basis, N has the Jordan normal form $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, i.e., there exists an invertible matrix S such that $N = SJS^{-1}$. Working in the basis given by the columns of S , i.e., replacing $C = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ by $S^{-1}CS$ and N by J , the condition $[C, N] = 0$ becomes $[S^{-1}CS, J] = 0$. Since $S^{-1}CS$ is again traceless (as the trace is invariant under conjugation), it suffices to show the claim for $N = J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$:

$$[C, N] = CN - NC = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} - \begin{pmatrix} c & -a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -c & 2a \\ 0 & c \end{pmatrix} = 0,$$

which forces $c = 0$ and $a = 0$, so $C = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = bN$. Therefore:

$$\det(C) = \det(bN) = b^2 \det(N) = 0.$$

Applying this to $C = \tilde{\xi}(0, \lambda_0)$, which is traceless (since $\tilde{\xi}$ has the form $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ by the Lax structure, as the tracelessness of $U(\lambda)$ is preserved by the Lax equation (3.9)) and commutes with N by Step 2, we conclude:

$$\det \tilde{\xi}(0, \lambda_0) = 0. \tag{5.22}$$

Since λ_0 was an arbitrary simple zero of $\Delta(\lambda)^2 - 4$, we have shown that every simple zero of $\Delta(\lambda)^2 - 4$ is a zero of $\det \tilde{\xi}(0, \lambda)$. It remains to show that $\det \tilde{\xi}(0, \lambda)$ has only finitely many zeros.

Since $\det \tilde{\xi}(x, \lambda)$ is independent of x (as shown in Section 3.4), we evaluate at $x = 0$. Since $\tilde{\xi}$ has only finitely many non-vanishing coefficients running from $n = 0$ to N , the entries of $\tilde{\xi}(0, \lambda)$ are Laurent polynomials in λ of the form:

$$\begin{aligned}\alpha(0, \lambda) &= \sum_{n=0}^N \alpha_n(0) \lambda^{-n}, \\ \beta(0, \lambda) &= \sum_{n=0}^N \beta_n(0) \lambda^{-n}, \\ \gamma(0, \lambda) &= \gamma_{-1} \lambda + \sum_{n=0}^N \gamma_n(0) \lambda^{-n}.\end{aligned}$$

Therefore $\det \tilde{\xi}(0, \lambda) = -\alpha(0, \lambda)^2 - \beta(0, \lambda)\gamma(0, \lambda)$ is a Laurent polynomial in λ of the form:

$$\det \tilde{\xi}(0, \lambda) = \sum_{k=-2N}^1 c_k \lambda^k.$$

Multiplying by λ^{2N} yields a polynomial of degree at most $2N+1$, which by the fundamental theorem of algebra has at most $2N+1$ zeros. Therefore $\det \tilde{\xi}(0, \lambda)$ has at most $2N+1$ zeros, and since:

$$\{\text{simple zeros of } \Delta(\lambda)^2 - 4\} \subseteq \{\text{zeros of } \det \tilde{\xi}(0, \lambda)\},$$

it follows that $\Delta(\lambda)^2 - 4$ has at most $2N+1$ simple zeros. Therefore L has only finitely many open spectral gaps. $\square$

($\Leftarrow$): Finitely many spectral gaps implies finite type.

Suppose the stable spectrum of L consists of only finitely many intervals, i.e., $\Delta(\lambda)^2 - 4$ has only finitely many simple zeros. Let $E_0, E_1, \dots, E_{2N}$ denote these simple zeros, and define the monic polynomial:

$$p(\lambda) := \prod_{m=0}^{2N} (\lambda - E_m) \in \mathbb{R}[\lambda]. \quad (5.23)$$

This polynomial is uniquely determined by its zeros and leading coefficient. Note that the number of simple zeros is odd, since by Theorem 5.3 the smallest eigenvalue λ_0 is always a simple zero of $\Delta(\lambda)^2 - 4$, and the remaining simple zeros come in pairs corresponding to the open gaps $(\lambda'_{2n-1}, \lambda'_{2n})$ and $(\lambda_{2n-1}, \lambda_{2n})$.

Step 1: $2M(\lambda) - \Delta(\lambda)I_2$ vanishes at all double zeros of $\Delta(\lambda)^2 - 4$.

Here, a *simple zero* of $\Delta(\lambda)^2 - 4$ is a zero of multiplicity one, and a *double zero* is a zero of multiplicity two. A double zero arises when two simple zeros coincide, corresponding to a closed spectral gap. By the Cayley-Hamilton theorem, $M(\lambda)$ satisfies its characteristic equation:

$$M(\lambda)^2 - \Delta(\lambda)M(\lambda) + I_2 = 0.$$

Completing the square:

$$\left(M(\lambda) - \frac{1}{2}\Delta(\lambda)I_2\right)^2 = \frac{\Delta(\lambda)^2 - 4}{4}I_2. \quad (5.24)$$

Since $\Delta(\lambda_0)^2 - 4 = 0$ at a double zero λ_0 , the right-hand side of (5.24) vanishes, giving:

$$\left(M(\lambda_0) - \frac{1}{2}\Delta(\lambda_0)I_2\right)^2 = 0.$$

By Lemma 5.8 (ii), $M(\lambda_0) = \pm I_2$ at a double zero λ_0 . Since $\Delta(\lambda_0) = \text{tr}(M(\lambda_0)) = \text{tr}(\pm I_2) = \pm 2$, we compute:

$$M(\lambda_0) - \frac{1}{2}\Delta(\lambda_0)I_2 = \pm I_2 - \frac{1}{2}(\pm 2)I_2 = \pm I_2 \mp I_2 = 0,$$

and therefore $2M(\lambda_0) - \Delta(\lambda_0)I_2 = 0$.

Step 2: An entire square root of the discriminant.

Define

$$f(\lambda) := \frac{\Delta(\lambda)^2 - 4}{p(\lambda)}.$$

Since $\Delta(\lambda)$ and $p(\lambda)$ are analytic in λ , the only possible singularities of $f(\lambda)$ are at the zeros of $p(\lambda)$, i.e., at the simple zeros of $\Delta(\lambda)^2 - 4$. We show these are removable and that every remaining zero of f has even order, by distinguishing two cases.

At a simple zero λ_0 of $\Delta(\lambda)^2 - 4$: By definition of p , λ_0 is a simple zero of $p(\lambda)$, so we can write $p(\lambda) = (\lambda - \lambda_0)\tilde{p}(\lambda)$ with $\tilde{p}(\lambda_0) \neq 0$. Since $\Delta(\lambda)^2 - 4$ also vanishes to first order at λ_0 , we write $\Delta(\lambda)^2 - 4 = (\lambda - \lambda_0)s(\lambda)$ with $s(\lambda_0) \neq 0$. Therefore

$$f(\lambda) = \frac{(\lambda - \lambda_0)s(\lambda)}{(\lambda - \lambda_0)\tilde{p}(\lambda)} = \frac{s(\lambda)}{\tilde{p}(\lambda)},$$

which is analytic at λ_0 , with $f(\lambda_0) = s(\lambda_0)/\tilde{p}(\lambda_0) \neq 0$.

At a double zero λ_0 of $\Delta(\lambda)^2 - 4$: Then $p(\lambda_0) \neq 0$, since λ_0 is not among the simple zeros $E_0, \dots, E_{2N}$ of $\Delta(\lambda)^2 - 4$ that make up the zeros of p . Writing $\Delta(\lambda)^2 - 4 = (\lambda - \lambda_0)^2 h(\lambda)$ with $h(\lambda_0) \neq 0$, we get

$$f(\lambda) = \frac{(\lambda - \lambda_0)^2 h(\lambda)}{p(\lambda)},$$

which is analytic at λ_0 with a zero of order exactly two.

Since these are all the zeros of $\Delta(\lambda)^2 - 4$, $f(\lambda)$ is an entire function, and every zero of f has even multiplicity. Moreover, $\Delta(\lambda)^2 - 4$ is an entire function of finite order in the sense of Definition 5.10. Write $\mu = \sqrt{\lambda}$, so that $|\lambda| = r$ corresponds to $|\mu| = \sqrt{r}$. Using $|\sin \mu| \leq \frac{1}{2}e^{|\text{Im} \mu|}$ and $|\text{Im} \mu| \leq |\mu|$, the asymptotic estimate $\Delta(\lambda)^2 - 4 = -4\sin^2 \mu + O(e^{2|\text{Im} \mu|}/|\mu|)$ established below in Step 3 gives, for a constant $C > 0$ and all sufficiently large $r = |\lambda|$,

$$|\Delta(\lambda)^2 - 4| \leq C e^{2|\mu|} = C e^{2\sqrt{r}}.$$

Hence $\ln M(r) \leq \ln C + 2\sqrt{r} = O(\sqrt{r})$, and therefore

$$\ln \ln M(r) = \ln(2\sqrt{r} + O(1)) = \frac{1}{2} \ln r + O(1),$$

which shows that $\Delta(\lambda)^2 - 4$, viewed as an entire function of λ , has order at most $\frac{1}{2}$. Since p is monic of degree $2N+1$, we have $|p(\lambda)| \geq |\lambda|^{2N+1} - C_1|\lambda|^{2N} = r^{2N+1}(1 - C_1/r) \geq \frac{1}{2}r^{2N+1}$ for a constant $C_1 > 0$ and all r large enough, so

$$|f(\lambda)| = \frac{|\Delta(\lambda)^2 - 4|}{|p(\lambda)|} \leq \frac{2C}{r^{2N+1}} e^{2\sqrt{r}}$$

for all λ with $|\lambda| = r$. Hence the maximum modulus of f satisfies $\ln M_f(r) \leq 2\sqrt{r} + O(\ln r)$, and since $\ln r = o(\sqrt{r})$, the same computation as above gives $\ln \ln M_f(r) = \frac{1}{2} \ln r + O(1)$. Therefore dividing by the polynomial p does not change the order, and f is likewise an entire function of order $\frac{1}{2}$.

Since f is entire of order $\rho(f) = \frac{1}{2}$ (Definition 5.10) and its genus $\mu(f)$ (Definition 5.11) is a non-negative integer, Borel's inequality from Theorem 5.12 forces $\mu(f) = 0$. This means simultaneously that the Weierstrass elementary factors in the Hadamard factorization of f have rank $s = 0$, i.e. are simply $E_0(w) = 1 - w$, and that the polynomial Q in the exponential factor satisfies $\deg Q = 0$, i.e. is a constant $c \in \mathbb{C}$. Writing $\{\lambda_n\}$ for the zeros of f away from the origin, listed with multiplicity, Theorem 5.12 therefore gives

$$f(\lambda) = \lambda^m e^c \prod_n \left(1 - \frac{\lambda}{\lambda_n}\right),$$

where m is the order of f at $\lambda = 0$, and the product converges locally uniformly since rank $s = 0$ is equivalent to $\sum_n 1/|\lambda_n| < \infty$.

By the case distinction above, every zero of f has multiplicity exactly two: at a simple zero of $\Delta(\lambda)^2 - 4$, f is regular and non-zero, while at a double zero of $\Delta(\lambda)^2 - 4$, f has a zero of order exactly two. In particular m is even, $m = 2m'$, and the list $\{\lambda_n\}$ consists of the distinct double zeros ν_j of $\Delta(\lambda)^2 - 4$, each occurring twice, so that

$$\prod_n \left(1 - \frac{\lambda}{\lambda_n}\right) = \prod_j \left(1 - \frac{\lambda}{\nu_j}\right)^2.$$

Substituting this back, f is exhibited as a perfect square of an entire function:

$$f(\lambda) = \left[\lambda^{m'} e^{c/2} \prod_j \left(1 - \frac{\lambda}{\nu_j}\right) \right]^2.$$

We define

$$\sqrt{f(\lambda)} := \lambda^{m'} e^{c/2} \prod_j \left(1 - \frac{\lambda}{\nu_j}\right),$$

for a fixed choice of $e^{c/2}$. This is entire, since it is built from the same convergent factors as f itself, and satisfies $(\sqrt{f})^2 = f$ by construction. By this construction, $\sqrt{f}$ has a zero of order exactly one at each double zero ν_j of $\Delta(\lambda)^2 - 4$, and is non-zero at each simple zero.

This square root is unique up to sign: if g_1, g_2 are entire with $g_1^2 = g_2^2 = f$, then $h := g_1/g_2$ satisfies $h^2 \equiv 1$ away from the zeros of f , and since g_1, g_2 vanish to the same (simple) order at each such zero, h extends to an entire function with $h^2 \equiv 1$; as $\mathbb{C}$ is connected, h is constant, so $h \equiv \pm 1$ and $g_1 = \pm g_2$. We fix the sign by normalizing $\sqrt{f}$ to be positive for sufficiently negative real λ .

We now define

$$\xi(x, \lambda) := \frac{1}{\sqrt{f(\lambda)}} F(x, \lambda) (2M(\lambda) - \Delta(\lambda)I_2) F^{-1}(x, \lambda).$$

For fixed λ with $\sqrt{f(\lambda)} \neq 0$, the matrix $F(x, \lambda) (2M(\lambda) - \Delta(\lambda)I_2) F^{-1}(x, \lambda)$ solves the Lax equation (3.9) by exactly the same argument as for η in Step 2 of the forward direction, and since $\sqrt{f(\lambda)}$ does not depend on x , $\xi(x, \lambda)$ solves (3.9) on its domain.

We show that $\xi(x, \lambda)$ in fact extends to an entire function of λ for every fixed x . At a simple zero λ_0 of $\Delta(\lambda)^2 - 4$ we have $f(\lambda_0) \neq 0$ by the computation above, so $\sqrt{f(\lambda_0)} \neq 0$ and $\xi(x, \lambda)$ is regular at λ_0 . At a double zero λ_0 of $\Delta(\lambda)^2 - 4$, by Step 1, $2M(\lambda) - \Delta(\lambda)I_2$ vanishes to order one at λ_0 . Since $\sqrt{f}$ has a zero of order exactly one there, the entrywise quotient $(2M - \Delta I_2)/\sqrt{f}$ is regular at λ_0 . Since $F(x, \lambda)$ and $F^{-1}(x, \lambda)$ are entire and invertible for all λ , $\xi(x, \lambda)$ is regular at λ_0 as well.

Since these are the only zeros of $\Delta(\lambda)^2 - 4$, $\xi(x, \lambda)$ is an entire function of λ for every fixed $x \in \mathbb{R}$.

Step 3: ξ is a matrix-valued polynomial in λ . We write $\lambda = \mu^2$ and use the asymptotic estimates of Theorem 1.3 of Pöschel and Trubowitz [20]; an independent derivation of the analogous asymptotics for $q \in L^2([0, 1], \mathbb{R})$, via the Volterra integral equations for y_1, y_2 , is given by Crutoi [4, Lemma 2.1, Lemma 2.2, Proposition 2.5]:

$$\begin{aligned} y_1(1, \lambda) &= \cos \mu + O\left(\frac{e^{|\operatorname{Im} \mu|}}{|\mu|}\right), & y_2(1, \lambda) &= \frac{\sin \mu}{\mu} + O\left(\frac{e^{|\operatorname{Im} \mu|}}{\mu^2}\right), \\ y_1'(1, \lambda) &= -\mu \sin \mu + O\left(e^{|\operatorname{Im} \mu|}\right), & y_2'(1, \lambda) &= \cos \mu + O\left(\frac{e^{|\operatorname{Im} \mu|}}{|\mu|}\right), \end{aligned}$$

as $|\mu| \rightarrow \infty$. As before,

$$2M(\lambda) - \Delta(\lambda)I_2 = \begin{pmatrix} y_1(1, \lambda) - y_2'(1, \lambda) & 2y_2(1, \lambda) \\ 2y_1'(1, \lambda) & y_2'(1, \lambda) - y_1(1, \lambda) \end{pmatrix}.$$

We now derive the asymptotics of $\Delta(\lambda)$ and $\Delta(\lambda)^2 - 4$ from the first and fourth estimates above (the ones for $y_1(1, \lambda)$ and $y_2'(1, \lambda)$, since $\Delta(\lambda) = y_1(1, \lambda) + y_2'(1, \lambda)$ by definition of the discriminant). Write

$$y_1(1, \lambda) = \cos \mu + \varepsilon_1(\mu), \quad y_2'(1, \lambda) = \cos \mu + \varepsilon_4(\mu),$$

for the error terms $\varepsilon_1, \varepsilon_4$ implicit in the first and fourth $O(\cdot)$ -estimates above; both satisfy $|\varepsilon_1(\mu)|, |\varepsilon_4(\mu)| = O(e^{|\operatorname{Im} \mu|}/|\mu|)$ as $|\mu| \rightarrow \infty$. Adding the two expansions gives

$$\Delta(\lambda) = y_1(1, \lambda) + y_2'(1, \lambda) = 2 \cos \mu + \underbrace{\varepsilon_1(\mu) + \varepsilon_4(\mu)}_{=: \varepsilon(\mu)} = 2 \cos \mu + O\left(\frac{e^{|\operatorname{Im} \mu|}}{|\mu|}\right), \quad (5.25)$$

where $\varepsilon(\mu) := \varepsilon_1(\mu) + \varepsilon_4(\mu)$ again satisfies $|\varepsilon(\mu)| = O(e^{|\operatorname{Im} \mu|}/|\mu|)$, being a sum of two terms of that order.

To square this expansion, recall the elementary bound $|\cos \mu| \leq e^{|\operatorname{Im} \mu|}$: writing $\cos \mu = \frac{1}{2}(e^{i\mu} + e^{-i\mu})$ and $|e^{\pm i\mu}| = e^{\mp \operatorname{Im} \mu} \leq e^{|\operatorname{Im} \mu|}$ gives $|\cos \mu| \leq \frac{1}{2}(e^{|\operatorname{Im} \mu|} + e^{|\operatorname{Im} \mu|}) = e^{|\operatorname{Im} \mu|}$. Squaring $\Delta(\lambda) = 2 \cos \mu + \varepsilon(\mu)$ and expanding the binomial,

$$\Delta(\lambda)^2 = 4 \cos^2 \mu + 4 \cos \mu \varepsilon(\mu) + \varepsilon(\mu)^2.$$

We estimate the two correction terms separately, using $|\varepsilon(\mu)| = O(e^{|\operatorname{Im} \mu|}/|\mu|)$:

$$|4 \cos \mu \varepsilon(\mu)| = O\left(e^{|\operatorname{Im} \mu|} \cdot \frac{e^{|\operatorname{Im} \mu|}}{|\mu|}\right) = O\left(\frac{e^{2|\operatorname{Im} \mu|}}{|\mu|}\right), \quad |\varepsilon(\mu)^2| = O\left(\frac{e^{2|\operatorname{Im} \mu|}}{|\mu|^2}\right).$$

Since $1/|\mu|^2 = o(1/|\mu|)$ as $|\mu| \rightarrow \infty$, the quadratic term $\varepsilon(\mu)^2$ is of strictly smaller order than the cross term $4 \cos \mu \varepsilon(\mu)$, hence is absorbed into it, and

$$\Delta(\lambda)^2 = 4 \cos^2 \mu + O\left(\frac{e^{2|\operatorname{Im} \mu|}}{|\mu|}\right).$$

Using the identity $4 \cos^2 \mu = 4 - 4 \sin^2 \mu$, we obtain

$$\Delta(\lambda)^2 - 4 = -4 \sin^2 \mu + O\left(\frac{e^{2|\operatorname{Im} \mu|}}{|\mu|}\right). \quad (5.26)$$

We now estimate ξ directly on the circles $|\mu| = (K + \frac{1}{2})\pi$, $K \in \mathbb{N}$. By Lemma 2.1 of [20], for K sufficiently large,

$$|\sin^2 \mu| \geq \frac{1}{16} e^{2|\operatorname{Im} \mu|}.$$

Together with $\Delta(\lambda)^2 - 4 = -4 \sin^2 \mu + O(e^{2|\operatorname{Im} \mu|}/|\mu|)$, this gives for a constant $C_1 > 0$ and K large enough,

$$|\Delta(\lambda)^2 - 4| \geq 4 \cdot \frac{1}{16} e^{2|\operatorname{Im} \mu|} - \frac{C_1}{|\mu|} e^{2|\operatorname{Im} \mu|} \geq \frac{1}{8} e^{2|\operatorname{Im} \mu|}$$

on $|\mu| = (K + \frac{1}{2})\pi$. Since p is monic of degree $2N + 1$, write

$$p(\lambda) = \prod_{m=0}^{2N} (\lambda - E_m) = \lambda^{2N+1} \prod_{m=0}^{2N} \left(1 - \frac{E_m}{\lambda}\right).$$

For $|\lambda| > 2 \max_m |E_m|$, each factor satisfies $|1 - \frac{E_m}{\lambda}| \leq 1 + \frac{|E_m|}{|\lambda|} \leq \frac{3}{2}$, so the product is bounded by a constant $C_2 := (\frac{3}{2})^{2N+1}$, independent of λ . Hence

$$|p(\lambda)| \leq C_2 |\lambda|^{2N+1} = C_2 |\mu|^{4N+2}$$

on these circles, once K is large enough that $|\mu| = (K + \frac{1}{2})\pi$ exceeds $2 \max_m |E_m|$. Therefore

$$|f(\lambda)| = \frac{|\Delta(\lambda)^2 - 4|}{|p(\lambda)|} \geq \frac{e^{2|\operatorname{Im} \mu|}}{8C_2 |\mu|^{4N+2}}, \quad |\sqrt{f(\lambda)}| \geq \frac{e^{|\operatorname{Im} \mu|}}{C_3 |\mu|^{2N+1}}$$

for a constant $C_3 = \sqrt{8C_2}$.

It remains to bound $\|F(x, \lambda)(2M(\lambda) - \Delta(\lambda)I_2)F^{-1}(x, \lambda)\|$ from above, uniformly in $x \in [0, 1]$.

For $x_0 \in \mathbb{R}$, define the shifted potential $q_{x_0}(x) := q(x + x_0)$ and set $\tilde{F}(x, \lambda) := F(x + x_0, \lambda)F^{-1}(x_0, \lambda)$. We claim that $\tilde{F}(\cdot, \lambda)$ is the fundamental solution of q_{x_0} , i.e. that it solves $\frac{d}{dx}Y(x) = U_{x_0}(x, \lambda)Y(x)$ with $U_{x_0}(x, \lambda) := \begin{pmatrix} 0 & 1 \\ q_{x_0}(x) - \lambda & 0 \end{pmatrix}$ and $Y(0) = I_2$. Indeed, $\tilde{F}(0, \lambda) = F(x_0, \lambda)F^{-1}(x_0, \lambda) = I_2$, and by the chain rule together with $\frac{d}{dx}F(x, \lambda) = U(x, \lambda)F(x, \lambda)$,

$$\begin{aligned} \frac{d}{dx}\tilde{F}(x, \lambda) &= \frac{d}{dx}F(x + x_0, \lambda) \cdot F^{-1}(x_0, \lambda) \\ &= U(x + x_0, \lambda) F(x + x_0, \lambda) F^{-1}(x_0, \lambda) = U(x + x_0, \lambda) \tilde{F}(x, \lambda). \end{aligned}$$

Since $U(x + x_0, \lambda) = \begin{pmatrix} 0 & 1 \\ q(x + x_0) - \lambda & 0 \end{pmatrix} = U_{x_0}(x, \lambda)$ by definition of q_{x_0} , the matrix $\tilde{F}(x, \lambda)$ solves the correct equation with the correct initial condition. By uniqueness of solutions of this linear initial value problem, $\tilde{F}(\cdot, \lambda)$ is indeed the fundamental solution of q_{x_0} , and its monodromy matrix is by definition $M_{x_0}(\lambda) := \tilde{F}(1, \lambda)$.

It remains to identify $\tilde{F}(1, \lambda)$. Since q is 1-periodic, $U(x + 1, \lambda) = U(x, \lambda)$ for all x , so $x \mapsto F(x + 1, \lambda)$ solves the same equation $\frac{d}{dx}Y = U(x, \lambda)Y$ as $F(x, \lambda)$ itself, with initial value $F(1, \lambda) = M(\lambda)$ at $x = 0$. By uniqueness, applied now to the original potential q ,

$$F(x + 1, \lambda) = F(x, \lambda) M(\lambda) \quad \text{for all } x \in \mathbb{R}.$$

Evaluating at $x = x_0$ gives $F(x_0 + 1, \lambda) = F(x_0, \lambda)M(\lambda)$, and therefore

$$M_{x_0}(\lambda) = \tilde{F}(1, \lambda) = F(x_0 + 1, \lambda)F^{-1}(x_0, \lambda) = F(x_0, \lambda) M(\lambda) F^{-1}(x_0, \lambda).$$

Since q is a smooth periodic potential, the family $(q_{x_0})_{x_0 \in [0, 1]}$ is compact in the space of periodic L^2 potentials, so the Basic Estimates (Theorem 1.3 of [20]) hold uniformly for this family, and by cyclicity of the trace, Δ does not depend on x_0 . Applying the Basic Estimates to q_{x_0} exactly as above gives

$$\|F(x, \lambda)(2M(\lambda) - \Delta(\lambda)I_2)F^{-1}(x, \lambda)\| \leq C_4 |\mu| e^{|\operatorname{Im} \mu|}$$

uniformly for $x \in [0, 1]$ and λ on the circles $|\mu| = (K + \frac{1}{2})\pi$, for a constant $C_4 > 0$. Combining the two estimates,

$$\|\xi(x, \lambda)\| \leq \frac{C_4 |\mu| e^{|\operatorname{Im} \mu|}}{e^{|\operatorname{Im} \mu|} / (C_3 |\mu|^{2N+1})} = C_3 C_4 |\mu|^{2N+2} = C |\lambda|^{N+1}$$

uniformly in $x \in [0, 1]$, on the circles $|\sqrt{\lambda}| = (K + \frac{1}{2})\pi$, for a constant $C > 0$ independent of K .

Since $\xi(x, \cdot)$ is entire by Step 2, write its Taylor expansion around $\lambda = 0$ entrywise as $\xi(x, \lambda) = \sum_{n=0}^{\infty} A_n(x) \lambda^n$ with matrix coefficients $A_n(x) \in \mathbb{C}^{2 \times 2}$. By Cauchy's estimate for Taylor coefficients (a standard consequence of Cauchy's integral formula, see e.g. [3]), applied entrywise to $\xi(x, \cdot)$ on the circle $|\lambda| = R_K := (K + \frac{1}{2})^2 \pi^2$ (corresponding to $|\sqrt{\lambda}| = (K + \frac{1}{2})\pi$), each entry $(A_n(x))_{ij}$ satisfies

$$|(A_n(x))_{ij}| \leq \frac{\max_{|\lambda|=R_K} |\xi_{ij}(x, \lambda)|}{R_K^n} \leq \frac{C R_K^{N+1}}{R_K^n} = C R_K^{N+1-n},$$

using the bound $\|\xi(x, \lambda)\| \leq C |\lambda|^{N+1}$ established above and the equivalence, up to a fixed constant, of the operator norm and the entrywise bounds on $\mathbb{C}^{2 \times 2}$. For $n > N + 1$ the exponent $N + 1 - n$ is negative, so the right-hand side tends to 0 as $K \rightarrow \infty$ (since $R_K \rightarrow \infty$). Hence $(A_n(x))_{ij} = 0$ for all $n > N + 1$ and all entries i, j , i.e. $A_n(x) = 0$ for $n > N + 1$. Therefore $\xi(x, \lambda) = \sum_{n=0}^{N+1} A_n(x) \lambda^n$ is a matrix-valued polynomial in λ of degree at most $N + 1$, for every $x \in \mathbb{R}$.

Step 4: ξ satisfies the Lax equation and q is of finite type.

By Step 2, $\xi(x, \lambda)$ satisfies the Lax equation (3.9) for every x . It is traceless, since

$$\operatorname{tr}(2M(\lambda) - \Delta(\lambda)I_2) = 2 \operatorname{tr}(M(\lambda)) - 2\Delta(\lambda) = 2\Delta(\lambda) - 2\Delta(\lambda) = 0,$$

and the trace is unaffected by conjugation with $F(x, \lambda)$ or by the scalar factor $1/\sqrt{f(\lambda)}$. Hence $\xi(x, \lambda)$ has the general traceless form

$$\xi(x, \lambda) = \begin{pmatrix} \alpha(x, \lambda) & \beta(x, \lambda) \\ \gamma(x, \lambda) & -\alpha(x, \lambda) \end{pmatrix},$$

where α, β, γ are smooth functions of x satisfying the component equations (3.10)-(3.12) derived in Section 3.3, since ξ satisfies the Lax equation (3.9). By Step 3, $\xi(x, \lambda)$ is, for each fixed x , a polynomial in λ of degree at most $N + 1$, and hence has only finitely many non-vanishing Laurent coefficients.

Furthermore, ξ does not vanish identically: at $x = 0$,

$$\xi(0, \lambda) = \frac{2M(\lambda) - \Delta(\lambda)I_2}{\sqrt{f(\lambda)}},$$

and $2M(\lambda) - \Delta(\lambda)I_2 \not\equiv 0$, since at a simple zero λ_0 of $\Delta(\lambda)^2 - 4$ we have $\Delta(\lambda_0) = \pm 2$, so $\frac{1}{2}\Delta(\lambda_0)I_2 = \pm I_2$, while by Lemma 5.8 (i), $M(\lambda_0) \neq \pm I_2 = \frac{1}{2}\Delta(\lambda_0)I_2$.

Therefore ξ satisfies condition (i) of Theorem 4.8, and by Theorem 4.8, q is of finite type.

Thus q is of finite type if and only if L has only finitely many open spectral gaps. $\square$

Remark 5.15. In the periodic case $q \in C^\infty(\mathbb{R}, \mathbb{R})$ with $q(x+1) = q(x)$, Definition 4.9 is equivalent to the stable spectrum of L consisting of only finitely many intervals, by Theorem 5.14.

Chapter 6

Spectral Theory for L^1 Potentials

The theory developed in Chapters 2-5 rests throughout on the assumption that the potential q is smooth, i.e., $q \in C^\infty(\mathbb{R}, \mathbb{R})$. This assumption was used in an essential way: the coefficients β_n of the formal Laurent series are differential polynomials in q , and the characterization of finite-type potentials via Theorem 4.8 requires q to be infinitely differentiable.

However, from the point of view of functional analysis, the natural setting for the spectral theory of the Hill operator is the much larger space $L^1([0, 1], \mathbb{R})$ of integrable potentials, extended periodically to $\mathbb{R}$. This raises the question how much of the spectral theory developed in Chapter 5 remains valid for L^1 potentials.

The motivation for this question comes from the following long-term goal, suggested by the structure of Theorem 5.14: one expects that periodic finite-type potentials are *dense* in $L^1([0, 1], \mathbb{R})$. To make this statement precise and prove it, one needs a version of the spectral theory that is valid for L^1 potentials. The density statement itself, however, is not established in this thesis and remains open. In particular, the fundamental estimates for the solutions y_1 and y_2 of Hill's equation must be established in this more general setting.

The purpose of this chapter is to carry out this extension. We follow the approach of Pöschel and Trubowitz [20], who develop the spectral theory for $q \in L^2([0, 1], \mathbb{R})$, and show that their arguments extend, with appropriate modifications, to the case $q \in L^1([0, 1], \mathbb{R})$.

The key observation is that the integral representations of y_1 and y_2 , which form the foundation of the entire theory, make sense for L^1 potentials, since the kernel $\frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}}$ is bounded. The main technical difference is that the Schwarz inequality, which is used repeatedly in [20] to estimate the terms of the power series, must be replaced by the Hölder inequality with exponents $p = 1$ and $q = \infty$:

$$\left| \int_0^x f(t)g(t) dt \right| \leq \|f\|_{L^1([0,x])} \|g\|_{L^\infty([0,x])},$$

which holds for all $f \in L^1([0, 1])$ and $g \in L^\infty([0, 1])$. In our setting, $f = q$ is the potential

and g is the bounded kernel $\frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}}$, which satisfies $\|g\|_{L^\infty} \leq \frac{e^{|\operatorname{Im} \sqrt{\lambda}|}}{|\sqrt{\lambda}|}$.

This substitution is carried out uniformly throughout the chapter. We clarify its use once here rather than restating it at every theorem. In each of the theorems of [20] adapted below, the Cauchy-Schwarz inequality enters only in the estimation of integrals of the form $\int_0^x K(x, t, \lambda) q(t) h(t) dt$ against the bounded kernel $K(x, t, \lambda) = \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}}$, where q is the potential and h is the relevant auxiliary function. Replacing this estimate by the Hölder inequality with exponents $p = 1$, $q = \infty$, bounding the integral by $\|K(x, \cdot, \lambda)\|_{L^\infty} \|q\|_{L^1}$ instead of $\|K(x, \cdot, \lambda)\|_{L^2} \|q\|_{L^2}$, is the only place at which the argument of [20] uses the Hilbert space structure of L^2 . The remaining steps of each proof (variation of constants, the inductive estimate on the terms of the Neumann series, the resulting uniform convergence on bounded sets, and differentiation with respect to x , λ and q) do not use the inner product on L^2 and therefore transfer without modification to $q \in L^1$. We record this correspondence once here. In the remainder of the chapter we indicate, for each theorem, only where this single substitution enters the argument of [20], rather than reproducing the full proof.

The chapter is organized as follows. In Section 6.1, we construct the fundamental solutions y_1 and y_2 for L^1 potentials and establish their basic properties. In Section 6.2, we prove the basic estimates (Theorem 6.7), which are the L^1 analogues of the estimates of Pöschel and Trubowitz [20, Theorem 3], together with the properties of y_1 and y_2 established in Section 6.3 (Corollary ??) that guarantee the fundamental solutions form a basis of the solution space. In Section 6.3, we use these fundamental solutions to construct explicit solutions of the Lax equation (3.9) for every $q \in L^1([0, 1], \mathbb{C})$ and every fixed $\lambda \in \mathbb{C}$ (Proposition 6.8). The question of how this fixed- λ construction relates to the finite-maximum property of the coefficients β_n from Section 3.3 is taken up in Section 7.1 of Chapter 7.

6.1 A Fundamental Solution

The purpose of this section is to construct the fundamental solutions $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$ of the eigenvalue equation

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq 1, \quad (6.1)$$

for potentials $q \in L^1([0, 1], \mathbb{C})$ and $\lambda \in \mathbb{C}$, and to establish their basic properties. Throughout this chapter, we work on the interval $[0, 1]$ and extend q periodically to $\mathbb{R}$. We recall the standard notation

$$c_\lambda(x) = \cos \sqrt{\lambda}x, \quad s_\lambda(x) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}},$$

and note that both functions are entire in λ for each fixed x . The starting point is the inhomogeneous equation

$$-y'' = \lambda y - f(x), \quad 0 \leq x \leq 1, \quad (6.2)$$

for $f \in L^1([0, 1], \mathbb{C})$ and $\lambda \in \mathbb{C}$. Lemma 6.1 shows that this equation has a unique solution for given initial conditions, expressed via the variation of constants formula. This will then serve as the basis for constructing the fundamental solutions y_1 and y_2 of the full equation (6.1) as power series in q .

Lemma 6.1 (Variation of Constants Formula). *Let $f \in L^1([0, 1], \mathbb{C})$, $\lambda \in \mathbb{C}$, and $a, b \in \mathbb{C}$. The unique solution of*

$$-y'' = \lambda y - f(x), \quad 0 \leq x \leq 1,$$

satisfying $y(0) = a$, $y'(0) = b$, is given by

$$y(x) = a \cos \sqrt{\lambda} x + b \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} f(t) dt.$$

Proof. The proof follows that of Pöschel and Trubowitz [20, Lemma 1], with L^2 replaced by L^1 throughout. We verify that the formula makes sense for $f \in L^1$. The integral $\int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} f(t) dt$ is well-defined for $f \in L^1([0, 1], \mathbb{C})$, since the kernel $\frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}}$ is bounded: for fixed $x \in [0, 1]$ and $\lambda \in \mathbb{C}$, writing $\lambda = \mu^2$,

$$\left| \frac{\sin \mu(x-t)}{\mu} \right| = \left| \frac{e^{i\mu(x-t)} - e^{-i\mu(x-t)}}{2i\mu} \right| \leq \frac{e^{|\operatorname{Im} \mu| |x-t|}}{|\mu|} \leq \frac{e^{|\operatorname{Im} \mu|}}{|\mu|}.$$

Therefore, by the Hölder inequality with $p = 1$ and $q = \infty$:

$$\left| \int_0^x \frac{\sin \mu(x-t)}{\mu} f(t) dt \right| \leq \frac{e^{|\operatorname{Im} \mu|}}{|\mu|} \int_0^x |f(t)| dt \leq \frac{e^{|\operatorname{Im} \mu|}}{|\mu|} \|f\|_{L^1([0, 1])}.$$

Setting $y_f(x) = \int_0^x \frac{\sin \mu(x-t)}{\mu} f(t) dt$, we verify directly that y_f is absolutely continuous. By the addition formula $\sin \mu(x-t) = \sin(\mu x) \cos(\mu t) - \cos(\mu x) \sin(\mu t)$, we can write

$$y_f(x) = \frac{\sin(\mu x)}{\mu} F_1(x) - \frac{\cos(\mu x)}{\mu} F_2(x),$$

where

$$F_1(x) := \int_0^x \cos(\mu t) f(t) dt, \quad F_2(x) := \int_0^x \sin(\mu t) f(t) dt.$$

Since μ is fixed and t ranges over the compact interval $[0, 1]$, the functions $t \mapsto \cos(\mu t)$ and $t \mapsto \sin(\mu t)$ are bounded (e.g. by $e^{|\operatorname{Im} \mu|}$, by the same estimate used above), so $\cos(\mu \cdot) f$ and $\sin(\mu \cdot) f$ lie in $L^1([0, 1], \mathbb{C})$ whenever $f \in L^1([0, 1], \mathbb{C})$. By the fundamental theorem of calculus for the Lebesgue integral, the indefinite integral of an L^1 -function is absolutely continuous, hence F_1 and F_2 are absolutely continuous on $[0, 1]$. Since $x \mapsto \sin(\mu x)/\mu$ and $x \mapsto \cos(\mu x)$ are entire functions of x , in particular Lipschitz on the bounded interval $[0, 1]$, and since sums and products of an absolutely continuous function with a Lipschitz function are again absolutely continuous, it follows that y_f is absolutely continuous on

$[0, 1]$.

By the product rule (valid a.e., since F_1, F_2 are absolutely continuous),

$$y'_f(x) = \cos(\mu x)F_1(x) + \sin(\mu x)F_2(x) \quad \text{a.e.},$$

the two terms involving $f(x)$ cancelling exactly. Since the right-hand side is continuous everywhere on $[0, 1]$ (as F_1, F_2 are continuous), the fundamental theorem of calculus for absolutely continuous functions shows y'_f exists everywhere and equals this continuous expression, i.e. $y_f \in C^1([0, 1], \mathbb{C})$. Differentiating once more (now only a.e., since F_1, F_2 are merely absolutely continuous),

$$y''_f(x) = -\mu [\sin(\mu x)F_1(x) - \cos(\mu x)F_2(x)] + [\cos^2(\mu x) + \sin^2(\mu x)]f(x) = -\mu^2 y_f(x) + f(x)$$

a.e., which means $-y''_f = \lambda y_f - f(x)$ almost everywhere. Furthermore $y_f(0) = y'_f(0) = 0$. The general solution is then $y(x) = a \cos \mu x + b \frac{\sin \mu x}{\mu} + y_f(x)$. Uniqueness follows as in [20]: if u is another solution with the same initial conditions, then $u - y$ satisfies $-(u - y)'' = \lambda(u - y)$ with zero initial conditions, hence $u - y \equiv 0$. $\square$

Lemma 6.1 provides the solution formula for the inhomogeneous equation $-y'' = \lambda y - f$ with a given right-hand side $f \in L^1$. To construct the fundamental solutions y_1 and y_2 of the full eigenvalue equation $-y'' + qy = \lambda y$, we use this formula iteratively: writing $f = qy$ and substituting the solution back into the right-hand side, we obtain the integral equations of Theorem 6.2. The following theorem shows that these integral equations have unique solutions, given by uniformly convergent power series in q .

Theorem 6.2 (Convergence of the Neumann Series). *For each $q \in L^1([0, 1], \mathbb{C})$ and $\lambda \in \mathbb{C}$, the integral equations*

$$\begin{aligned} y_1(x, \lambda, q) &= \cos \sqrt{\lambda} x + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t) y_1(t, \lambda, q) dt, \\ y_2(x, \lambda, q) &= \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t) y_2(t, \lambda, q) dt, \end{aligned}$$

have unique solutions $y_1, y_2 \in C^1([0, 1], \mathbb{C})$ satisfying the initial conditions:

$$y_1(0) = 1, \quad y'_1(0) = 0, \quad y_2(0) = 0, \quad y'_2(0) = 1.$$

These solutions are given by the uniformly convergent series:

$$\begin{aligned} y_1(x, \lambda, q) &= \sum_{n=0}^{\infty} y_1^{(n)}(x, \lambda, q), \\ y_2(x, \lambda, q) &= \sum_{n=0}^{\infty} y_2^{(n)}(x, \lambda, q), \end{aligned}$$

where

$$\begin{aligned} y_1^{(0)}(x, \lambda) &= \cos \sqrt{\lambda} x, \quad y_2^{(0)}(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}}, \\ y_1^{(n)}(x, \lambda, q) &= \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t) y_1^{(n-1)}(t, \lambda, q) dt, \\ y_2^{(n)}(x, \lambda, q) &= \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t) y_2^{(n-1)}(t, \lambda, q) dt. \end{aligned}$$

The series converge uniformly on bounded subsets of $[0, 1] \times \mathbb{C} \times L^1([0, 1], \mathbb{C})$, and the solutions are entire functions of λ for each fixed x and q .

Proof. We first record the elementary bounds needed throughout the proof. Writing $\mu := \sqrt{\lambda} = a + ib$ ($a, b \in \mathbb{R}$), and using $\cos(\mu t) = \cos(at) \cosh(bt) - i \sin(at) \sinh(bt)$, we get

$$|\cos(\mu t)|^2 = \cos^2(at) \cosh^2(bt) + \sin^2(at) \sinh^2(bt) = \cosh^2(bt) - \sin^2(at) \leq \cosh^2(bt) \leq e^{2|b|t}$$

for $t \geq 0$, so $|c_\lambda(t)| = |\cos \sqrt{\lambda} t| \leq e^{|\operatorname{Im} \sqrt{\lambda}|t}$. Since $s_\lambda(t) = \int_0^t c_\lambda(\tau) d\tau$, and $e^{|\operatorname{Im} \sqrt{\lambda}|\tau}$ is increasing in τ ,

$$|s_\lambda(t)| \leq \int_0^t |c_\lambda(\tau)| d\tau \leq \int_0^t e^{|\operatorname{Im} \sqrt{\lambda}|\tau} d\tau \leq t e^{|\operatorname{Im} \sqrt{\lambda}|t} \leq e^{|\operatorname{Im} \sqrt{\lambda}|t}$$

for $t \in [0, 1]$.

Existence. Set $u(x) := \|q\|_{L^1([0, x])} = \int_0^x |q(t)| dt$. We show by induction that

$$|y_1^{(n)}(x, \lambda, q)| \leq e^{|\operatorname{Im} \sqrt{\lambda}|x} \frac{u(x)^n}{n!}, \quad n \geq 0,$$

and likewise for $y_2^{(n)}$.

For $n = 0$, $|y_1^{(0)}(x, \lambda)| = |\cos \sqrt{\lambda} x| \leq e^{|\operatorname{Im} \sqrt{\lambda}|x}$ by the crude bound above, matching $u(x)^0/0! = 1$ (and $|y_2^{(0)}(x, \lambda)| = |s_\lambda(x)| \leq e^{|\operatorname{Im} \sqrt{\lambda}|x}$ likewise).

For the inductive step, assume the bound holds for $n - 1$. Using the crude bound on $s_\lambda(x - t)$ and the inductive hypothesis on $y_1^{(n-1)}(t, \lambda, q)$,

$$\begin{aligned} |y_1^{(n)}(x, \lambda, q)| &\leq \int_0^x |s_\lambda(x-t)| |q(t)| |y_1^{(n-1)}(t, \lambda, q)| dt \\ &\leq \int_0^x e^{|\operatorname{Im} \sqrt{\lambda}|(x-t)} |q(t)| e^{|\operatorname{Im} \sqrt{\lambda}|t} \frac{u(t)^{n-1}}{(n-1)!} dt \\ &= e^{|\operatorname{Im} \sqrt{\lambda}|x} \cdot \frac{1}{(n-1)!} \int_0^x |q(t)| u(t)^{n-1} dt. \end{aligned}$$

Since $q \in L^1([0, 1])$, the function u is absolutely continuous with $u'(t) = |q(t)|$ almost everywhere (fundamental theorem of calculus for the Lebesgue integral), and hence so is

$u(t)^n$, being the composition of the absolutely continuous monotone function u with the smooth function $x \mapsto x^n$; therefore

$$\int_0^x |q(t)| u(t)^{n-1} dt = \int_0^x u'(t) u(t)^{n-1} dt = \frac{1}{n} \int_0^x (u(t)^n)' dt = \frac{u(x)^n}{n}.$$

Combining the two displays gives

$$|y_1^{(n)}(x, \lambda, q)| \leq e^{|\operatorname{Im}\sqrt{\lambda}|x} \cdot \frac{1}{(n-1)!} \cdot \frac{u(x)^n}{n} = e^{|\operatorname{Im}\sqrt{\lambda}|x} \frac{u(x)^n}{n!},$$

completing the induction (the argument for $y_2^{(n)}$ is identical).

Summing over $n \geq 0$,

$$|y_1(x, \lambda, q)| \leq e^{|\operatorname{Im}\sqrt{\lambda}|x} \sum_{n=0}^{\infty} \frac{u(x)^n}{n!} = e^{|\operatorname{Im}\sqrt{\lambda}|x+u(x)} \leq e^{|\operatorname{Im}\sqrt{\lambda}|x+\|q\|_{L^1}},$$

and likewise for y_2 .

On any bounded subset of $[0, 1] \times \mathbb{C} \times L^1([0, 1], \mathbb{C})$, say $|\lambda| \leq R$ and $\|q\|_{L^1} \leq M$, the constant $C := \max_{|\lambda| \leq R} e^{|\operatorname{Im}\sqrt{\lambda}|}$ is finite, and $|y_1^{(n)}(x, \lambda, q)| \leq C M^n/n!$ uniformly on the set. Since $\sum_n M^n/n! = e^M < \infty$, the Weierstrass M -test gives uniform convergence of $\sum_n y_1^{(n)}(x, \lambda, q)$ (and likewise for y_2) on the set.

Since y_1 is continuous (hence bounded) and $q \in L^1([0, 1])$, the function $f := qy_1 \in L^1([0, 1], \mathbb{C})$, so y_1 is exactly the solution given by Lemma 6.1 with $a = 1, b = 0$ and this f ; that lemma gives $y_1 \in C^1([0, 1], \mathbb{C})$, confirming the regularity claimed above (the same lemma also gives $-y_1'' + qy_1 = \lambda y_1$ a.e., which we record separately in Remark 6.3).

It remains to check that this limit actually solves the integral equation. Summing the recursive definition $y_1^{(n)}(x, \lambda, q) = \int_0^x s_\lambda(x-t)q(t)y_1^{(n-1)}(t, \lambda, q) dt$ over $n \geq 1$ and using the uniform convergence just established to interchange sum and integral,

$$\sum_{n \geq 1} y_1^{(n)}(x, \lambda, q) = \int_0^x s_\lambda(x-t)q(t) \sum_{n \geq 1} y_1^{(n-1)}(t, \lambda, q) dt = \int_0^x s_\lambda(x-t)q(t) y_1(t, \lambda, q) dt,$$

i.e. $y_1(x, \lambda, q) = \cos \sqrt{\lambda}x + \int_0^x s_\lambda(x-t)q(t)y_1(t, \lambda, q) dt$, so y_1 indeed solves the integral equation (and likewise y_2).

Each term $y_1^{(n)}(\cdot, \lambda, \cdot)$ is entire in λ for fixed x, q (by induction: $y_1^{(0)} = c_\lambda$ is entire, and integrating an entire function of λ against $q(t)dt$ preserves holomorphy), so y_1 , being a locally uniform limit of entire functions, is itself entire in λ .

Uniqueness. Suppose $y_1, \tilde{y}_1$ both solve the integral equation for y_1 . Set $v := y_1 - \tilde{y}_1$, then v is continuous on $[0, 1]$, $v(0) = 0$, and

$$v(x) = \int_0^x s_\lambda(x-t)q(t)v(t) dt.$$

Using the crude bound $|s_\lambda(x-t)| \leq e^{|\operatorname{Im}\sqrt{\lambda}|(x-t)}$,

$$|v(x)| \leq \int_0^x e^{|\operatorname{Im}\sqrt{\lambda}|(x-t)} |q(t)| |v(t)| dt.$$

Setting $\tilde{v}(x) := |v(x)|e^{-|\operatorname{Im}\sqrt{\lambda}|x}$, this becomes $\tilde{v}(x) \leq \int_0^x |q(t)| \tilde{v}(t) dt$.

Since $|q| \in L^1([0, 1])$ and $\tilde{v}(0) = 0$, we verify directly that $\tilde{v} \equiv 0$.

Set $\Phi(x) := \int_0^x |q(t)| \tilde{v}(t) dt$. Since $\tilde{v}$ is continuous and $|q| \in L^1$, Φ is absolutely continuous with $\Phi'(x) = |q(x)|\tilde{v}(x)$ almost everywhere, and $\tilde{v}(x) \leq \Phi(x)$ for all x . Because $|q(x)| \geq 0$, this gives

$$\Phi'(x) = |q(x)|\tilde{v}(x) \leq |q(x)|\Phi(x) \quad \text{a.e.},$$

so, multiplying by the absolutely continuous integrating factor $e^{-\int_0^x |q(s)|ds} > 0$,

$$\left[\Phi(x) e^{-\int_0^x |q(s)|ds} \right]' = [\Phi'(x) - |q(x)|\Phi(x)] e^{-\int_0^x |q(s)|ds} \leq 0 \quad \text{a.e.}$$

Hence $x \mapsto \Phi(x)e^{-\int_0^x |q(s)|ds}$ is non-increasing on $[0, 1]$, and since it vanishes at $x = 0$ (as $\Phi(0) = 0$), it is ≤ 0 for all x , and by $\Phi \geq 0$ it is also ≥ 0 , so $\Phi \equiv 0$. Then $\tilde{v}(x) \leq \Phi(x) = 0$, and since $\tilde{v} \geq 0$, we conclude $\tilde{v} \equiv 0$, hence $v \equiv 0$, i.e. $y_1 = \tilde{y}_1$.

The argument for y_2 is identical. $\square$

Theorem 6.2 establishes that y_1 and y_2 exist, are unique, and depend continuously and holomorphically on x, λ, q . Before proceeding, we record two further basic properties of y_1, y_2 that will be used repeatedly throughout the rest of this chapter: that they actually solve the differential equation itself, not merely the integral equation, and that they satisfy the classical Wronskian identity.

Remark 6.3 (y_1, y_2 solve the differential equation). The fundamental solutions y_1, y_2 of Theorem 6.2 satisfy

$$y_i''(x, \lambda, q) = (q(x) - \lambda) y_i(x, \lambda, q), \quad i = 1, 2,$$

for almost every $x \in [0, 1]$.

Indeed, by the proof of Theorem 6.2, y_1 solves the integral equation $y_1(x, \lambda, q) = \cos \sqrt{\lambda}x + \int_0^x s_\lambda(x-t)q(t)y_1(t, \lambda, q) dt$ as a whole (and likewise y_2). Since $y_1 \in C^1([0, 1], \mathbb{C})$ is in particular bounded and $q \in L^1([0, 1])$, the function $f := qy_1 \in L^1([0, 1], \mathbb{C})$, so this is exactly the equation of Lemma 6.1 with $a = 1, b = 0$ and this f . Therefore, that lemma gives $-y_1'' + qy_1 = \lambda y_1$, i.e. $y_1'' = (q - \lambda)y_1$ almost everywhere on $[0, 1]$.

The same argument applies to y_2 .

Remark 6.4 (Wronskian Identity). The fundamental solutions y_1 and y_2 satisfy the *Wronskian identity*:

$$W(y_1, y_2)(x) := y_1(x, \lambda) y_2'(x, \lambda) - y_1'(x, \lambda) y_2(x, \lambda) = 1$$

for all $x \in [0, 1]$. Indeed, $y_1, y_2 \in C^1([0, 1], \mathbb{C})$ (Theorem 6.2), so y_1', y_2' are absolutely continuous and $W(y_1, y_2)$ is C^1 . By the product rule, for almost every $x \in [0, 1]$

$$W(y_1, y_2)'(x) = y_1 y_2'' - y_1'' y_2 = y_1(q - \lambda)y_2 - (q - \lambda)y_1 y_2 = 0,$$

using $y_i'' = (q - \lambda)y_i$ a.e. from Remark 6.3. Since $W(y_1, y_2)$ is continuous with a.e. vanishing derivative, it is constant on $[0, 1]$ (FTC for absolutely continuous functions), and evaluating at $x = 0$ using $y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1$ gives $W(y_1, y_2)(x) \equiv 1$. Theorem 6.2 establishes the existence and uniqueness of the fundamental solutions y_1 and y_2 of the homogeneous equation $-y'' + qy = \lambda y$ with zero right-hand side. The next theorem extends this to the inhomogeneous equation $-y'' + qy = \lambda y - f$, combining the variation of constants formula of Lemma 6.1 with the fundamental solutions constructed in Theorem 6.2.

Theorem 6.5 (Inhomogeneous Equation with Potential). *Let $f \in L^1([0, 1], \mathbb{C})$, $\lambda \in \mathbb{C}$, and $a, b \in \mathbb{C}$. Then there exists a unique solution of the inhomogeneous equation*

$$-y'' + q(x)y = \lambda y - f(x), \quad 0 \leq x \leq 1,$$

satisfying $y(0) = a$ and $y'(0) = b$. It is given by

$$y(x) = ay_1(x, \lambda) + by_2(x, \lambda) + \int_0^x (y_1(t, \lambda)y_2(x, \lambda) - y_1(x, \lambda)y_2(t, \lambda)) f(t) dt,$$

where y_1 and y_2 are the fundamental solutions of Theorem 6.2.

Proof. The proof is analogous to that of Pöschel and Trubowitz [20, Theorem 2], replacing the kernel $s_\lambda(x - t) = s_\lambda(x)c_\lambda(t) - s_\lambda(t)c_\lambda(x)$ by $y_1(t, \lambda)y_2(x, \lambda) - y_1(x, \lambda)y_2(t, \lambda)$, and making use of the Wronskian identity $y_1(x, \lambda)y_2'(x, \lambda) - y_1'(x, \lambda)y_2(x, \lambda) = 1$. The integral is well-defined since $y_1, y_2 \in C^1([0, 1], \mathbb{C})$ are bounded, hence in L^∞ , and $f \in L^1$, so the Hölder inequality with $p = 1$ and $q = \infty$ applies. $\square$

As an immediate consequence of Theorem 6.5 we obtain the following corollary, which shows that the fundamental solutions y_1 and y_2 provide a complete description of all solutions of the homogeneous equation.

Corollary 6.6 (Representation of Solutions). *Every solution of the homogeneous equation $-y'' + q(x)y = \lambda y$ on $[0, 1]$ is uniquely given by:*

$$y(x) = y(0)y_1(x, \lambda) + y'(0) \cdot y_2(x, \lambda).$$

Moreover, if a solution has a double zero in $[0, 1]$, then it vanishes identically.

Proof. The proof is analogous to that of Pöschel and Trubowitz [20, Corollary 1]. The first statement follows immediately from Theorem 6.5 with $f = 0$. For the second statement, suppose y has a double zero at some point $x_0 \in [0, 1]$, i.e., $y(x_0) = y'(x_0) = 0$. Then the matrix equation:

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix} = 0,$$

and since the 2×2 -matrix is nonsingular by the Wronskian identity, we conclude $y(0) = y'(0) = 0$, hence $y \equiv 0$. $\square$

The fundamental solutions y_1 and y_2 constructed in this section form the foundation for the spectral theory of the Hill operator with L^1 potential. Together, they solve both the homogeneous and inhomogeneous eigenvalue equations, and any solution can be expressed as a linear combination of y_1 and y_2 by Corollary 6.6. In the following section, we establish quantitative estimates on the size of y_1 and y_2 as functions of λ and q . These *basic estimates* are the L^1 -analogues of the central estimates of Pöschel and Trubowitz [20, Theorem 3], and will serve as the key tool for the further development of the spectral theory in this chapter.

6.2 Basic Estimates

Having constructed the fundamental solutions y_1 and y_2 in Section 6.1, we now establish quantitative estimates on their size as functions of x , λ , and q . These estimates are the L^1 -analogues of the basic estimates of Pöschel and Trubowitz [20, Theorem 3], and replace the L^2 -norm $\|q\|_{L^2}$ appearing there by the L^1 -norm $\|q\|_{L^1}$. This substitution is made possible by the Hölder inequality with exponents $p = 1$ and $q = \infty$, as explained in the introduction of this chapter.

Theorem 6.7 (Basic Estimates). *On $[0, 1] \times \mathbb{C} \times L^1([0, 1], \mathbb{C})$ (with $\sqrt{\lambda} \neq 0$):*

$$\begin{aligned} \left| y_1(x, \lambda, q) - \cos \sqrt{\lambda} x \right| &\leq \frac{1}{|\sqrt{\lambda}|} \exp\left(|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}\right), \\ \left| y_2(x, \lambda, q) - \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \right| &\leq \frac{1}{|\lambda|} \exp\left(|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}\right), \end{aligned}$$

and

$$\begin{aligned} \left| y_1'(x, \lambda, q) + \sqrt{\lambda} \sin \sqrt{\lambda} x \right| &\leq \|q\|_{L^1} \exp\left(|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}\right), \\ \left| y_2'(x, \lambda, q) - \cos \sqrt{\lambda} x \right| &\leq \frac{\|q\|_{L^1}}{|\sqrt{\lambda}|} \exp\left(|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}\right). \end{aligned}$$

Proof. Throughout, write $u(x) := \|q\|_{L^1([0, x])} \leq \|q\|_{L^1}$. We use the crude and sharp elementary bounds on c_λ, s_λ , and the crude bound

$$|C_{n-1}(t, \lambda, q)| \leq e^{|\operatorname{Im} \sqrt{\lambda}|t} \frac{u(t)^{n-1}}{(n-1)!}, \quad |D_{n-1}(t, \lambda, q)| \leq e^{|\operatorname{Im} \sqrt{\lambda}|t} \frac{u(t)^{n-1}}{(n-1)!}, \quad n \geq 1,$$

on the Neumann terms of $y_1(x, \lambda, q) = c_\lambda(x) + \sum_{n \geq 1} C_n(x, \lambda, q)$ and $y_2(x, \lambda, q) = s_\lambda(x) + \sum_{n \geq 1} D_n(x, \lambda, q)$ (convention $C_0 = c_\lambda$, $D_0 = s_\lambda$), all established in the proof of Theorem 6.2, and in particular the key integral identity

$$\int_0^x |q(t)| u(t)^{n-1} dt = \frac{u(x)^n}{n}$$

(from $u' = |q|$ a.e. and the fundamental theorem of calculus), which recurs at every step below and which we do not rederive. We also use the elementary inequality $e^v - 1 \leq ve^v$ for $v \geq 0$ (immediate from comparing derivatives).

Step 1: the estimate for $y_1 - \cos \sqrt{\lambda}x$.

Here $C_1(x, \lambda, q) = \int_0^x s_\lambda(x-t)q(t)c_\lambda(t) dt$ and, for $n \geq 2$, $C_n(x, \lambda, q) = \int_0^x s_\lambda(x-t)q(t)C_{n-1}(t, \lambda, q) dt$. Bounding the single outer factor $s_\lambda(x-t)$ sharply and the inner term $C_{n-1}(t, \lambda, q)$ by the crude bound above (valid already for $n-1=0$, i.e. for c_λ itself), we get for every $n \geq 1$, using the identity recalled above:

$$|C_n(x, \lambda, q)| \leq \int_0^x \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|(x-t)} |q(t)| e^{|\operatorname{Im} \sqrt{\lambda}|t} \frac{u(t)^{n-1}}{(n-1)!} dt = \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|x} \frac{u(x)^n}{n!}.$$

Summing over $n \geq 1$:

$$|y_1(x, \lambda, q) - \cos \sqrt{\lambda}x| \leq \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|x} (e^{u(x)} - 1) \leq \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}},$$

which is the first estimate. All three remaining estimates follow the same pattern: outer kernel factor bounded either sharply or crudely as the case allows, inner Neumann term bounded by the already-established crude bound, the recalled integral identity applied, and the resulting series summed via $\sum_n v^n/n! = e^v - 1$ — so from here on we indicate only where a step deviates from Step 1.

Step 2: the estimate for $y_2 - \sin \sqrt{\lambda}x/\sqrt{\lambda}$.

The only difference to Step 1 is that the zeroth term here does admit the sharp bound. So for $n=1$ we may bound both factors of $D_1(x, \lambda, q) = \int_0^x s_\lambda(x-t)q(t)s_\lambda(t) dt$ sharply:

$$\begin{aligned} |D_1(x, \lambda, q)| &\leq \int_0^x |s_\lambda(x-t)| |q(t)| |s_\lambda(t)| dt \\ &\leq \int_0^x \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|(x-t)} |q(t)| \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|t} dt \\ &= \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} \int_0^x |q(t)| dt = \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} u(x). \end{aligned}$$

We now claim $|D_n(x, \lambda, q)| \leq \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} \frac{u(x)^n}{n!}$ for all $n \geq 1$.

The computation above is the case $n=1$ (note $u(x)^1/1! = u(x)$).

For the inductive step, fix $n \geq 2$ and assume the bound holds for $n-1$, i.e. $|D_{n-1}(t, \lambda, q)| \leq \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|t} \frac{u(t)^{n-1}}{(n-1)!}$. Bounding only the inner factor D_{n-1} by the inductive hypothesis and the outer factor crudely:

$$\begin{aligned} |D_n(x, \lambda, q)| &= \left| \int_0^x s_\lambda(x-t)q(t)D_{n-1}(t, \lambda, q) dt \right| \\ &\leq \int_0^x e^{|\operatorname{Im} \sqrt{\lambda}|(x-t)} |q(t)| \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|t} \frac{u(t)^{n-1}}{(n-1)!} dt \\ &= \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} \frac{1}{(n-1)!} \int_0^x |q(t)| u(t)^{n-1} dt = \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} \frac{u(x)^n}{n!}, \end{aligned}$$

using the integral identity recalled above. This proves the claim by induction.

Summing over $n \geq 1$ gives

$$|y_2(x, \lambda, q) - s_\lambda(x)| \leq \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} \sum_{n \geq 1} \frac{u(x)^n}{n!} = \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} (e^{u(x)} - 1) \leq \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}},$$

which is the second estimate.

Step 3: the estimate for $y_1' + \sqrt{\lambda} \sin \sqrt{\lambda} x$.

Differentiating the Volterra equation for y_1 (the boundary term vanishes since $s_\lambda(0) = 0$, and $s_\lambda'(x - t) = c_\lambda(x - t)$) gives

$$y_1'(x, \lambda, q) + \sqrt{\lambda} \sin \sqrt{\lambda} x = \sum_{n \geq 1} E_n(x, \lambda, q), \quad E_n(x, \lambda, q) := \int_0^x c_\lambda(x-t) q(t) C_{n-1}(t, \lambda, q) dt.$$

Here the outer factor $c_\lambda(x - t)$ never admits a sharp bound, so, unlike in Steps 1–2, every factor is bounded crudely, giving simply $|E_n(x, \lambda, q)| \leq e^{|\operatorname{Im} \sqrt{\lambda}|x} u(x)^n / n!$ for all $n \geq 1$ by the same computation as in Step 1.

Summing, and now using $e^{u(x)} - 1 \leq u(x)e^{u(x)}$ to produce the linear prefactor $\|q\|_{L^1}$:

$$|y_1'(x, \lambda, q) + \sqrt{\lambda} \sin \sqrt{\lambda} x| \leq e^{|\operatorname{Im} \sqrt{\lambda}|x} (e^{u(x)} - 1) \leq \|q\|_{L^1} e^{|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}},$$

which is the third estimate.

Step 4: the estimate for $y_2' - \cos \sqrt{\lambda} x$.

Analogously,

$$y_2'(x, \lambda, q) - \cos \sqrt{\lambda} x = \sum_{n \geq 1} F_n(x, \lambda, q), \quad F_n(x, \lambda, q) := \int_0^x c_\lambda(x-t) q(t) D_{n-1}(t, \lambda, q) dt.$$

As in Step 3 the outer factor admits no sharp bound, so the only sharp factors available come from D_{n-1} itself, exactly as in Step 2:

For $n = 1$ ($D_0 = s_\lambda$, sharp bound available), $|F_1(x, \lambda, q)| \leq \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|x} u(x)$.

For $n \geq 2$, using the sharp bound on D_{n-1} from Step 2, $|F_n(x, \lambda, q)| \leq \frac{1}{|\lambda|} e^{|\operatorname{Im} \sqrt{\lambda}|x} u(x)^n / n!$.

For $|\sqrt{\lambda}| \geq 1$ we have $1/|\lambda| \leq 1/|\sqrt{\lambda}|$, so summing over all $n \geq 1$ as before:

$$\sum_{n \geq 1} |F_n(x, \lambda, q)| \leq \frac{1}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|x} (e^{u(x)} - 1) \leq \frac{\|q\|_{L^1}}{|\sqrt{\lambda}|} e^{|\operatorname{Im} \sqrt{\lambda}|x + \|q\|_{L^1}},$$

which is the fourth estimate. □

Theorem 6.7 gives uniform bounds on y_1 and y_2 in terms of $\|q\|_{L^1}$. For L^2 potentials, Pöschel and Trubowitz [20, Theorem 4] refine these estimates under the additional assumption $q \in H^2$, using partial integration to extract more precise asymptotic information. Since L^1 potentials do not have any derivatives, such refinements are not available in our setting, and Theorem 6.7 is the best possible estimate for general $q \in L^1$.

6.3 Solutions of the Lax Equation for L^1 Potentials

The fundamental solutions y_1, y_2 constructed in Section 6.1 make it possible to produce, for every $q \in L^1([0, 1], \mathbb{C})$, explicit solutions of the Lax equation

$$\frac{d}{dx}\xi = [U(\lambda), \xi], \quad U(\lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix}, \quad (3.9)$$

at each fixed $\lambda \in \mathbb{C}$. This is the L^1 -analogue of Remark 3.3, and the argument is the same: it uses only that y_1, y_2 solve $-y'' + qy = \lambda y$ with Wronskian identically 1, not any smoothness of q beyond what Chapter 6 already provides.

Proposition 6.8 (Existence of Lax-equation solutions for L^1 potentials). *Let $q \in L^1([0, 1], \mathbb{C})$ and $\lambda \in \mathbb{C}$, and let y_1, y_2 be the fundamental solutions of Theorem 6.2. Define*

$$\beta := y_1 y_2, \quad \alpha := -\frac{1}{2}(y_1 y_2)' = -\frac{1}{2}(y_1' y_2 + y_1 y_2'), \quad \gamma := -y_1' y_2'.$$

Then $\beta \in C^1([0, 1], \mathbb{C})$, $\alpha, \gamma \in C([0, 1], \mathbb{C})$ are absolutely continuous, and

$$\xi(x, \lambda) := \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & -\alpha(x) \end{pmatrix}$$

solves the Lax equation (3.9) for almost every $x \in [0, 1]$.

Proof. Since $y_1, y_2 \in C^1([0, 1], \mathbb{C})$ by Theorem 6.2, the product $\beta = y_1 y_2$ is C^1 with $\beta' = y_1' y_2 + y_1 y_2' = -2\alpha$, so $\alpha \in C([0, 1], \mathbb{C})$. Likewise $\gamma = -y_1' y_2'$ is continuous, being a product of continuous functions.

In fact α and γ are absolutely continuous. By Lemma 6.1, applied to y_1 and y_2 via Remark 6.3, the derivatives y_1' and y_2' are absolutely continuous, and y_1, y_2 themselves are absolutely continuous as C^1 -functions on $[0, 1]$. Since a product of two bounded absolutely continuous functions on $[0, 1]$ is again absolutely continuous, both $y_1' y_2$ and $y_1 y_2'$ are absolutely continuous, and hence so is their sum $\beta' = -2\alpha$, i.e. α is absolutely continuous. Likewise $\gamma = -y_1' y_2'$, being a product of the absolutely continuous functions y_1' and y_2' , is absolutely continuous.

By construction (Remark 6.3), y_1 and y_2 satisfy $-y_i'' + qy_i = \lambda y_i$, i.e. $y_i'' = (q - \lambda)y_i$ for almost every $x \in [0, 1]$. Since α is absolutely continuous, the product rule applies almost everywhere to $\beta' = y_1' y_2 + y_1 y_2'$ as well, giving $\beta'' = y_1'' y_2 + 2y_1' y_2' + y_1 y_2''$ almost everywhere. Hence

$$\alpha' = -\frac{1}{2}\beta'' = -\frac{1}{2}((q - \lambda)y_1 y_2 + 2y_1' y_2' + y_1(q - \lambda)y_2) = -(q - \lambda)\beta - y_1' y_2' = \gamma - (q - \lambda)\beta$$

almost everywhere. Similarly,

$$\gamma' = -(y_1'' y_2' + y_1' y_2'') = -(q - \lambda)(y_1 y_2' + y_1' y_2) = -(q - \lambda)\beta' = 2(q - \lambda)\alpha$$

almost everywhere. Together, $\beta' = -2\alpha$, $\alpha' = \gamma - (q - \lambda)\beta$, and $\gamma' = 2(q - \lambda)\alpha$ almost everywhere are exactly the component equations of (3.9), which completes the proof. $\square$

Remark 6.9. By linearity of (3.9) (Remark 3.1), any $\mathbb{C}$ -linear combination of solutions of the form given in Proposition 6.8, built from different pairs of solutions of $-y'' + qy = \lambda y$, again solves (3.9). In particular, for every $q \in L^1([0, 1], \mathbb{C})$ and every $\lambda \in \mathbb{C}$, solutions of the Lax equation (3.9) exist for the full three-parameter family of traceless initial data $\xi(0, \lambda)$, exactly as in the smooth case treated in Chapter 3.

Chapter 7

Outlook

The results established in this thesis, i.e. the characterization of finite-type potentials via Theorem 4.8 and their spectral interpretation via Theorem 5.14, raise several natural questions that point towards deeper connections between integrable systems, spectral theory, and algebraic geometry. We discuss three directions, without developing either in full: the open question of how the fixed- λ Lax-equation solutions constructed in Section 6.3 relate to the classical Laurent-series construction of Chapter 3, a concrete conjecture suggested by our results, and a brief sketch of the classical algebro-geometric picture that finite-type potentials fit into.

7.1 Relating the Two Lax-Equation Constructions

Proposition 6.8 produces for arbitrary $q \in L^1([0, 1], \mathbb{C})$ an honest solution of the Lax equation at each *fixed* λ . It does not yet address the original question motivating Chapter 6, namely the existence of a solution whose Laurent coefficients β_n , defined via the recursion (3.14) of Section 3.3, have a set of nonvanishing indices with a finite maximum. That recursion is intrinsically tied to the smooth setting of Chapter 3: $\beta_1 = \frac{1}{2}q$ requires no derivatives of q , but β_2 requires $\beta_1''' = \frac{1}{2}q'''$, and so on, so that for q of limited regularity the recursion can only be carried out up to some finite order before it ceases to produce genuine functions. Relating this order to the fixed- λ construction of Proposition 6.8 (first for $q \in L^2$, as in the original formulation of the problem, and then further for $q \in L^1$) requires the refined large- λ asymptotics of y_1, y_2 analogous to Pöschel and Trubowitz's Theorem 4, which are not available at the level of generality of Theorem 6.6 alone. We leave this connection, and the resulting characterization of the finite-maximum property in terms of the regularity of q , for future work.

7.2 Density of Finite-Type Potentials in L^1

A fundamental question in the theory of integrable systems is whether the special solutions (in our case, the finite-type potentials) are representative of the general situation. The

following conjecture makes this precise:

The set of periodic finite-type potentials is dense in $L^1([0, 1], \mathbb{R})$ with respect to the L^1 -norm.

The significance of this conjecture is twofold. On the one hand, it would imply that every L^1 -potential can be approximated arbitrarily well by algebraic-geometric potentials, whose solutions of the KdV equation can be expressed explicitly in terms of Riemann theta functions. On the other hand, it would provide a functional-analytic justification for the central role of finite-type potentials in the theory of integrable systems.

A possible strategy for proving Conjecture 7.2, building on Chapter 6, would proceed as follows. Given any $q \in L^1([0, 1], \mathbb{R})$, one could try to construct a sequence of finite-type potentials q_n approximating q in L^1 by modifying the spectral data of q : since a potential is of finite type if and only if it has only finitely many open spectral gaps (Theorem 5.14), one would approximate q by closing all but finitely many of its spectral gaps. The spectral theory for L^1 potentials developed in Chapter 6 would be needed to make such an argument precise, since it ensures that the spectral data (eigenvalues, discriminant, monodromy matrix) are well-defined and behave continuously in L^1 . Concretely, this would require a substantially more detailed analysis of the isospectral manifold than we have carried out here, and we leave this analysis for future work.

7.3 Connection to Algebraic Geometry

The finite-type potentials studied in this thesis are, classically, also studied from an algebro-geometric point of view. We record here only the part of this picture that follows directly from our own results, and refer to Gesztesy and Holden [9] for the full construction (see also McKean and Trubowitz [16] for the analogous theory in the case of potentials with infinitely many spectral gaps).

To each finite-type potential q with N open spectral gaps and gap endpoints $E_0 < E_1 < \dots < E_{2N}$, one associates the polynomial $p(\lambda) = \prod_{m=0}^{2N} (\lambda - E_m)$ from the proof of Theorem 5.14, and the algebraic curve

$$\mathcal{C} = \{(\lambda, \mu) \in \mathbb{C}^2 : \mu^2 = p(\lambda)\},$$

a hyperelliptic curve of genus N ; this is precisely the same polynomial p whose zeros are the endpoints of the open spectral gaps in Theorem 5.14, so the spectral data computed in Chapter 5 already determines $\mathcal{C}$.

One observation from our own proof is worth recording. The construction of the entire function $\xi(\lambda)$ in the proof of Theorem 5.14 (Chapter 5) uses an entire square root of $\Delta(\lambda)^2 - 4$ obtained via the Hadamard factorization theorem, and this square root is closely related to the function $\mu = \sqrt{p(\lambda)}$ defining $\mathcal{C}$. This suggests (though we do not make this precise here) that the constructions of Chapters 3-5 might be more naturally carried out on the curve $\mathcal{C}$ itself rather than on the complex λ -plane. Making this idea precise would be a natural next step but is left for future work.

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