Martin Schmidt
Ross Ogilvie

# Partial Differential Equations
## Exercise sheet 12

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**42. Finite differences.** Let $u \in W^{1,2}(B(0,1))$ be a weak solution of

$$L_0 u := \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) = f \text{ in } B(0,1)$$

with $a_{ij} \in L^\infty(B(0,1))$ and $f \in L^2(B(0,1))$. Show that the finite difference

$$\partial_l^h u(x) := \frac{u(x+he_l) - u(x)}{h} \text{ for } x \in B(0, 1-|h|)$$

is a weak solution of

$$L_0 \partial_l^h u(x) = \partial_l^h f(x) - \sum_{i,j=1}^{n} \partial_i(\partial_l^h a_{ij}\partial_j u(x+he_l)),\ x \in B(0, 1-|h|).$$

**Solution.** $L_0$ is an operator in divergence form. $u$ is a weak solution of $L_0 u = f$ when

$$-\int_\Omega A\nabla u \cdot \nabla v = \langle f, v\rangle$$

for all $v \in W_0^{1,2}(\Omega)$. So we will compute the left hand side of the above for $\partial_l^h u$ and see if we can make it look like an element of $W_0^{1,2}(\Omega)^*$.

$$\begin{aligned}
-\int_\Omega A\nabla(\partial_l^h u)\cdot\nabla v &= \frac{1}{h}\int_\Omega \Big[A(x)\nabla u(x) - A(x)\nabla u(x+he_l)\Big]\cdot\nabla v(x) \\
&= \frac{1}{h}\int_\Omega \Big[A(x)\nabla u(x) - A(x+he_l)\nabla u(x+he_l) \\
&\qquad + A(x+he_l)\nabla u(x+he_l) - A(x)\nabla u(x+he_l)\Big]\cdot\nabla v(x) \\
&= \frac{1}{h}\int_\Omega \Big[-f(x) + f(x+he_l)\Big]v + \int_\Omega \partial_l^h A(x)\nabla u(x+he_l)\cdot\nabla v(x) \\
&= \langle \partial_l^h f, v\rangle + \int_\Omega \partial_l^h A(x)\nabla u(x+he_l)\cdot\nabla v(x) \\
&= \langle \partial_l^h f - \nabla\cdot[\partial_l^h A(x)\nabla u(x+he_l)], v\rangle.
\end{aligned}$$

This last expression is not to be taken literally, since we not know if the divergence actually exists, but in the sense of the pairing preceding Definition 4.1.

**43. An Interpolation inequality.**

Let $K = \overline{B(0,2)}$ and $(X, \|\cdot\|)$ be a Banach space that contains $C^1(K)$. In other words, there exists a continuous, injective linear map $I : C^1(K) \hookrightarrow X$. Examples are $X = L^2(K)$, which is similar to Theorem 4.11, and $X = C^0(K)$, which is similar to how how Lemma 3.44 (Interpolation of Sobolev spaces) is used in Theorem 4.34.

Show that a constant $C(n) < \infty$ exists such that

$$\|u\|_{C^2(K)} \le C(n)\left(\|D^2 u\|_{L^\infty(K)} + \|I(u)\|_X\right).$$

[Hint. Show that the embedding $C^2(K) \to C^1(K) \hookrightarrow X$ satisfies the assumptions of Ehrling's Lemma 3.3, ie that $T : C^2(K) \to C^1(K)$ is continuous and compact. Use the embedding theorems for space of continuous functions.]

**Solution.** Following the hint, we check that the assumptions of Ehrling's Lemma are satisfied. That $I$ is continuous and injective is assumed in the question. The inclusion $T : C^2(K) \to C^1(K)$ can be factored as the inclusions $R : C^2(K) \to C^{1,1}(K)$ and $S : C^{1,1}(K) \to C^1(K)$. Proposition 3.15 shows that $R$ is continuous and compact and Proposition 3.13 shows the same for $S$.

Now if we apply Ehrling's Lemma with $\varepsilon = 1/2$ we get

$$\|u\|_{C^1(K)} \leq 0.5\|u\|_{C^2(K)} + C'\|I(u)\|_X.$$

To get the inequality we desire, we apply the re-absorption trick

$$\begin{aligned}
\|u\|_{C^2(K)} &= \|u\|_{C^1(K)} + \|D^2u\|_\infty \leq 0.5\|u\|_{C^2(K)} + C'\|I(u)\|_X + \|D^2u\|_\infty \\
0.5\|u\|_{C^2(K)} &\leq C'\|I(u)\|_X + \|D^2u\|_\infty \\
\|u\|_{C^2(K)} &\leq 2C'\|I(u)\|_X + \|D^2u\|_\infty.
\end{aligned}$$

**44. The interior Schauder Estimate.**

At which place in the proof of the interior Schauder estimate 4.11 should we make a modification to instead prove the inequality

$$\|u\|_{C^{2,\alpha}(B(0,1))} \leq C(\Lambda, n, \alpha) \left(\|Lu\|_{C^{0,\alpha}(B(0,2))} + \|u\|_{L^1(B(0,2))}\right)$$

for $u \in C^{2,\alpha}(B(0,2))$? (Note in particular the last term on the right.)

**Solution.** As hinted at in the previous exercise, a sort of interpolation result is used in the proof of theorem 4.11. In the script we took $X = L^2(K)$ (top of page 75). If we want to end up with the $L^1$ norm in the estimate instead, clearly we should take $X = L^1(K)$. We should check that the inclusion $I : C^1(K) \to L^1(K)$ is a continuous injection. The injectivity is clear: if two $C^1$ functions are almost everywhere equal, they are equal. For continuity

$$\int_K |f| \leq \mu(K)\,\|f\|_\infty \leq \mu(K)\,\|f\|_{C^1}$$

does the job, since $K$ is finite volume.