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**Partial Differential Equations**
**Exercise sheet 12**

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**42. Finite differences.** Let $u \in W^{1,2}(B(0,1))$ be a weak solution of

$$L_0 u := \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) = f \text{ in } B(0,1)$$

with $a_{ij} \in L^\infty(B(0,1))$ and $f \in L^2(B(0,1))$. Show that the finite difference

$$\partial_l^h u(x) := \frac{u(x+he_l)-u(x)}{h} \quad \text{for } x \in B(0,1-|h|)$$

is a weak solution of

$$L_0\partial_l^h u(x) = \partial_l^h f(x) - \sum_{i,j=1}^{n} \partial_i(\partial_l^h a_{ij}\partial_j u(x+he_l)),\ x \in B(0,1-|h|).$$

**43. An Interpolation inequality.**

Let $K = \overline{B(0,2)}$ and $(X, \|\cdot\|)$ be a Banach space that contains $C^1(K)$. In other words, there exists a continuous, injective linear map $I : C^1(K) \hookrightarrow X$. Examples are $X = L^2(K)$, which is similar to Theorem 4.11, and $X = C^0(K)$, which is similar to how how Lemma 3.44 (Interpolation of Sobolev spaces) is used in Theorem 4.34.

Show that a constant $C(n) < \infty$ exists such that

$$\|u\|_{C^2(K)} \le C(n)\left(\|D^2u\|_{L^\infty(K)} + \|I(u)\|_X\right).$$

[Hint. Show that the embedding $C^2(K) \to C^1(K) \hookrightarrow X$ satisfies the assumptions of Ehrling's Lemma 3.3, ie that $T : C^2(K) \to C^1(K)$ is continuous and compact. Use the embedding theorems for space of continuous functions.]

**44. The interior Schauder Estimate.**

At which place in the proof of the interior Schauder estimate 4.11 should we make a modification to instead prove the inequality

$$\|u\|_{C^{2,\alpha}(B(0,1))} \le C(\Lambda, n, \alpha)\left(\|Lu\|_{C^{0,\alpha}(B(0,2))} + \|u\|_{L^1(B(0,2))}\right)$$

for $u \in C^{2,\alpha}(B(0,2))$? (Note in particular the last term on the right.)