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Partial Differential Equations
Exercise sheet 11

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38. Divergence and rotation.

(a) Let $\Omega \subset \mathbb{R}^2$ be open and L a (non-elliptic) differential operator on Ω, defined by

$$Lu := \partial_1(\partial_2 u) - \partial_2(\partial_1 u).$$

Show that every $u \in W^{1,2}(\Omega)$ is a solution of $Lu = 0$ in the weak sense.

(b) Let $\Omega \subset \mathbb{R}^n$ be open. We say that a vector field $f = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ with $f_k \in L^2(\Omega)$, $k \in \{1, \dots, n\}$ is a weak solution the differential equation $\nabla \cdot f = 0$ when

$$\int_\Omega (f \cdot \nabla\phi)\, d\mu = 0 \quad \text{for all} \quad \phi \in W_0^{1,2}(\Omega).$$

Now set $n = 3$. The *curl* (also called the *rotation*) of a vector field $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ with $u_k \in W^{1,2}(\Omega)$, $k \in \{1, 2, 3\}$ is defined to be

$$\nabla \times u := (\partial_2 u_3 - \partial_3 u_2,\, \partial_3 u_1 - \partial_1 u_3,\, \partial_1 u_2 - \partial_2 u_1),$$

in analogy to the cross product $\times$.

Show that the curl $f := \nabla \times u$ of u is a weak solution of $\nabla \cdot f = 0$.

Solution.

(a) We have a weak solution if

$$-\int \partial_2 u \partial_1 \phi - \partial_1 u \partial_2 \phi = 0$$

for all $v \in W_0^{1,2}$ (Exercise 36(c)). But we saw that the two terms are equal in Exercise 26 "An integration by parts".

(b) Following the definitions, we must compute

$$\begin{aligned}\int (\nabla \times u) \cdot \nabla\phi &= \int (\partial_2 u_3 - \partial_3 u_2)\partial_1\phi + (\partial_3 u_1 - \partial_1 u_3)\partial_2\phi + (\partial_1 u_2 - \partial_2 u_1)\partial_3\phi \\ &= \int (\partial_3 u_1 \partial_2 - \partial_2 u_1 \partial_3\phi) + (\partial_1 u_2 \partial_3\phi - \partial_3 u_2 \partial_1\phi) + (\partial_2 u_3 \partial_1\phi - \partial_1 u_3 \partial_2\phi) = 0.\end{aligned}$$

39. On Friedrich's Theorem on the interior.

Consider the real, open intervals $I := (-2, 2)$ and $J := (-1, 1)$. We choose a function $a \in L^\infty(I) \setminus W^{1,2}(J)$ wth $a \geq 1$, and let

$$u : I \to \mathbb{R}, \qquad u(t) := \int_0^t \frac{1}{a(x)}\, \mathrm{d}x.$$

(a) Show that $u \in W^{1,2}(I)$ and $u \notin W^{2,2}(J)$.

(b) Show that u is a weak solution of $(au')' = 0$ on I.

(c) Why does this not contradict Friedrich's theorem on the interior?

Solution.

(a) We have the bound on the function

$$|u(x)| \leq \left| \int_0^x 1 \right| \leq |x|,$$

since $a \geq 1$, from which the $L^2(I)$-norm can be bounded. Likewise $|\partial u(x)| = |a(x)^{-1}| \leq 1$ shows $u \in W^{1,2}(I)$.

The second derivative, if it existed, would have to be $\partial^2 u = -a^{-2}\partial a$. To see this, we need to extend the chain rule (Proposition 3.29) a little:

Let u, f be as in Proposition 3.29, except Ω is bounded and $f(0) = f_0 \in \mathbb{R}$. Then the conclusion still holds.

Proof. Let $g(y) = f(y) - f_0$, so that $g(0) = 0$. $f(u)$ is still $L^p(\Omega)$ because both $g(u)$ and f_0 are (the latter using the fact that Ω is bounded). For the derivatives, on one hand $\nabla(g(u)) = g'(u)\nabla u = f'(u)\nabla u$ and on the other $\nabla(g(u)) = \nabla(f(u)) + 0$. This shows the weak derivative exists and is L^p.

Since $f(y) = y^{-1}$ is Lipschitz for $y \geq 1$, our claim about $\partial^2 u$ holds. But then $\partial a = a^2 \partial^2 u$ would be $L^2(J)$, contradicting the assumption of the question.

(b) This is an elliptic PDE in divergence form, except in dimension one. It's elliptic because of the bound $a\xi\xi \geq \xi^2$ for $\xi \in \mathbb{R}$. u is a weak solution if

$$-\int_I au'v' = 0$$

for all $v \in W_0^{1,2}(I)$. But $au' = 1$ and by approximating v by test functions and applying integration by parts we see that the integral indeed vanishes.

(c) Friedrich's theorem on the interior says that under the additional assumptions that a, b are Lipschitz then the weak solution is $W^{2,2}_{loc}$. Here $b = 0$, so the only way out is if a is not Lipschitz. We know from Proposition 3.27 that $C^{0,1}(J) = W^{1,\infty}(\Omega)$ and since J is bounded $L^\infty \subset L^2$. Therefore the assumption that $a \notin W^{1,2}(J)$ stands in contradiction to a being Lipschitz.

40. On the Cacciopoli inequality at the boundary.

Complete the proof of the Cacciopoli inequality at the boundary (Theorem 4.7) from the lecture notes by adapting the proof of Theorem 4.5.

Solution. Adding a few more details to the proof of Theorem 4.7. We really consider $\tilde{u} = u - \varphi$, which is a weak solution of $-\mathcal{L}(\tilde{u}, v) = \langle f, v\rangle + \mathcal{L}(\varphi, v)$. We have seen in a previous exercise that the right hand side also defines an element $\tilde{f} \in W_0^{1,2}(\Omega)^*$. Thus we consider $L\tilde{u} = \tilde{f}$. This is the same idea as in the proof of the existence of weak solutions of the Dirichlet problem, but it is good to see it a second time.

Next we follow the proof and define $v = \tilde{u}\eta^2 \in W_0^{1,2}(B(0,2)_+)$ for a smooth cutoff function $\eta \in C_0^\infty(B(0,2))$ that is 1 on $B(0,1)$. We want to say that $\Omega = B(0,2)_+$ and $\Omega' = B(0,1)_+$ and just apply Cacciopoli's inequality 4.5. But this doesn't apply since Ω' and Ω share part of their boundary and therefore Ω' is not compactly contained within Ω. Instead we have to repeat the calculation with the integrals. The difference to watch out for is that η is not zero on $B(0,2)_0$ in Theorem 4.7, whereas it is zero on $\partial\Omega$ in Theorem 4.5. But this plays no role in the integral calculation.

Hence we still get

$$\|\tilde{u}\|_{W^{1,2}(\Omega')} \leq C(\Lambda, n)(\|\tilde{f}\|_{W_0^{1,2}(\Omega)^*} + \|\tilde{u}\|_{L^2(\Omega)}).$$

The constant only depends on Λ and n because Ω and Ω' are fixed spaces in this theorem. Finally we have to deal with the tildes, but this follows from

$$\|\tilde{u}\|_{W^{1,2}(\Omega')} \geq \|u\|_{W^{1,2}(\Omega')} - \|\varphi\|_{W^{1,2}(\Omega')} \geq \|u\|_{W^{1,2}(\Omega')} - \|\varphi\|_{W^{1,2}(\Omega)}$$
$$\|\tilde{u}\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \|\varphi\|_{W^{1,2}(\Omega)}$$

and for the operator norm, from Equation 4.4 in the script

$$\| - \mathcal{L}(\varphi, \cdot)\|_{W_0^{1,2}(\Omega)^*} \leq C(n)\Lambda\|\varphi\|_{W^{1,2}(\Omega)}.$$