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Partial Differential Equations
Exercise sheet 11

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38. Divergence and rotation.

(a) Let $\Omega \subset \mathbb{R}^2$ be open and L a (non-elliptic) differential operator on Ω, defined by

$$Lu := \partial_1(\partial_2 u) - \partial_2(\partial_1 u).$$

Show that every $u \in W^{1,2}(\Omega)$ is a solution of $Lu = 0$ in the weak sense.

(b) Let $\Omega \subset \mathbb{R}^n$ be open. We say that a vector field $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$ with $f_k \in L^2(\Omega)$, $k \in \{1, \ldots, n\}$ is a weak solution the differential equation $\nabla \cdot f = 0$ when

$$\int_\Omega (f \cdot \nabla\phi)\, d\mu = 0 \quad \text{for all} \quad \phi \in W_0^{1,2}(\Omega).$$

Now set $n = 3$. The *curl* (also called the *rotation*) of a vector field $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ with $u_k \in W^{1,2}(\Omega)$, $k \in \{1, 2, 3\}$ is defined to be

$$\nabla \times u := (\partial_2 u_3 - \partial_3 u_2,\, \partial_3 u_1 - \partial_1 u_3,\, \partial_1 u_2 - \partial_2 u_1),$$

in analogy to the cross product $\times$.

Show that the curl $f := \nabla \times u$ of u is a weak solution of $\nabla \cdot f = 0$.

39. On Friedrich's Theorem on the interior.

Consider the real, open intervals $I := (-2, 2)$ and $J := (-1, 1)$. We choose a function $a \in L^\infty(I) \setminus W^{1,2}(J)$ wth $a \geq 1$, and let

$$u : I \to \mathbb{R}, \qquad u(t) := \int_0^t \frac{1}{a(x)}\, \mathrm{d}x.$$

(a) Show that $u \in W^{1,2}(I)$ and $u \notin W^{2,2}(J)$.

(b) Show that u is a weak solution of $(au')' = 0$ on I.

(c) Why does this not contradict Friedrich's theorem on the interior?

40. On the Cacciopoli inequality at the boundary.

Complete the proof of the Cacciopoli inequality at the boundary (Theorem 4.7) from the lecture notes by adapting the proof of Theorem 4.5.