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Partial Differential Equations
Exercise sheet 10

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35. The dual to $W_0^{1,2}$.

At the beginning of Chapter 4 we are introduced to the dual space $W_0^{1,2}(\Omega)^*$. In this question we explore this a little more.

(a) Show that the formula for $\langle g + \nabla \cdot f, v\rangle$ given at the start of Chapter 4 does indeed define a distribution for $v \in C_0^\infty(\Omega)$. Why does it extend to $W_0^{1,2}(\Omega)$?

(b) Give an example of two different choices g, f, g', f' that give the same distribution.

(c) In Chapter 3 we saw that $W_0^{1,2}(\Omega)$ was a Hilbert space. Recall the definition of its inner product. What is the relation between this inner product and the norm on $W^{1,2}$.

(d) Optional: Use the Riesz representation theorem to show that all extending distributions have this form.

Solution.

(a) Let $K \Subset \Omega$ and suppose that $v \in C_0^\infty(K)$. Then

$$|\langle g + \nabla \cdot f, v\rangle| \le \int_K |gv| + \int_K |f \cdot \nabla v| \le \|v\|_\infty \int_K |g| + \sum \|\partial_i v\|_\infty \int_K |f_i|$$
$$\le C\|g\|_2 \|v\|_{K,0} + \sum C\|f_i\|_2 \|\partial_i v\|_{K,e_i}$$

This shows it is a distribution.

The product of two L^2 function is L^1 by Hölder's inequality, so the formula is well-defined for $v \in W^{1,2}$. In fact for $v, w \in C_0^\infty$

$$|\langle g + \nabla \cdot f, v - w\rangle| \le \|g\|_2 \|v - w\|_2 + \sum \|f_i\|_2 \|\partial_i v - \partial_i w\|_2 \le \max\{\|g\|_2, \|f_i\|_2\} \|v - w\|_{W^{2,2}}$$

so it is uniformly continuous with respect to the operator norm on $\mathcal{L}(W^{1,2}(\Omega), \mathbb{R})$ and so has a unique continuous extension to $W_0^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}$.

A note: On the left hand side, the expression $\nabla \cdot f$ is not meant to be taken literally, since f is only L^2, but it is meant as a memorisation aid that integration by parts should be applied. It does hold literally when f is sufficiently regular, as we will see in the next question.

(b) If f is smooth then

$$\langle 0 + \nabla \cdot f, v\rangle = \int_\Omega -f \cdot \nabla v = \int_\Omega \nabla \cdot f v = \langle \nabla \cdot f + 0, v\rangle.$$

(c)

$$(u, v) = \sum_{|\gamma| \le 1} \int_\Omega \partial^\gamma u \partial^\gamma v = \int_\Omega uv + \nabla u \cdot \nabla v$$

Note that the norm induced by this inner product is

$$\|u\|_{\text{Hilbert}} = \sqrt{(u, u)} = \left(\sum_{|\gamma| \le 1} \|\partial^\gamma u\|_2^2 \right)^{1/2} \ne \sum_{|\gamma| \le 1} \|\partial^\gamma u\|_2.$$

So the norm induced by the inner product is different to the Sobolev norm. However the two norms are equivalent, so what is continuous with respect to one norm is continuous with respect to the other.

(d) We have seen that the pairing is a continuous linear functional on the Hilbert space $W_0^{1,2}(\Omega)$. Therefore the Riesz representation theorem tells us that there is an element $u \in W_0^{1,2}(\Omega)$ with $(u, v) = \langle g + \nabla \cdot f, v\rangle$ for all $v \in W_0^{1,2}(\Omega)$. Examining the left hand side we see that $g = u$ and $f = -\nabla u$.

36. On the weak solutions of elliptic differential equations.

Let $\Omega \subset \mathbb{R}^n$ be open, $u \in W^{1,2}(\Omega)$, $f \in W_0^{1,2}(\Omega)^*$, and L be an elliptic differential operator in the sense of Definition 4.1.

(a) State what it means for u to be a weak solution of $Lu \geq f$.

(b) Show that the following is a distribution:

$$C_0^\infty(\Omega) \ni \phi \mapsto -\mathcal{L}(u, \phi).$$

(c) Suppose that u is a weak solution of both

$$Lu \geq f \quad \text{and} \quad Lu \leq f, \tag{$*$}$$

in the sense of Definition 4.1. Show for all $v \in W_0^{1,2}(\Omega)$ that $-\mathcal{L}(u, v) = \langle f, v\rangle$. Note: there is no requirement for v to be non-negative.

(d) How should we interpret Lu as a distribution, if it is not L^1_{loc}? Hence prove that if $(*)$ holds then $Lu = f$ in the sense of distributions.

(e) Suppose now that $u \in W^{1,2}_{\text{loc}}(\Omega)$, $f \in L^2_{\text{loc}}(\Omega)$ such that $\triangle u \geq f$ and $\triangle u \leq f$ hold in the weak sense. Show for all $\phi \in C_0^\infty(\Omega)$ that

$$\triangle(\phi u) = (\triangle\phi)u + 2\nabla\phi \cdot \nabla u + f\phi$$

holds in the sense of distributions.

Solution.

(a) From Definition 4.1, to an elliptic operator L in divergence form we associate a bilinear form $\mathcal{L}$. We say that u is a solution of $Lu \geq f$ in the weak sense if for all non-negative $v \in W_0^{1,2}(\Omega)$ we have $-\mathcal{L}(u, v) \geq \langle f, v\rangle$. The pairing $\langle f, v\rangle$ is nothing other than the action of the dual, but we write it with this notation because we have a special description of some of these dual elements and their action given in the formula before Definition 4.1.

(b) It is clearly linear, so let's show it is continuous as distribution directly from the form of $\mathcal{L}$. Choose a compact subset $K \subset \Omega$ and a test function $\phi \in C_0^\infty(K)$. Then

$$|-\mathcal{L}(u,\phi)| \leq \sum \int_K |a_{ij}\partial_i u \partial_j \phi| + \sum \int_K |b_i u \partial_j \phi| + \sum \int_K |c_i \partial_i u \phi| + \int_K |du\phi|.$$

Taking the leading term for example

$$\int_K |a_{ij}\partial_i u \partial_j \phi| \leq \Lambda \|\phi\|_{K,1} \int_K |\partial_i u| \leq \Lambda \|\partial_i u\|_{L^1(K)} \|\phi\|_{K,1} \leq \Lambda \|\partial_i u\|_{L^2(K)} \|\phi\|_{K,1},$$

because K is a compact set so we know that $L^2(K) \subset L^1(K)$ (apply Hölder's inequality with the function times 1). All the other terms are bound similarly.

(c) Given any $v \in W_0^{1,2}$ we can write it in positive and negative parts $v = v^+ - v^-$, where $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$. We know from Exercise 27 that both parts are also $W^{1,2}$. To see that they belong to $W_0^{1,2}$, first take a sequence v_n in C_0^∞ that converges to v, take the positive part v_n^+ and then mollify $v_{n,\varepsilon}^+$. For sufficiently small ε the mollification $v_{n,\varepsilon}^+$ also belongs to $C\infty_0(\Omega)$. A standard diagonal argument now shows v^+ lies in their limit.

Given this splitting, the question is straightforward

$$-\mathcal{L}(u,v) = -\mathcal{L}(u,v^+) + \mathcal{L}(u,v^-) = \langle f, v^+ \rangle - \langle f, v^- \rangle = \langle f, v \rangle.$$

(d) The important idea here, not explicit in the lecture script, how to understand Lu as a distribution. Really, it doesn't seem well defined with only the conditions given in Definition 4.1. But when u and the coefficients are sufficiently differentiable then it is well defined and equal to $-\mathcal{L}(u,\cdot)$ as a distribution. Taking a sequence of operators and functions $L_m u_m$ we can take a limit and there is a unique extension of the association of Lu with a distribution, namely the one given by $-\mathcal{L}(u,\cdot)$.

With this understanding, the equality follows immediately from the previous parts, since we have just seen that $-\mathcal{L}(u,\cdot)$ acts the same as f on $W_0^{1,2}(\Omega) \supset C_0^\infty(\Omega)$.

(e) The essential point here (that is new to this course) is that the product of $u \in W^{1,2}$ with a test function is again $W^{1,2}$ and the product rule holds (Proposition 3.28 Product rule for Sobolev functions with $p = r = 2$ and $q = \infty$). We must however avoid second derivatives of u, which may not exist. If $\operatorname{supp} \psi = K$ then $u \in W^{1,2}(K)$, ie the fact that they are only locally Sobolev is not a problem. Hence we calculate

$$\begin{aligned}
\int_K ((\triangle\phi)u + 2\nabla\phi\cdot\nabla u + f\phi)\psi &= \int_K u\psi(\nabla\cdot\nabla\phi) + 2\psi\nabla\phi\cdot\nabla u + f\phi\psi \\
&= \int_K \nabla\cdot(u\psi\nabla\phi) - \psi\nabla u\cdot\nabla\phi - u\nabla\psi\cdot\nabla\phi + 2\psi\nabla\phi\cdot\nabla u - \nabla u\cdot\nabla(\phi\psi) \\
&= \int_K -u\nabla\psi\cdot\nabla\phi + \nabla u\cdot[\psi\nabla\phi - \nabla(\phi\psi)] \\
&= -\int_K [u\nabla\phi + \nabla u\phi]\cdot\nabla\psi \\
&= -\triangle(\phi u, \psi).
\end{aligned}$$

We can justify the vanishing of the divergence term by taking a smooth approximation, to which we can apply the divergence theorem.

37. Weak solutions of the Poisson equation.

In the following we demonstrate an example of functions $u, f \in C^0(\Omega)$ such that $\triangle u = f$ in the weak sense, but $u \notin C^2(\Omega)$. Let $\Omega = B(0, \frac{1}{2}) \subset \mathbb{R}^2$ and $u(x,y) := (x^2 - y^2)\log|\log(r)|$ with $r = (x^2+y^2)^{1/2}$.

(a) Show that $u \in C^2(B(0,\frac{1}{2})\backslash\{0\})$ and $\lim_{r\to 0} u(x,y) = 0$. In other words, u extends to a continuous function on $B(0,\frac{1}{2})$.

(b) Compute the following derivatives of u on $B(0,\frac{1}{2})\backslash\{0\}$

$$\begin{aligned}\frac{\partial}{\partial x}u(x,y) &= 2x\log|\log(r)| + (x^3 - y^2x)\frac{1}{r^2\log(r)},\\ \frac{\partial^2}{\partial x^2}u(x,y) &= 2\log|\log(r)| + (5x^2 - y^2)\frac{1}{r^2\log(r)} - (x^4 - x^2y^2)\frac{2\log(r)+1}{r^4(\log(r))^2}.\end{aligned}$$

(c) Argue that $\frac{\partial^2}{\partial y^2}u(x,y) = -\frac{\partial^2}{\partial x^2}u(y,x)$ and hence

$$\triangle u = (x^2 - y^2)\left(\frac{4}{r^2\log(r)} - \frac{1}{r^2(\log(r))^2}\right).$$

Conclude therefore that $\lim_{r\to 0}\triangle u(x,y) = 0$.

(d) Let $f \in C(B(0,\frac{1}{2}))$ be the continuous extension of $\triangle u$ on $B(0,\frac{1}{2})$. Prove that $\triangle u = f$ weakly on $B(0,\frac{1}{2})$.

Solution.

(a) $\ln|\ln r|$ is well-defined and smooth except for $r=0$ and $\ln r = 0$, ie $r=1$.

$$|u| \le r^2\ln|\ln r| \to \frac{\frac{1}{r|\ln r|}}{-2r^{-3}} = \frac{r^2}{-2\ln r} \to 0.$$

(b) Damn it, Jim. I'm a doctor, not a calculator.

(c) One could do this calculation directly, but this is a way to avoid more calculation. Consider the reflection $t(x,y) = (y,x)$ and note that $u \circ t = -u$. Then $\partial_y u = -\partial_y(u\circ t) = -\partial_x u \circ t$ and $\partial_y^2 u = -\partial_y(\partial_x u \circ t) = -\partial_x^2 u \circ t$. The Laplacian now follows from part (b).

$$|\triangle u| \le \frac{4}{\log(r)} - \frac{1}{(\log(r))^2} \to 0.$$

(d) We need to show $-\triangle(u,\phi) = \langle f, \phi \rangle$ for all test functions $\phi \in C_0^\infty(B(0,1/2))$. Let $\Omega_\varepsilon := B(0,1/2) \setminus B(0,\varepsilon)$. On Ω_eps we have $\triangle u = f$ since everything is smooth. So

$$\int_{\Omega_\varepsilon} \nabla u \cdot \nabla \phi + f\phi = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \phi + \nabla \cdot \nabla u \, \phi = \int_{\partial\Omega_\varepsilon} \nabla u \cdot \nabla \phi + \nabla \cdot \nabla u \, \phi$$
$$= -\int_{\partial B(0,\varepsilon)} (\phi \nabla u) \cdot N \, d\sigma.$$

When plug everything in, you get a factor times an integral $\int_0^{2\pi} \cos 2\theta \, d\theta = 0$. Taking the limit now as $\varepsilon \to 0$ shows that $\triangle u = f$ in the weak sense.