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Partial Differential Equations
Exercise sheet 9

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31. Another approach to Sobolev inequalities.

Sobolev inequalities compare the "size" of ∇u with that of u. Therefore we want to express u in terms of its gradient.

(a) Let Ω be bounded and $u \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^n)$ and take polar coordinates $(r, v) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ on $\mathbb{R}^n$. Show:

$$u(x) = -\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \partial_r(u(x + rv)) \,\mathrm{d}r \,\mathrm{d}\sigma(v).$$

[Hint. First compute $-\int_0^\infty \partial_r(u(x + rv)) \,\mathrm{d}r$.]

(b) Prove further that

$$u(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\langle x - y, \nabla u(y)\rangle}{|y - x|^n} \,\mathrm{d}y \text{ and } |u(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \,\mathrm{d}y.$$

(c) Find a bound on u in terms of $\|\nabla u\|_p$ for $p > n$.

Solution.

(a) The inner integral we can integrate exactly

$$-\int_0^\infty \partial_r(u(x + rv)) \,\mathrm{d}r = -\Big[u(x + rv)\Big]_0^\infty = 0 + u(x).$$

This is then constant with respect to v and so we can bring it outside the outer integral.

(b) For $y = x + rv$ we have

$$\partial_r(u(x + rv)) = \nabla u \cdot \partial_r(x + rv) = \nabla u \cdot v = \nabla u \cdot \frac{y - x}{r}.$$

So

$$\begin{aligned}
u(x) &= -\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \partial_r(u(x + rv)) \,\mathrm{d}r \,\mathrm{d}\sigma(v) \\
&= -\frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \nabla u(y) \cdot \frac{y - x}{r^n} r^{n-1} \,\mathrm{d}r \,\mathrm{d}\sigma(v) \\
&= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \nabla u(y) \cdot \frac{x - y}{|x - y|^n} \, dy \\
|u(x)| &\leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy
\end{aligned}$$

(c) Since the function has compact support, the support lies in some large ball $B(0, R)$. We apply Hölder's inequality to the right hand side.

$$\int_{B(0,R)} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy \leq \|\nabla u\|_p \||x - y|^{1-n}\|_q.$$

We see

$$\||x - y|^{1-n}\|_q^q = (n\omega_n)^q \int_0^R |x - y|^{p(1-n)} \, dy < \infty$$

for $q < n/(n - 1) \Leftrightarrow p > n$. So we get $|u(x)| < C\|\nabla u\|_p$.

32. The Sobolev conjugate.

Suppose that for compactly supported smooth functions we have an inequality

$$\|u\|_q \leq C\|\nabla u\|_p.$$

By considering the rescaled functions $u_\lambda(x) := u(\lambda x)$ show that this inequality is only possible for $q^{-1} = p^{-1} - n^{-1}$.

Solution. We have $\|u_\lambda\|_q = \lambda^{-n/q}\|u\|_q$ and $\|\nabla u_\lambda\|_p = \lambda^{1-n/p}\|\nabla u\|_p$. If the inequality also holds for u_λ then

$$\|u\|_q \leq C\lambda^{1-n/q+n/q}\|\nabla u\|_p.$$

Considering the limits $\lambda \to 0$ and $\lambda \to \infty$, this can only hold if the exponent is zero: $1-n/q+n/q$.

33. The Sobolev embedding theorem.

Show that $W^{1,1}((0,1)) \hookrightarrow C([0,1])$ is a continuous embedding.

[Hint. One needs to show that $\|u\|_\infty \leq \|u\|_1 + \|u_1\|_1$ holds. Therefore define, for $(u, u_1) \in W^{1,1}((0,1))$, the function $U := \int_{x_0}^{x} u_1(t)\,\mathrm{d}t$ and prove: $U \in W^{1,1}((0,1)) \cap C([0,1])$ and $U - u \equiv \text{const}$. It then follows that $|u|$ obtains a minimum $x_0 \in [0,1]$. Finally, one can show $|u(x) - u(x_0)| \leq \|u_1\|_1$ and estimate $\|u\|_\infty$ with the triangle inequality.]

Solution. We follow the hint. U is continuous because

$$|U(x) - U(y)| \leq \int_x^y |u_1| \to 0,$$

and clearly $W^{1,1}$ with $U' = u_1$. Hence $(U - u)' = u_1 - u_1 = 0$ and it follows that $U - u$ is a constant. Already we see that u is continuous. Let x_0 be the minimum of $|u|$, so

$$\|u\|_1 = \int_0^1 |u| \geq \int_0^1 |u(x_0)| = |u(x_0)|.$$

We have

$$|u(x) - u(x_0)| = |U(x) - U(x_0)| \leq \|u_1\|_1.$$

On the other hand

$$|u(x) - u(x_0)| \geq |u(x)| - |u(x_0)|.$$

Finally

$$\|u\|_\infty \leq |u(x_0)| + \sup|u(x) - u(x_0)| \leq \|u\|_1 + \|u_1\|_1.$$

This shows that the embedding is continuous.

34. The Garding inequality.

The Garding inequality, Equation (4.5) in the script, is needed to apply the Lax-Milgram theorem. Here we prove a special case. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and $L : C_0^2(\Omega) \to C_0(\Omega)$ the elliptic operator

$$(Lu)(x) = -\operatorname{div}(A(x)\nabla u(x)) + c(x)u(x)$$

given in divergence form. Let $K > 0$ and $c(x) \geq K \ \ \forall x \in \Omega$. Show that L obeys the inequality

$$\langle Lu, u\rangle_{L^2(\Omega)} \geq C \cdot \|u\|^2_{W^{1,2}(\Omega)} \ \text{ (for a constant } C > 0).$$

Solution.

$$\begin{aligned}\langle Lu, u\rangle_{L^2(\Omega)} &= \int_\Omega Lu\, u = \int_\Omega -\operatorname{div}(A\nabla u)\, u + cu^2 = \int_\Omega -\operatorname{div}(uA\nabla u) + (A\nabla u)\cdot \nabla u + cu^2 \\ &= \int_\Omega (A\nabla u)\cdot \nabla u + cu^2.\end{aligned}$$

Now we use the fact that L is elliptic, so $v \cdot Av = v^T Av \geq \Lambda^{-1}\|v\|^2$. We continue

$$\langle Lu, u\rangle_{L^2(\Omega)} \geq \int_\Omega \Lambda^{-1}|\nabla u|^2 + Ku^2 = \Lambda^{-1}\|\nabla u\|_2^2 + K\|u\|_2^2.$$

Finally, Cauchy-Schwarz inequality gives $\sum 1\, |v_i| \leq \sqrt{n}\left(\sum |v_i|^2\right)^{0.5}$

$$\begin{aligned}\langle Lu, u\rangle_{L^2(\Omega)} &\geq \min\{\Lambda^{-1}, K\}\left(\|\nabla u\|_2^2 + \|u\|_2^2\right) \geq \min\{\Lambda^{-1}, K\} n^{-1}\left(\|u\|_2 + \sum \|\partial_i u\|_2\right)^2 \\ &= \min\{\Lambda^{-1}, K\} n^{-1}\|u\|^2_{W^{1,2}}.\end{aligned}$$