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Partial Differential Equations
Exercise sheet 8

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27. Approximation by truncated Sobolev functions.

(a) Let $\Omega \subset \mathbb{R}^n$ be open and $u, v \in W^{1,p}(\Omega)$. Show for $w(x) := \min\{u(x), v(x)\}$ that w also lies in $W^{1,p}(\Omega)$. Determine the weak derivatives of w.
[Hint. Use the identities $\min\{a, b\} = \min\{a - b, 0\} + b$ and $\min\{a, 0\} = 0.5(a - |a|)$, and apply Propositions 3.29 (Chain Rule) and 3.30.]

(b) Show using (a) that $\max\{u(x), v(x)\} \in W^{1,p}(\Omega)$ too.

(c) Prove that when $u \in W^{1,p}(\Omega)$ then $w := \min\{u, 1\} \in W^{1,p}(\Omega)$. Calculate the first derivatives of w.

(d) Show $L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.
[Hint. For $u \in W^{1,p}(\Omega)$ consider the sequence of truncations $u_n := \max\{-n, \min\{u, n\}\}$.]

Solution.

(a) We can simplify if we write $\min\{u, v\} = \min\{u - v, 0\} + v$. Thus we need only consider the case $v = 0$. The function $f(y) = \min\{y, 0\}$ is a Lipschitz function with $f(0) = 0$, so the chain rule for Sobolev functions (Proposition 3.29) tells us that $w \in W^{1,p}$.

It remains to determine the derivative. The proof of Proposition 3.30 also works with the L^p norm. But even not assuming this, we know that the weak derivative of $\nabla|u| = \operatorname{sign}(u)\nabla u$ and $\|\nabla|u|(x)\| \leq \|\nabla u(x)\|$ then shows that it is L^p. The derivative is therefore (not assuming $v = 0$)

$$\begin{aligned}\nabla w &= \nabla\left(v + \frac{1}{2}[u - v - |u - v|]\right) = \frac{1}{2}\nabla\left[u + v - |u - v|\right] \\ &= \frac{1}{2}\left[\nabla u + \nabla v - \operatorname{sign}(u - v)\left(\nabla u - \nabla v\right)\right].\end{aligned}$$

(b) $\max\{u, v\} = -\min\{-u, -v\}$.

(c) The difficulty here is that for an Ω with infinite area the constant functions are not L^p and we can not use part (a). But $|\min\{u(x), 1\}| \leq |u(x)|$ show us that $w \in L^p$ anyway. If we consider the function restricted to a bounded set $\Omega' \subset \Omega$ then we can apply part (a) and see that the weak derivative is given by

$$\nabla w = \chi_{\{u \geq 1\}} \nabla u.$$

Since $\|\nabla w(x)\| \leq \|\nabla u(x)\|$ this is also L^p.

(d) As suggested by the hint, we consider $u_n := \max\{-n, \min\{u, n\}\}$. Since $\min\{u, n\} = n\min\{u/n, 1\}$ and $\max\{-n, v\} = -n\min\{-v/n, 1\}$ for $n > 0$, part (c) applies to u_n. It remains to show convergence in norms. But $f_n(x) := |u(x) - u_n(x)| = (|u(x)| - n)\chi_{|u(x)|>n}$ is a decreasing sequence of functions converging pointwise to zero and $f_0 = |u| \in L^p$, so by the dominated convergence theorem we get $\|u - u_n\|_p \to 0$. A similar result holds for the derivative with $\partial_i u_n = \partial_i u \chi_{|u|<n}$.

28. More Sobolev functions.

Let $p^{-1} + q^{-1} = 1$, $n > q$ and $\Omega = B(0,1) \subset \mathbb{R}^n$. Choose $u \in C^1(\Omega \backslash \{0\})$ such that

$$\int_{\Omega\backslash\{0\}} |u(x)|^p \,\mathrm{d}\mu < \infty \quad \text{and} \quad \int_{\Omega\backslash\{0\}} |\nabla u(x)|^p \,\mathrm{d}\mu < \infty.$$

(a) Choose any $\psi \in C^\infty(\mathbb{R})$ with $\psi(r) = 1$ for $r \geq 1$, $\psi(r) = 0$ for $r \leq \frac{1}{2}$, and $0 \leq \psi(r) \leq 1$. Let $\psi_k(x) = \psi(k|x|)$. Show that $\psi_k \to 1$ in $W^{1,q}(\Omega)$.

(b) Define $u_k := u\psi_k$. Show that $\|\partial_i u - \partial_i u_k\|_1 \to 0$ as $k \to \infty$.

(c) Complete the proof that $u \in W^{1,p}(\Omega)$ and $\partial_j u$ is its weak derivative.

(d) Let $u : \Omega \setminus \{0\} \to \mathbb{R}$ be defined by $u(x) := \|x\|^\gamma$. Show $\partial^\alpha u(x) = P_\alpha(x)\|x\|^{\gamma - 2|\alpha|}$, where P_α is a homogeneous degree $|\alpha|$ polynomial. The exact form of P_α is unimportant.

(e) Using u from the previous part show that u belongs to $W^{k,p}(\Omega)$ for $\gamma > k - \frac{n}{p}$.

Solution.

(a) These functions converge pointwise almost everywhere to 1 and are bounded, so the dominated convergence theorem shows they converge in L^q. We know that outside of $B(0,k^{-1})$ that ψ_k' is zero, so they converge pointwise to zero. However it remains to show the derivatives converge to zero in L^q. $\partial_i \psi_k = \psi'(k|x|)kx_i|x|^{-1}$, so

$$\begin{aligned} \|\partial_i \psi_k\|_q^q = \int_\Omega |\psi'(k|x|)|^q k^q |x_i|^q |x|^{-q} &= \int_{B(0,1/k)} |\psi'(k|x|)|^q k^q |x_i|^q |x|^{-q} \\ &\leq \int_{B(0,1/k)} \|\psi'\|_\infty^q k^q = \|\psi'\|_\infty^q k^q \times \omega_k k^{-n} \to 0 \end{aligned}$$

using the assumption that $n > q$.

(b) ψ_k is identically zero on $B(0,(2k)^{-1})$ so $u_k := u\phi_k$ is $C^1(\Omega)$:

$$\begin{aligned} \|\partial_1 u - \partial_1 u\psi_k - u\partial_1\psi_k\|_1 &\leq \|\partial_1 u - \partial_1 u\psi_k\|_1 + \|u\partial_1\psi_k\|_1 \\ &\leq \|\partial_1 u\|_p \|1 - \psi_k\|_q + \|u\|_p \|\partial_i \psi_k\|_q \to 0. \end{aligned}$$

(c) If we try to directly compute the distributional derivative

$$\begin{aligned} -\int_\Omega u\partial_i\phi &= -\int_{B_\varepsilon} u\partial_i\phi - \int_{\Omega\backslash B_\varepsilon} u\partial_i\phi \\ &= -\int_{B_\varepsilon} u\partial_i\phi + \int_{\partial B_\varepsilon} u\phi\frac{x_i}{r}\, d\sigma + \int_{\Omega\backslash B_\varepsilon} \partial_i u\phi \end{aligned}$$

then we see that we have no good way to estimate the surface integral term. However

$$\left|\int_\Omega u_k\partial_i\phi - \int_\Omega u\partial_i\phi\right| \leq \|\partial_i\phi\|_\infty \|u\psi_k - u\|_1 \leq \|\partial_i\phi\|_\infty \|u\|_p \|\psi_k - 1\|_q \to 0$$

and

$$\left|\int_\Omega \partial_i u_k \phi - \int_\Omega \partial_i u \phi\right| \leq \|\phi\|_\infty \|\partial_i u_k - \partial_i u\|_1 \to 0$$

due to part (b). Together this shows

$$-\int_\Omega u \partial_i \phi = \lim -\int_\Omega u_k \partial_i \phi = \lim \int_\Omega \partial_i u_k \phi = \int_\Omega \partial_i u \phi,$$

i.e $\partial_i u$ is the weak derivative of u. By the given assumptions both are L^p.

(d) As a base case, we have $\partial^0 u = u = 1 \times |x|^{\gamma - 0}$. Inductively

$$\begin{aligned}\partial_i P_\alpha(x)|x|^{\gamma-|\alpha|} &= P'_\alpha(x)|x|^{\gamma-|\alpha|} + (\gamma - |\alpha|)P_\alpha(x)|x|^{\gamma-|\alpha|-1} x_i |x|^{-1} \\ &= \left[P'(x)_\alpha |x|^2 + (\gamma - |\alpha|)P_\alpha(x) x_i\right] |x|^{\gamma-|\alpha|-2}\end{aligned}$$

and $P_{\alpha+e_i} := P'_\alpha(x)|x|^2 + (\gamma - |\alpha|)P_\alpha(x)x_i$ is indeed homogeneous of degree $|\alpha + e_i|$.

(e) For any homogeneous polynomial of degree $|\alpha|$ we can bound its growth crudely for all x using the triangle inequality

$$|P_\alpha(x)| = \left|\sum_{|\beta|=|\alpha|} C_\beta x^\beta\right| \leq \sum_{|\beta|=|\alpha|} |C_\beta|\,|x^\beta| \leq |x|^{|\alpha|} \sum_{|\beta|=|\alpha|} |C_\beta| = C|x|^{|\alpha|}.$$

(For non-homogeneous polynomial, we can only bound their growth for sufficiently large x.)

So for $\partial^\alpha u$ to belong to L^p it is sufficient for

$$\|\partial^\alpha u\|_p^p = \int_\Omega |P_\alpha|^p \, |x|^{p(\gamma - 2|\alpha|)} \leq C \int_\Omega |x|^{p(\gamma-|\alpha|)} = C n \omega_n \int_0^1 r^{p(\gamma-|\alpha|)} \, r^{n-1} \, dr$$

to be finite, which holds when $p(\gamma - |\alpha|) + n - 1 > -1$. This is equivalent to $\gamma > |\alpha| - n/p$. If $\gamma > k - n/p$ then this holds for all $|\alpha| \leq k$ and all derivatives of u up and including order k belong to L^p.

29. An inequality for functions in $W_0^{2,2}(\Omega)$.

Let $\Omega \Subset \mathbb{R}^n$ be open and bounded, and $u \in W_0^{2,2}(\Omega)$. *Prove* the following inequality:

$$\|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}^{1/2} \cdot \|\triangle u\|_{L^2(\Omega)}^{1/2}.$$

[Hint. Consider $u \in C_0^\infty(\Omega)$ and integrate $\int_\Omega |\nabla u|^2 \, \mathrm{d}\mu$ by parts.]

Solution. For smooth functions note the identity $|\nabla u|^2 = \nabla \cdot (u \nabla u) - u \triangle u$. Let $u_k \in C_0^\infty(\Omega)$ converge to $u \in W_0^{2,2}(\Omega)$. Then

$$\begin{aligned}\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2 &= \lim \int_\Omega |\nabla u_k|^2 = \lim \int_\Omega \nabla \cdot (u_k \nabla u_k) - u_k \triangle u_k \\ &= \lim \int_{\partial\Omega} u_k \nabla u_k \cdot N \, d\sigma - \int_\Omega u_k \triangle u_k = \lim 0 - \int_\Omega u_k \triangle u_k \\ &\leq \lim \|u_k\|_2 \, \|\triangle u_k\|_2 = \|u\|_2 \, \|\triangle u\|_2.\end{aligned}$$

30. The Divergence theorem for Lipschitz continuous vector fields.

Let $\Omega \Subset \mathbb{R}^n$ be an open and bounded subset with boundary $\partial\Omega \in C^{0,1}$. We will show that the divergence theorem also holds for $f = (f_1, \ldots, f_n) \in (C^{0,1}(\Omega))^n$:

$$\int_\Omega \nabla \cdot f \,\mathrm{d}\mu = \int_{\partial\Omega} f \cdot N \,\mathrm{d}\sigma. \tag{$*$}$$

Firstly we must modify Definition 1.7 appropriately. Concretely: We choose a finite open covering of coordinate charts $\{V_l\}_{l=1}^N$ and appropriate diffeomorphisms $\Phi_l : U_l \to V_l$, for open subsets $U_l \subset \mathbb{R}^{n-1}$. Next take a partition of unity $(h_l)_{l=1}^N$ and define

$$\int_{\partial\Omega} f \cdot N \mathrm{d}\sigma = \sum_{l=1}^N \int_{U_l} h_l (f \cdot N) \circ \Phi_l \sqrt{\det(\Phi_l')^t \Phi_l'} \mathrm{d}\mu. \tag{$**$}$$

(a) *Show*: $\partial\Omega$ is continuously differentiable when, after a permutation of coordinates, Φ_l has the form $\Phi_l(y) = (y, \varphi_l(y))$, with $\varphi_l \in C^1(U_l, \mathbb{R})$.

(b) *Show*: When $\partial\Omega$ is continuously differentiable and Φ_l has the form as in (a), then $(**)$ becomes

$$\int_{\partial\Omega} f \cdot N \,\mathrm{d}\sigma = \sum_{l=1}^N \int_{U_l} h_l f(y, \varphi_l(y)) \cdot (\nabla_y \varphi_l(y), -1) \,\mathrm{d}^{n-1} y. \tag{$***$}$$

(c) Let $A \in O(n, \mathbb{R})$ be an orthogonal matrix and f a smooth function.

Show: For $f_A = A \cdot f \circ A^{-1}$ the normal vector N_A of the transformed domain $\Omega_A = A[\Omega]$ satisfies the equation $N_A(x) = A \cdot N(A^{-1}x)$ and the divergence theorem $(*)$ holds for (f_A, Ω_A), if and only if it folds for (f, Ω).

(d) Let $\varphi \in C^{0,1}(B^{n-1}(0,\rho))$ with $\|\varphi\|_\infty < M$ and $f \in \left(W_0^{1,\infty}(B^{n-1}(0,\rho) \times (-M, M))\right)^n$. Then the following holds

$$\int_{B^{n-1}(0,\rho)} \int_{\varphi(y)}^M \nabla \cdot f(y,t) \,\mathrm{d}^{n-1} y \, dt = \int_{B^{n-1}(0,\rho)} f(y, \varphi(y)) \cdot (\nabla_y \varphi, -1) \,\mathrm{d}^{n-1} y.$$

[Hint: Approximationssatz 3.33]

(e) *Show* that for $f = (f_1, \ldots, f_n) \in (C^{0,1}(\Omega))^n$ the divergence theorem $(*)$ hold.

[Hint: Show first that the expression in (c) holds also for $f \in (C^{0,1}(\Omega))^n$ and $\partial\Omega \in C^{0,1}$. Then use (d).]