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Partial Differential Equations
Exercise sheet 7

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23. The dual space of $L^p(\mathbb{R}^n)$.

Let $1 < p < \infty$ (we exclude $p = 1$ for this exercise). The Banach space $L^p(\mathbb{R}^n)$ has the norm

$$\|\cdot\| : L^p(\mathbb{R}^n) \to \mathbb{R}, \ \ f \mapsto \|f\|_p = \left(\int_{\mathbb{R}^n} |f|^p \mathrm{d}\mu\right)^{1/p}.$$

We will show that for q with $\frac{1}{p} + \frac{1}{q} = 1$ the map

$$j : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)' = \mathcal{L}(L^p(\mathbb{R}^n), \mathbb{R}), \ \ g \mapsto j(g) \text{ with } j(g)(f) = \int_{\mathbb{R}^n} fg\,\mathrm{d}\mu$$

is a linear isometry, i.e. $\|g\|_q = \|j(g)\|$ holds. One can then show that for $1 \le p < \infty$ the dual space of $L^p(\mathbb{R}^n)$ is isometrically isomorphic to $L^q(\mathbb{R}^n)$.

(a) Show, with the help of the Hölders inequality that $j : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)'$ is Lipschitz continuous with Lipschitz constant $L \le 1$.

(b) Given a function g, find a function f_g such that $|j(g)(f_g)| = \|f_g\|_p \cdot \|g\|_q$.

(c) Show that j is an isometry.

(d) Optional: Use the Radon-Nikodym theorem to prove that j is surjective.

(e) Finish the proof that $L^p(\mathbb{R}^n)'$ and $L^q(\mathbb{R}^n)$ are isometrically isomorphic.

(f) Optional: Extend this result to the case $p = 1$ and $q = \infty$.

(g) What is the connection to distributions and Proposition 3.22?

Solution.

(a) Hölder's inequality is $\|fg\|_1 \le \|f\|_p \|g\|_q$ for such p and q. Since j and $j(g)$ are linear we compute

$$\|j\|_{op} = \sup_{\|g\|_q=1} \|j(g)\|_{op} = \sup_{\|g\|_q=1} \sup_{\|f\|_p=1} |j(g)(f)| \le \sup_{\|g\|_q=1} \sup_{\|f\|_p=1} \|fg\|_1 \le 1.$$

This shows that j is Lipschitz with constant ≤ 1.

(b)

$$f_g(x) = \begin{cases} \|g\|_q^{1-q} \, |g(x)|^q \, g(x)^{-1} & \text{for } g(x) \neq 0 \\ 0 & \text{for } g(x) = 0 \end{cases}$$

Then

$$\left|\int_\Omega f_g g\right| = \left|\int_{g\neq 0} \|g\|_q^{1-q} \, |g(x)|^q\right| = \|g\|_q^{1-q} \int_{g \neq 0} |g(x)|^q = \|g\|_q$$

and

$$\|f_g\|_p^p = \int_{g\neq 0} \|g\|_q^{p(1-q)} \, |g(x)|^{p(q-1)} = \int_{g \neq 0} \|g\|_q^{-q} \, |g(x)|^q = 1.$$

(c) As in part (a), we have that

$$\|j(g)\|_{op} = \sup_{\|f\|_p=1} |j(g)(f)| \le \|g\|_q.$$

Conversely, part (b) shows that equality is obtained and $\|j(g)\|_{op} = \|g\|_q$.

(d) Take an element $\kappa \in L^p(\mathbb{R}^n)'$. This defines a measure $\tilde{\kappa}$ on $\mathbb{R}^n$ via $A \mapsto \kappa(\chi_A)$. If A is a Lebesgue null set, then $\chi_A = 0$ in $L^p(\mathbb{R}^n)$, so $\tilde{\kappa}(A) = 0$. This shows that $\tilde{\kappa}$ is absolutely continuous with respect to the Lebesgue measure. The Radon-Nikodym theorem then gives us a measurable function g with

$$\tilde{\kappa}(A) = \int_A g \, d\mu.$$

In other words

$$\kappa(\chi_A) = \int \chi_A g \, d\mu.$$

Using simple functions (linear combinations of indicator functions) and their limits, this relationship extends to all measurable functions. It remains to show that $g \in L^q$. But this follows using the function $f = |g(x)|^q g(x)^{-1}$ similar to part (b), since then $\infty > \kappa(f) = \|g\|_q^q$. In summary, for any $\kappa \in L^p(\mathbb{R}^n)'$ we have found a $g \in L^q(\mathbb{R}^n)$ with $j(g) = \kappa$.

(e) The isometry property shows that j is injective. Part (c) proved that j was surjective. Part (a) showed that it was continuous. Hence it is an isomorphism of Banach spaces.

(f) Hölder's inequality also holds in this case, so part (a) is unchanged.

Part (b) doesn't work at all. Instead look to the definition of the essential supremum $\|g\|_\infty = \sup\{a \in \mathbb{R} \mid \mu(g^{-1}[(a, \infty)]) \neq 0\}$. Take an increasing sequence a_n converging to the supremum. Take a nonzero but finite measure subset A_n of $g^{-1}[(a_n, \infty)]$. Finally take a sequence of functions $f_{g,n} = \chi_{A_n}$. The equality of part (b) holds in the limit, which is enough to show that j is an isometry in part (c).

The modification for part (d) is probably similar, but I haven't thought about it.

(g) $j(g)$ is very similar to the distribution F_g; they have the same formula but are defined on different spaces, L^p and C_0^∞ respectively. We know from Proposition 3.19 (or more generally Proposition 3.24) that test function are dense in L^p. So when the associated distribution F_g is bounded with respect to the operator norm on $\mathcal{L}(L^p, \mathbb{R})$ then it extends to a continuous operator on L^p. The condition on Proposition 3.22 uses the L^q norm instead of the operator norm, but we have just seen that these are isometric.

24. Sobolev Functions.

(a) Write the definition of a Sobolev space using distributions.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$x \mapsto \begin{cases} 1 + x & \text{für } -1 \leq x \leq 0 \\ 1 - x & \text{für } \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Describe the first derivative of the distribution $F : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto F(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x$.

(ii) Show that the second derivative of the distribution $F(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\mathrm{d}x$ is a linear combination of Dirac distributions.

(iii) Show: $f \in W^{1,1}(\mathbb{R})$, but $f \notin W^{2,1}(\mathbb{R})$.

(c) Let $\Omega = \mathbb{R}^n$ and $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n)$ so that $\partial^\alpha u = 0$ for all α with $|\alpha| = 2$ in the weak sense. Show that u is affine, i.e. $u(x) = a \cdot x + b$ *a.e.* with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

[Hint. Proposition 3.22.]

(d) Let $\Omega = B(0, 0.5) \subset \mathbb{R}^2$ and $u(x) = \left(\ln \frac{1}{\|x\|}\right)^{1/4}$. Show that $u \in W^{1,2}(\Omega)$ but that it is not continuous.

Solution.

(a) The definition given in the script is

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \forall |\gamma| \le k \, \exists u_\gamma \in L^p(\Omega) \, \forall \varphi \in C_0^\infty(\Omega) : \int_\Omega u_\gamma \varphi = (-1)^\gamma \int_\Omega u \partial^\gamma \varphi \right\}.$$

We can recognise that

$$\int_\Omega u_\gamma \varphi = F_{u_\gamma}(\varphi), \qquad (-1)^\gamma \int_\Omega u \partial^\gamma \varphi = (-1)^\gamma F_u(\partial^\gamma \varphi). = \partial^\gamma F_u(\varphi).$$

So the definition becomes

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \forall |\gamma| \le k \, \exists u_\gamma \in L^p(\Omega) : F_{u_\gamma} = \partial^\gamma F_u \right\}.$$

In other words, we require that the γ-derivative of the distribution corresponding to u corresponds to some L^p function, for all γ up to and including order k.

(b) **(i)**

$$\begin{aligned}
\partial F(\phi) &= -\int_{-1}^0 (1+x)\partial\phi - \int_0^1 (1-x)\partial\phi \\
&= -[(1+x)\phi]_{-1}^0 + \int_{-1}^0 \phi - [(1-x)\phi]_0^1 + \int_0^1 (-1)\phi \\
&= -\phi(0) + \int_{-1}^0 \phi + \phi(0) - \int_0^1 \phi \\
&= \int_{\mathbb{R}} (\chi_{[-1,0]} - \chi_{[0,1]})\phi
\end{aligned}$$

(ii)

$$\begin{aligned}
\partial^2 F(\phi) &= -\int_{-1}^0 \partial\phi + \int_0^1 \partial\phi = -\phi(0) + \phi(-1) + \phi(1) - \phi(0) \\
&= \phi(-1) - 2\phi(0) + \phi(1)
\end{aligned}$$

(iii) f and $\chi_{[-1,0]} - \chi_{[0,1]}$ are both $L^1(\mathbb{R})$ functions so $f \in W^{1,1}$. However we know that there is no $L^1(\mathbb{R})$ function g with $F_g = \delta_{-1} - 2\delta_0 + \delta_1$ so $f \notin W^{2,1}$.

(c) Consider u_{e_i}. By assumption, $\nabla u_{e_i} = 0$ and so using Proposition 3.23 we know that $u_{e_i} = a_i$, a constant. Consider the function $v(x) = a \cdot x$. Now we see that $\partial^i(u - v) = a_i - a_i = 0$. So we can again apply Proposition 3.23 to get that $u = v + b = a \cdot x + b$.

(d) It is not continuous because as $x \to 0$ we have $u \to (\infty)^{0.25}$.

$$\|u\|_2^2 = \int_0^{2\pi} \int_0^{0.5} (-\ln r)^{0.5} r \, dr \, d\theta$$

Observe that $r^2 \ln r \to 0$ as $r \to 0$ so the integrand is a continuous function and the integral is therefore finite.

For the derivative, we try to see if the distributional derivative comes from a L^1_{loc} function, and then if that function is L^2.

$$\int_\Omega u\partial_1\phi = \left(\int_{B_\varepsilon} + \int_{\Omega\setminus B_\varepsilon}\right) u\partial_1\phi = \int_{B_\varepsilon} u\partial_1\phi + \int_{\Omega\setminus B_\varepsilon} \nabla \cdot \begin{pmatrix} u\phi \\ 0 \end{pmatrix} - \partial_1 u \, \phi$$
$$= \int_{B_\varepsilon} u\partial_1\phi - \int_{\partial B_\varepsilon} u\phi \frac{x}{r} \, d\sigma - \int_{\Omega\setminus B_\varepsilon} \partial_1 u \, \phi$$

We have three integrals to consider. The first integral vanishes by Hölder's inequality $\|u\partial_1\phi\|_{L^1(B_\varepsilon)} \le \|u\|_{L^2(B_\varepsilon)} \|\partial_1\phi\|_{L^2(B_\varepsilon)} \to 0$. The second integral self-cancels, but would vanish for power reasons even if it didn't:

$$\int_{\partial B_\varepsilon} u\phi \frac{x}{r} \, d\sigma = (-\ln \varepsilon)^{0.25} \varepsilon \int_0^{2\pi} \cos\theta \, d\theta = 0.$$

And the final integral shows us that the distributional derivative of u is associated to the function $\partial_1 u$ for $x \neq 0$, provided this function is locally integrable. Since $\partial_1 u = 0.25(-\ln\|x\|)^{-0.75} x \|x\|^{-1}$ and

$$\|\partial_1 u\|_2^2 \le 0.0625 \times 2\pi \int_0^{0.5} (-\ln r)^{-1.5} r \, dr < \infty$$

we see that $\partial_1 u \in L^2$. The same clearly holds for $\partial_2 u$ as well.

Note that the criterion in Proposition 3.22 is not so useful, because you have to do all the same analysis, including showing that $\partial_1 u \in L^2$. Then you can apply Hölder's inequality to get

$$\left|\partial^1 F(\phi)\right| \le \|\partial_1 u\|_p \, \|\phi\|_q,$$

and set $M = \|\partial_1 u\|_p$.

25. Another "fundamental lemma" for L^1_{loc}-functions

Let $\Omega \subseteq \mathbb{R}^n$ be open and connected. Show that for $u \in L^1_{loc}(\Omega)$ if

$$\int_\Omega u(x) \nabla\phi(x) \, dx = 0 \text{ for all } \phi \in C_0^\infty(\Omega),$$

then u is constant on Ω. [Hint. Modify the proof of Proposition 3.23.]

Solution. Consider the ball $B(x_0, 2\rho) \subset \Omega$. For $\varepsilon < \rho$ the mollification $u_\varepsilon = \lambda_\varepsilon * u$ is a smooth function on $B(x_0, \rho)$ and especially $y \mapsto \lambda_\varepsilon(x-y) \in C_0^\infty(B(x_0, 2\rho)) \subset C_0^\infty(\Omega)$ for all $x \in B(x_0, \rho)$. Therefore

$$\nabla u_\varepsilon(x) = \int_\Omega \nabla \lambda_\varepsilon(x-y) u(y)\, dy = 0,$$

showing that u_ε is a constant for each ε. We know that $u_\varepsilon \to u$ in L^1_{loc} (Proposition 3.18) so u is a constant on $B(x_0, \rho)$. A locally constant functions on a connected set is constant.

The lesson here was that we didn't really need to know that $u \in W^{1,1}_{loc}$ in Proposition 3.18, only that $u \in L^1_{loc}$ and that its distributional derivatives are zero.

26. An integration by parts.

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in W_0^{1,2}(\Omega)$, $v \in W^{1,2}(\Omega)$. *Prove*

$$\int_\Omega u_{e_i} v_{e_j}\, \mathrm{d}\mu = \int_\Omega u_{e_j} v_{e_i}\, \mathrm{d}\mu$$

[Hint. Approximate u with functions from $C_0^\infty(\Omega)$.]

Solution. The choice of $p = 2$ guarantees that the integrals exist with Hölder's inequality. $W_0^{1,2}(\Omega)$ is by definition the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. So let $u_n \to u$ for smooth compactly supported functions. But then

$$\left| \int_\Omega u_{e_i} v_{e_j} - \int_\Omega \partial_i u_n v_{e_j} \right| \le \int_\Omega |u_{e_i} - \partial_i u_n|\, |v_{e_j}| \le \|u_{e_i} - \partial_i u_n\|_2\, \|v_{e_j}\|_2 \to 0.$$

So we can approximate the integral with u_{e_i} by one with $\partial_i u_n$. Since these are test functions and v is Sobolev

$$\int_\Omega \partial_i u_n v_{e_j} = -\int_\Omega \partial_j \partial_i u_n v = \int_\Omega \partial_j u_n v_{e_i}.$$

Putting it all together for clarity

$$\begin{aligned}
\left| \int_\Omega u_{e_i} v_{e_j} - \int_\Omega u_{e_j} v_{e_i} \right| &= \left| \int_\Omega u_{e_i} v_{e_j} - \int_\Omega \partial_i u_n v_{e_j} + \int_\Omega \partial_j u_n v_{e_i} - \int_\Omega u_{e_j} v_{e_i} \right| \\
&\le \left| \int_\Omega u_{e_i} v_{e_j} - \int_\Omega \partial_i u_n v_{e_j} \right| + \left| \int_\Omega \partial_j u_n v_{e_i} - \int_\Omega u_{e_j} v_{e_i} \right| \\
&\to 0 + 0.
\end{aligned}$$