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Partial Differential Equations
Exercise sheet 6

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18. A detail from the proof of Schauder's fixed point theorem.

Optional: Let K and $\tilde{K}$ be bounded, closed, convex subets of $\mathbb{R}^n$ with non-empty interiors. Prove that K and $\tilde{K}$ are homeomorphic.

Solution. Because homeomorphic is a transitive relation, it is enough to prove this for $\tilde{K} = \overline{B(0,1)}$.

We can assume that $0 \in \operatorname{int} K$. Let $v \in \partial B(0,1)$. By the properties of K we know that $\{t \in \mathbb{R}_{\geq 0} \mid tv \in K\}$ is an interval of the form $[0, d]$. Or put differently $\mathbb{R}_{\geq 0} v \cap K = [0, d]v$. The idea is to rescale these rays by d to make K into a sphere.

Let us consider the 'furthest distance' function. Define $d : \partial B(0,1) \to \mathbb{R}_{>0}$ by $d(v) = \max\{t \in \mathbb{R}_{>0} \mid tv \in K\}$. Because K is bounded, d must have an upper bound R. We know that this function is strictly positive because 0 is in the interior of K. Moreover, d must be bounded from below by a positive constant r because otherwise we would have a sequence v_k with $d(v_k) \to 0$. But then the elements $d(v_k)v_k \in \partial K$ converge to 0, which contradicts the fact that 0 is in the interior of K.

We now show that d is continuous. Suppose that d were not continuous. That means there is a sequence $v_n \to v$ in $\partial B(0,1)$ such that $|d(v_k) - d(v)| > C$ for some positive constant C. On the other hand, consider $d(v_k)v_k \in \partial K$. Since ∂K is compact, there is a subsequence converging to $x \in \partial K$. We know that $x \neq 0$ so write $x = d(\hat{x})\hat{x}$. By normalising (a continuous function), we see that $\hat{x} = v$. For this subsequence we have

$$\begin{aligned} \|d(v_k)v_k - d(v)v\| &= \|d(v_k)v_k - d(v_k)v + d(v_k)v - d(v)v\| \\ &\geq \Big| d(v_k) \|v_k - v\| - |d(v_k) - d(v)| \Big| \end{aligned}$$

Because d is bounded, we know that $d(v_k) \|v_k - v\|$ converges to 0. For large k therefore this inequality cannot hold since $|d(v_k) - d(v)| > C$. This is a contradiction.

Now we can define the homeomorphism $\varphi : K \to \overline{B(0,1)}$ by $\varphi(0) = 0$ and $\varphi(x) = d(\hat{x})^{-1}x$. Since $x \mapsto \hat{x}$ is continuous away from $x = 0$ and d is strictly positive, φ is continuous away from $x = 0$. If we have a sequence $x_k \to 0$ then $\|\varphi(x_k)\| \leq r^{-1}\|x_k\|$ shows that $\varphi(x_k) \to 0$. Hence φ is continuous. It has an inverse $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(x) = d(\hat{x})x$, which is also continuous by essentially the same argument.

19. Peano's existence theorem.

In this question we use Schauder' fixed point theorem to prove an existence theorem for ODEs. We will prove: Let $R = \{(x, w) \in \mathbb{R}^2 \mid |x| \leq a, |w| \leq b\}$ be a closed rectangle and $F : R \to \mathbb{R}$ a continuous function. Let c be the maximum of $|F|$. Then for $0 < h \leq \min\{a, b/c\}$ the following ODE has at least one solution $u : (-h, h) \to \mathbb{R}$

$$u' = F(x, u), \qquad u(0) = 0.$$

(a) In Schauder's theorem what conditions must X and G obey? Let $X = C([-h, h])$ and $G = \{u \in X \mid \|u\|_\infty \leq b$. Prove that they have the required conditions.

(b) Consider $T : G \to X$ given by

$$(Tu)(x) = \int_0^x F(y, u(y))\, dy.$$

Why is this a well defined operator on G? Show that $T[G] \subseteq G$. Hence T is actually an operator $G \to G$.

(c) Prove T is continuous. [Hint. F is uniformly continuous.]

(d) Prove T is a compact operator.

[Hint. Arzela-Ascoli theorem: Consider a sequence of continuous functions $u_n : [-h, h] \to \mathbb{R}$. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence that converges in X.]

(e) Finish the proof of Peano's ODE existence theorem.

Solution.

(a) X must be a Banach space and G must be closed and convex. $C([-h, h])$ is a Banach space with the supremum norm. Because the norm is always a continuous function $G = \|\cdot\|_\infty^{-1}[[0, b]]$ shows G is closed. If $u, v \in G$ then for $t \in [0, 1]$

$$\|tu + (1-t)v\|_\infty \leq t\|u\|_\infty + (1-t)\|v\|_\infty \leq tb + (1-t)b = b$$

shows that G is convex.

(b) By the definition of G, $u(y) \in [-b, b]$ so $F(y, u(y))$ is well-defined. The fundamental theorem of calculus gives that Tu is continuous (in fact differentiable).

$$\|Tu\|_\infty \leq \sup_{x \in [-h,h]} \left| \int_0^x |F(y, u(y))|\, dy \right| \leq \sup_{x \in [-h,h]} c|x| \leq ch \leq b.$$

(c) Choose $\varepsilon > 0$. Since F is continuous on the compact set R, there exists $\delta > 0$ such that for all $|w - w'| < \delta$ we have $|F(y, w) - F(y, w')| < \varepsilon/h$. So if $\|u - v\|_\infty < \delta$ then $|F(y, u(y)) - F(y, v(y))| < \varepsilon/h$. It follows

$$\|Tu - Tv\|_\infty = \sup_{x \in [-h,h]} \left| \int_0^x |F(y, u(y)) - F(y, v(y))|\, dy \right| \leq (\varepsilon/h)h = \varepsilon.$$

This shows continuity.

(d) We know that $\overline{T[G]}$ is closed and bounded, but unfortunately that is not enough to establish that is it compact in X, since X is infinite dimensional. We need to prove that every sequence in $\overline{T[G]}$ has a convergent subsequence. By a standard diagonal argument, it is enough to show that this holds for sequences in $T[G]$.

Let (Tu_k) be a sequence in $T[G]$. We want to apply the Arzela-Ascoli theorem. The sequence is uniformly bounded by b since G is. It is uniformly equicontinuous since

$$|Tu_k(x) - Tu_k(x')| \leq \left| \int_{x'}^x |F(y, u_k(y))|\, dy \right| \leq c|x - z|.$$

The Arzela-Ascoli says that (Tu_k) has a convergent subsequence.

(e) To summarise, we have a closed and convex set G and a continuous operator $T : G \to G$ that is compact. Therefore by Schauder's fixed point theorem, there is a $u \in G$ with $u = Tu$. This means

$$u(x) = \int_0^x F(y, u(y))\, dy.$$

Differentiating shows that u obeys the ODE and $u(0) = \int_0^0 = 0$ shows the initial condition.

20. Properties of Hölder continuous functions.

Let $\Omega \subset \mathbb{R}^n$ be open.

(a) Give the definitions for a function u to be α-Hölder continuous and to belong to $C^{0,\alpha}(\Omega)$.

(b) Why is $\text{höl}_{\Omega,\alpha}$ not a norm?

(c) Show a Hölder continuous function is uniformly continuous.

(d) Suppose that $\alpha > 1$. Show that $u \in C^{0,\alpha}(\Omega)$ is differentiable and that $\nabla u \equiv 0$. This shows if Ω is connected and $\alpha > 1$ that $C^{0,\alpha}(\Omega)$ only contains the constant functions. For this reason we only consider $0 < \alpha \leq 1$.

(e) Suppose that $u : [a, b] \to \mathbb{R}$ is continuously differentiable. Show that it is Hölder continuous for all $0 < \alpha \leq 1$.

Solution.

(a) A function u is called α-Hölder continuous if

$$\text{höl}_{\Omega,\alpha}(u) := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha}$$

is finite. u belongs to $C^{0,\alpha}(\Omega)$ if u is continuous, bounded, and α-Hölder continuous.

(b) It is homogeneous and obeys the triangle inequality. But all constant functions have $\text{höl} = 0$. Therefore it is not positive definite.

(c) Choose $\varepsilon > 0$. We know that for all $x, y \in \Omega$ that $|u(x) - u(y)| \leq \text{höl}(u)\|x - y\|^\alpha$. Set $\delta = (\varepsilon/\text{höl}(u))^{1/\alpha}$. Then for all $\|x - y\| < \delta$ we have

$$|u(x) - u(y)| \leq \text{höl}(u)\delta^\alpha = \varepsilon.$$

(d) Choose some point $x \in \Omega$ and consider the i-partial derivative

$$\left|\frac{\partial u}{\partial x_i}\right| = \lim_{h \to 0} \frac{|u(x + he_i) - u(x)|}{|h|} \leq \lim_{h \to 0} \text{höl}(u)|h|^{\alpha - 1} \to 0.$$

This shows that all the partial derivatives of u are zero (in particular, u is differentiable).

(e) We apply the mean value theorem: for any two $x < y \in [a, b]$ there is a $c(x, y) \in [x, y]$ with

$$\left|\frac{u(y) - u(x)}{x - y}\right| \leq |f'(c)| \leq \|f'\|_\infty.$$

Thus u is 1-Hölder continuous. But

$$\text{höl}_1(u)\,|b - a|^{1-\alpha} \geq \frac{|u(x) - u(y)|}{|x - y|}|x - y|^{1-\alpha} = \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

shows that $\text{höl}_\alpha(u) < \infty$ and hence f is also α-Hölder continuous for all $0 < \alpha \leq 1$

21. Hölder-continuous functions on closed sets.

Optional: Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. These exercise considers the relationship between $C^{0,\alpha}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ (the latter is not defined in the script, but it has an obvious definition). Let $0 < \alpha \leq 1$ and $u \in C^{0,\alpha}(\Omega)$.

(a) Give a function $f : \overline{\Omega} \to \mathbb{R}$ that belongs to $C(\Omega)$ but not $C(\overline{\Omega})$, either for general Ω or a particular choice.

(b) Show that there is a unique function $\tilde{u} \in C(\overline{\Omega})$ with $\tilde{u}|_\Omega = u$. [Hint. Use uniform continuity.]

(c) Prove that $\text{höl}_{\overline{\Omega},\alpha}\tilde{u} = \text{höl}_{\Omega,\alpha}u$.

(d) What can you then say about the relationship between $C^{0,\alpha}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$?

Solution.

(a) Put aside the trivial case $\Omega = \mathbb{R}^n$, which is both open and closed. If $a \in \partial\Omega$, then consider $f(x) = \sin\|x - a\|^{-1}$ and $f(a) = b$. This is continuous on Ω, but not on $\overline{\Omega}$ for any value of b.

(b) Let $x \in \partial\Omega$ and let (x_n) be a sequence in Ω converging to x. We will show that $(u(x_n))$ is a Cauchy sequence. Choose any $\varepsilon > 0$. By uniform continuity, there is a $\delta > 0$ such that $|u(x) - u(y)| < \varepsilon$ for all $|x - y| < \delta$. Since (x_n) converges, choose a large N so that $|x_n - x_m| < \delta$ for all $n, m > N$. Thus also $|u(x_n) - u(x_m)| < \varepsilon$.

Define $\tilde{u}(x) = \lim u(x_n)$ for $x \in \partial\Omega$ and $\tilde{u}(x) = u(x)$ otherwise. If $y_n \to x$ is another sequence then for any $\varepsilon > 0$ there is an N with $x_n, y_n \in B(x, \delta/2)$ for all $n > N$ with the δ from the uniform continuity of u. This forces $\|x_n - y_n\| < \delta$ and $|u(x_n) - u(y_n)| < \varepsilon$. This shows $\lim u(x_n) = \lim u(y_n)$ and the definition of $\tilde{u}$ is independent of the choice of sequence. $\tilde{u}$ is continuous on $\overline{\Omega}$ either because of u (for points in Ω) or by a standard diagonal argument (for boundary points).

To prove uniqueness, if v is another continuous extension, then $w = u - v$ is also continuous. On Ω it is zero. For a point on the boundary $w(x) = \lim w(x_n) = \lim 0 = 0$. Thus $u = v$.

(c) By the definition of supremum, $\text{höl}_{\overline{\Omega},\alpha}(\tilde{u}) \geq \text{höl}_{\Omega,\alpha}(u)$. For the converse, take any two points $x \neq y \in \overline{\Omega}$. There are sequences $x_n \to x$ and $y_n \to y$ in Ω. Then

$$\frac{|\tilde{u}(x) - \tilde{u}(y)|}{\|x - y\|^\alpha} = \lim \frac{|u(x_n) - u(y_n)|}{\|x_n - y_n\|^\alpha} \leq \lim \text{höl}_{\Omega,\alpha}(u) = \text{höl}_{\Omega,\alpha}(u)$$

shows $\text{höl}_{\overline{\Omega},\alpha}(\tilde{u}) \leq \text{höl}_{\Omega,\alpha}(u)$.

(d) We have shown that any Hölder continuous function on Ω extends to a Hölder continuous function on $\overline{\Omega}$ with the same Hölder constant. The natural definition for $C^{k,\alpha}(\overline{\Omega})$ is the subset of $C^k(\overline{\Omega})$ such that the function and derivatives up to kth-order are bounded and α-Hölder continuous. But then $C^{k,\alpha}(\overline{\Omega}) = C^{k,\alpha}(\Omega)$. It is for this reason that in the script we only define Hölder spaces on open sets.

22. Examples of Hölder continuous functions.

(a) For $0 < b \leq 1$ define $f_b : (0,1) \to \mathbb{R}$ by $x \mapsto x^b$. To which Hölder spaces does f_b belong? Compute its Hölder constants höl_α.
[Hint. Consider the function $h(z) = (1 - z^b)(1 - z)^{-\alpha}$.]

(b) Now define $g_b : (0, \infty) \to \mathbb{R}$ by $x \mapsto x^b$. To which Hölder spaces does g_b belong? Compute its Hölder constants höl_α.

(c) Define $h : [0, 0.5] \to \mathbb{R}$ by $h(0) = 0$ and $h(x) = (\ln x)^{-1}$ otherwise. Show that this function is continuous but not Hölder continuous. Can you explain why?

(d) Explain parts (a) and (b) with respect to Proposition 3.13.

Solution.

(a) We begin with computing the constants. Consider the function $H : [0,1]^2 \setminus \{x = y\} \to \mathbb{R}$ with

$$H(x,y) := \frac{|x^b - y^b|}{|x - y|^\alpha}.$$

This function is symmetrical, so it is enough to consider $y > x$. We write

$$H(x,y) = y^{b-\alpha} \frac{1 - (x/y)^b}{(1 - x/y)^\alpha}$$

Consider the function $h(z) = (1 - z^b)(1 - z)^{-\alpha}$ for $z \in [0,1]$. For $z \to 1$ we have

$$\lim h(z) = \lim \frac{-bz^{b-1}}{\alpha(1 - z)^{\alpha - 1}} = \lim \frac{-b}{\alpha} \frac{(1 - z)^{1-\alpha}}{z^{1-b}} = 0$$

so this function is continuous. If we look for turning points

$$h'(z) = \frac{-bz^{b-1}(1 - z) + \alpha(1 - z^b)}{(1 - z)^{\alpha+1}}$$

we find there are none. Hence $0 = h(1) \leq h(z) \leq h(0) = 1$. We see now that

$$\text{höl}_\alpha f_\beta = \sup H(x, y) = \sup_{y \in (0,1)} y^{b-\alpha}.$$

For $\alpha \leq b$ this is 1. For $\alpha > b$ this is ∞.

f_b is always continuous and bounded. Therefore it belongs to $C^{0,\alpha}((0, 1))$ whenever $\alpha > b$.

(b) The same calculation as in the previous part shows that

$$\text{höl}_\alpha g_b = \sup_{y \in (0,\infty)} y^{b-\alpha}.$$

This time the Hölder constant is only finite for $\alpha = b$, in which case it is 1.

It belongs to no Hölder spaces $C^{0,\alpha}((0, \infty))$ however because it is not bounded.

(c)

$$\lim_{x \to 0} h(x) = 1/\infty = 0.$$

It is not Hölder continuous because for $x = 0$

$$\frac{-(\ln y)^{-1}}{y^\alpha} = -\frac{1}{y^\alpha \ln y} \to \frac{1}{0} = \infty$$

as $y \to 0$.

(d) In part (a), the domain $\Omega = (a, b)$ is bounded. Therefore the fact that $f_b \in C^{0,b}(\Omega)$ implies that it also belongs to $C^{0,\alpha}(\Omega)$ for $\alpha < b$. The theorem does not apply to part (b) because $(0, \infty)$ is not bounded.