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Partial Differential Equations
Exercise sheet 6

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18. A detail from the proof of Schauder's fixed point theorem.

Optional: Let K and $\tilde{K}$ be bounded, closed, convex subets of $\mathbb{R}^n$ with non-empty interiors. Prove that K and $\tilde{K}$ are homeomorphic.

19. Peano's existence theorem.

In this question we use Schauder' fixed point theorem to prove an existence theorem for ODEs. We will prove: Let $R = \{(x, w) \in \mathbb{R}^2 \mid |x| \le a, |w| \le b\}$ be a closed rectangle and $F : R \to \mathbb{R}$ a continuous function. Let c be the maximum of $|F|$. Then for $0 < h \le \min\{a, b/c\}$ the following ODE has at least one solution $u : (-h, h) \to \mathbb{R}$

$$u' = F(x, u), \qquad u(0) = 0.$$

(a) In Schauder's theorem what conditions must X and G obey? Let $X = C([-h, h])$ and $G = \{u \in X \mid \|u\|_\infty \le b$. Prove that they have the required conditions.

(b) Consider $T : G \to X$ given by

$$(Tu)(x) = \int_0^x F(y, u(y))\, dy.$$

Why is this a well defined operator on G? Show that $T[G] \subseteq G$. Hence T is actually an operator $G \to G$.

(c) Prove T is continuous. [Hint. F is uniformly continuous.]

(d) Prove T is a compact operator.
[Hint. Arzela-Ascoli theorem: Consider a sequence of continuous functions $u_n : [-h, h] \to \mathbb{R}$. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence that converges in X.]

(e) Finish the proof of Peano's ODE existence theorem.

20. Properties of Hölder continuous functions.

Let $\Omega \subset \mathbb{R}^n$ be open.

(a) Give the definitions for a function u to be α-Hölder continuous and to belong to $C^{0,\alpha}(\Omega)$.

(b) Why is $\text{höl}_{\Omega,\alpha}$ not a norm?

(c) Show a Hölder continuous function is uniformly continuous.

(d) Suppose that $\alpha > 1$. Show that $u \in C^{0,\alpha}(\Omega)$ is differentiable and that $\nabla u \equiv 0$. This shows if Ω is connected and $\alpha > 1$ that $C^{0,\alpha}(\Omega)$ only contains the constant functions. For this reason we only consider $0 < \alpha \le 1$.

(e) Suppose that $u : [a, b] \to \mathbb{R}$ is continuously differentiable. Show that it is Hölder continuous for all $0 < \alpha \le 1$.

21. Hölder-continuous functions on closed sets.

Optional: Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. These exercise considers the relationship between $C^{0,\alpha}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ (the latter is not defined in the script, but it has an obvious definition). Let $0 < \alpha \leq 1$ and $u \in C^{0,\alpha}(\Omega)$.

(a) Give a function $f : \overline{\Omega} \to \mathbb{R}$ that belongs to $C(\Omega)$ but not $C(\overline{\Omega})$, either for general Ω or a particular choice.

(b) Show that there is a unique function $\tilde{u} \in C(\overline{\Omega})$ with $\tilde{u}|_\Omega = u$. [Hint. Use uniform continuity.]

(c) Prove that $\text{höl}_{\overline{\Omega},\alpha}\tilde{u} = \text{höl}_{\Omega,\alpha}u$.

(d) What can you then say about the relationship between $C^{0,\alpha}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$?

22. Examples of Hölder continuous functions.

(a) For $0 < b \leq 1$ define $f_b : (0, 1) \to \mathbb{R}$ by $x \mapsto x^b$. To which Hölder spaces does f_b belong? Compute its Hölder constants höl_α.
[Hint. Consider the function $h(z) = (1 - z^b)(1 - z)^{-\alpha}$.]

(b) Now define $g_b : (0, \infty) \to \mathbb{R}$ by $x \mapsto x^b$. To which Hölder spaces does g_b belong? Compute its Hölder constants höl_α.

(c) Define $h : [0, 0.5] \to \mathbb{R}$ by $h(0) = 0$ and $h(x) = (\ln x)^{-1}$ otherwise. Show that this function is continuous but not Hölder continuous. Can you explain why?

(d) Explain parts (a) and (b) with respect to Proposition 3.13.