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Partial Differential Equations
Exercise sheet 4

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12. Spherical Means of Distributions.

The purpose of this question is to provide some context into the definition of the weak mean value property and Weyl's lemma. We will essentially prove the *co-area formula.*

Let $\Psi : U \times [-T, T] \subset \mathbb{R}^n \to O \subset \mathbb{R}^n$ be a diffeomorphism. This is a smooth invertible function whose inverse function is also smooth. In particular, for each t we know that $u \mapsto \Psi(u, t)$ is an $(n-1)$-dimensional submanifold. Denote these by $Y_t := \Psi[U \times \{t\}]$. Suppose further that $\partial_{u_i}\Psi \cdot \partial_t \Psi = 0$ for $i = 1, \ldots, n-1$ and $\|\partial_t \Psi\| = 1$.

(a) Check that spherical coordinates obey the assumptions on Ψ.

(b) Optional: Suppose we have vectors such that $b \cdot a_i = 0$ for $i = 1, \ldots, n-1$ and $\|b\| = 1$. Show that

$$|\det(a_1, \ldots, a_{n-1}, b)|^2 = |\det(a_1, \ldots, a_{n-1})^T (a_1, \ldots, a_{n-1})|.$$

Hint: Use the Gram matrix. Geometrically this is clear: the right hand side is the n-volume of a unit length right-prism and the left hand side is the $(n-1)$-volume of its cross-section.

(c) Argue that

$$\int_O f \, d\mu = \int_{U \times [-T,T]} f \circ \Psi \, |\det \Psi'| \, du \, dt = \int_{[-T,T]} \left(\int_{Y_t} f \, d\sigma \right) dt$$

(d) Consider the 'generalised mollifier' $\chi_\varepsilon : O \to \mathbb{R}$ defined by $\chi_\varepsilon(x) = \phi_\varepsilon(t(x))$ where ϕ_ε is a mollifier on $\mathbb{R}$. Complete the argument to show that

$$\lim_{\varepsilon \to 0} \int_O f \, \chi_\varepsilon \, d\mu = \int_{Y_0} f \, d\sigma.$$

This tempts us to define the integral of F on Y_0 to be $\lim_{\varepsilon \to 0} F(\chi_\varepsilon)$ when this exists. This is similar to how we cannot always find the 'value' of a distribution at a point, but when we can it is $\lim_{\varepsilon \to 0} F(\phi_\varepsilon)$.

(e) Let us denote the functions f and g_x in the weak mean value property and Weyl's lemma by a common notation

$$g_{x,\psi}(y) := \frac{\psi(|x-y|)}{n\omega_n |x-y|^{n-1}}.$$

Use the above results to interpret the expression $U(g_{x,\psi})$. What is the significance of $\int \psi = 0$ compared with $\int_\psi = 1$?

Solution.

(a) For spherical coordinates, you need to exclude the half plane where longitude is zero. But this doesn't change the values of any of the integrals.

(b) The insight here is that the Gramm-Schmidt process (without normalising the vectors) does not change the volume of the parallelepiped.

(c) The first equality is just applying change of variables, since $O = \Psi[U \times [-T, T]]$. We can then apply Fubini's theorem to change it into nested integrals. Let $\Phi_t(u) = \Psi(u, t)$ be a parameterisation of Y_t. Then by the previous part

$$|\det \Psi'| = |\det(\partial_{u_1}\Psi, \ldots, \partial_{u_{n-1}}\Psi, \partial_t \Psi)| = \sqrt{|\det {\Phi'_t}^T \Phi'_t|}$$

since $\Phi'_t = (\partial_{u_1}\Psi, \ldots, \partial_{u_{n-1}}\Psi)$. Finally we can see

$$\int_U f \circ \Psi \, |\det \Psi'| \, du = \int_U f \circ \Phi_t \sqrt{|\det {\Phi'_t}^T \Phi'_t|} \, du = \int_{Y_t} f \, d\sigma$$

(d) Because of its definition with $t(x)$ we can simplify $\chi_\varepsilon \circ \Psi(u, t) = \phi_\varepsilon(t)$. This is a constant on each submanifold Y_t and so can be brought outside the inner integral

$$\int_O f \, \chi_\varepsilon \, d\mu = \int_{[-T,T]} \phi_\varepsilon(t) \left(\int_{Y_t} f \, d\sigma \right) dt$$

What we have now is an integral of the form $\int_{\mathbb{R}} \phi_\varepsilon(t) I(t) \, dt$ for a mollifier ϕ_ε. We know in the limit as $\varepsilon \to 0$ that this integral tends to $I(0)$. Therefore

$$\lim_{\varepsilon \to 0} \int_O f \, \chi_\varepsilon \, d\mu = I(0) = \int_{Y_0} f \, d\sigma$$

(e) Suppose $U = F_u$ is a distribution induced by a function u. Using spherical coordinates Ψ centred on x and modified so that $t = \|x\| - r$ for a constant $r > 0$, we see that $\psi(|y - x|) = \phi_\varepsilon(|x - y| - r)$ is a generalised mollifier converging to $\partial B(x, r) = Y_0$.

$$\lim_{\varepsilon \to 0} U(g_{x,\psi}) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{u(y)}{n\omega_n |x - y|^{n-1}} \phi_\varepsilon(|x - y| - r) \, dy = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y) \, d\sigma(y).$$

In other words, this is exactly the spherical mean of U.

If U has the weak mean value property then let $\psi(t) = \phi_\varepsilon(t - r) - \phi_\varepsilon(t - R)$. This ensures that $\int \psi = 1 - 1 = 0$. Taking the limit as $\varepsilon \to 0$ we get

$$0 = U(g_{x,\psi}) = M(u, x, r) - M(u, x, R).$$

This shows that u has the mean value property. We have already seen in the script (bottom of page 22) that if u has the mean value property then U has the weak mean value property. In summary, for distributions that come from functions, the mean value and weak mean value properties are equivalent.

The advantage of the defining the weak mean value property as in the script is that you avoid limits (instead it has to hold for a family): $U(g_{x,\psi})$ is well-defined for all distributions, but the limit does not always exist.

In the proof of Weyl's lemma, for a harmonic distribution U (so it has the weak mean value property) we have the definition $u(x) = U(g_{x,\psi})$ for any test function ψ with $\int \psi = 1$. It is claimed that this does not depend on the choice of ψ. Indeed if χ is another choice then

$$U(g_{x,\psi}) - U(g_{x,\chi}) = U(g_{x,\psi-\chi}) = 0$$

because $\int \psi - \chi = 1 - 1 = 0$. We could choose $\psi(t) = \psi_\varepsilon(t-r)$ if we like, and therefore we see that $u(x)$ is defined to be the 'spherical mean' centred at x of the distribution U.

We see that the condition $\int \psi = 1$ corresponds to taking a spherical mean and $\int \psi = 0$ corresponds to taking the difference of two spherical means.

13. A detail in the proof of the Poisson Representation Formula (Poissonschen Darstellungsformel).

We denote by $K(x, y)$ the Poisson kernel as in Section 2.3 of the lecture notes. This has the following properties (do *not* prove these properties again, refer to the lecture notes):

(i) $K(x, y) > 0$ for $x \in B(0, 1),\ y \in \partial B(0, 1)$.

(ii) $\int_{\partial B(0,1)} K(x, y) \mathrm{d}\sigma(y) = 1$ for $x \in B(0, 1)$.

(iii) For all $x_0 \in \partial B(0, 1)$, in the limit $x \to x_0$, $x \in B(0, 1)$ the map $y \mapsto K(x, y)$ converges uniformly to 0 with respect to y on compact subsets of $\partial B(0, 1) \setminus \{x_0\}$.

Let a continuous function $u \in C(\partial B(0, 1))$ be given. We define

$$\tilde{u} : B(0, 1) \to \mathbb{R},\ x \mapsto \int_{\partial B(0,1)} K(x, y)u(y)\mathrm{d}\sigma(y)\ . \qquad (*)$$

Show that the function $\tilde{u}$ can be extended continuously to the boundary $\partial B(0, 1)$ and that the extension on $\partial B(0, 1)$ agrees with u.

[Hint: For any given $x_0 \in \partial B(0, 1)$ consider $x \in B(0, 1)$ in a neighbourhood of x_0 and break the integral $(*)$ into a piece close to x_0 and the "rest". Use the properties of K given above to show that the "rest" is well behaved with respect to the limit and goes to zero. For the part close to x_0 use the continuity of u to approximate the function values of $u(y)$ and $u(x_0)$.]

Solution. Choose a boundary point $x_0 \in \partial B(0, 1)$. We must show that

$$\lim_{x \to x_0} \tilde{u}(x) = u(x_0) = \int_{\partial B(0,1)} K(x, y)u(x_0)\ d\sigma(y),$$

(using property (ii)), which is equivalent to

$$\lim_{x \to x_0} \int_{\partial B(0,1)} K(x, y)[u(y) - u(x_0)]\ d\sigma(y) = 0.$$

Let $B = B(0, 1)$ and $B_\delta = B(x_0, \delta)$ for some $\delta > 0$. Suppose that $x \in B \cap B_\delta$. As suggested in the hint, we split the integral $\partial B = (\partial B \cap B_\delta) \cup (\partial B \setminus B_\delta)$. The second component $\partial B \setminus B_\delta$ is a compact subset of $\partial B \setminus \{x_0\}$, so the uniform convergence allows us to bring the limit inside the integral

$$\lim_{x \to x_0} \int_{\partial B \setminus B_\delta} K(x, y)[u(y) - u(x_0)]\ d\sigma(y) = \int_{\partial B \setminus B_\delta} K(x_0, y)[u(y) - u(x_0)]\ d\sigma(y) = 0,$$

since $K(x_0, y) = 0$ by the third property. By continuity of $u \in C(\overline{B})$, for all $\varepsilon > 0$ there is a $\delta > 0$ so that $|u(y) - u(x_0)| < \varepsilon$ for all $y \in \partial B \cap B_\delta$. This implies

$$\left| \int_{\partial B \cap B_\delta} K(x,y)[u(y) - u(x_0)] \, d\sigma(y) \right| \leq \varepsilon \int_{\partial B \cap B_\delta} K(x,y) \, d\sigma(y) \leq \varepsilon,$$

using (i) and (ii). Together these two estimates imply that

$$\lim_{x \to x_0} \left| \int_{\partial B(0,1)} K(x,y)[u(y) - u(x_0)] \, d\sigma(y) \right| \leq \varepsilon + 0$$

for any given $\varepsilon > 0$. This is only possible if the limit is zero.

14. A detail in the proof of the Weak Maximum Principle.

Let H be a real $n \times n$-matrix with

$$H = H^t \quad \text{and} \quad x^t H x \leq 0 \ \ \forall x.$$

We will show that there is a matrix D such that $H = -D \cdot D^t$.

(a) Optional: Show that the eigenvalues of a real symmetric $n \times n$ matrix are real.

(b) Consider the map $f : \partial B(0,1) \to \mathbb{R}$ defined by $x \mapsto x^T H x$. Let v be a maximum point of f. Show that $Hv = f(v)v$. [Hint. Consider a path $\alpha : (-\varepsilon, \varepsilon) \to \partial B(0,1)$ with $\alpha(0) = v$.]

(c) Suppose that v is an eigenvector of H. Let $v^\perp := \{x \in \mathbb{R}^n \mid x \cdot v = 0\}$ be the orthogonal complement. Show that $Hv^\perp \subset v^\perp$.

(d) Prove inductively that there exists a matrix O and real numbers λ_i such that $H = O \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\, O^T$.

(e) Finally, show there is a matrix D with $H = -D\, D^T$.

Solution.

(a) Since H is real and symmetric, $\bar{H}^T = H$. Suppose we have a complex eigenvalue λ. For complex vectors $\|v\|^2 = \bar{v}^T v$ with matrix multiplication.

$$\lambda \|v\| = \bar{v}^T (\lambda v) = \bar{v}^T (Hv) = \overline{(\bar{H}^T v)}^T v = \overline{(Hv)}^T v = \overline{(\lambda v)}^T v = \bar{\lambda} \|v\|^2.$$

This shows that $\lambda = \bar{\lambda}$.

(b) We will prove something slightly more general. As in the hint, consider a path α through a point $v \in \partial B(0,1)$. Then $f \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a function of one variable. If $(f \circ \alpha)'(0) = 0$ for all paths α through v, we say that v is a critical point of f. This generalises the notion of critical points to submanifolds. Clearly a local maximum or local minimum is a critical point.

Let v be a critical point of f. By the chain rule $\nabla f(v) \cdot \alpha'(0) = 0$. If w is any unit length vector that is perpendicular to v, then $\alpha(t) = \cos tv + \sin tw$ is a path in $\partial B(0,1)$ with $\alpha(0) = v$ and $\alpha'(0) = w$. This shows us that $\nabla f(v)$ must be parallel to v at a critical point.

On the other hand

$$\partial_i f = \partial_i \sum_{j,k} x_j H_{jk} x_k = \sum_{j,k} \partial_i x_j H_{jk} x_k + \sum_{j,k} x_j H_{jk} \partial_i x_k = \sum_k H_{ik} x_k + \sum_j x_j H_{ji} = (2Hx)_i,$$

since H is symmetric. At a critical point $2Hv = \nabla f(v) = cv$. This shows that v is an eigenvector. If we multiply this equation by v^T from the left, we further have $2f(v) = c\|v\|^2 = c$. This gives the desired result $Hv = f(v)v$.

(c) Let $w \in v^\perp$. Then $v^T(Hw) = (Hv)^T w = \lambda v^T w = 0$. This shows us that $Hw \in v^\perp$ too.

(d) Because f is continuous and $\partial B(0,1)$ is compact, we know that f has a maximum v_1. We know from part (b) that v_1 is an eigenvector with eigenvalue $\lambda_1 = f(v_1)$. Identify $v_1^\perp$ with $\mathbb{R}^{n-1}$ via an orthonormal basis. We know from part (c) that H restricts to give a linear operator on this subspace, and it will again be symmetric. We repeat the steps to get another eigenvector v_2. Inductively this gives use an orthonormal basis of eigenvectors v_i with eigenvalues $\lambda_i = f(v_i)$. The matrix O has these orthonormal vectors as its columns.

(e) By assumption $f(x) \leq 0$ so all the eigenvalues are non-positive. $D = O\,\mathrm{diag}(\sqrt{|\lambda_1|}, \ldots, \sqrt{|\lambda_n|})$.