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Partial Differential Equations
Exercise sheet 4

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12. Spherical Means of Distributions.

The purpose of this question is to provide some context into the definition of the weak mean value property and Weyl's lemma. We will essentially prove the *co-area formula*.

Let $\Psi : U \times [-T, T] \subset \mathbb{R}^n \to O \subset \mathbb{R}^n$ be a diffeomorphism. This is a smooth invertible function whose inverse function is also smooth. In particular, for each t we know that $u \mapsto \Psi(u, t)$ is an $(n-1)$-dimensional submanifold. Denote these by $Y_t := \Psi[U \times \{t\}]$. Suppose further that $\partial_{u_i}\Psi \cdot \partial_t \Psi = 0$ for $i = 1, \ldots, n-1$ and $\|\partial_t \Psi\| = 1$.

(a) Check that spherical coordinates obey the assumptions on Ψ.

(b) Optional: Suppose we have vectors such that $b \cdot a_i = 0$ for $i = 1, \ldots, n-1$ and $\|b\| = 1$. Show that

$$|\det(a_1, \ldots, a_{n-1}, b)|^2 = |\det(a_1, \ldots, a_{n-1})^T (a_1, \ldots, a_{n-1})|.$$

Hint: Use the Gram matrix. Geometrically this is clear: the right hand side is the n-volume of a unit length right-prism and the left hand side is the $(n-1)$-volume of its cross-section.

(c) Argue that

$$\int_O f \, d\mu = \int_{U\times[-T,T]} f \circ \Psi \, |\det \Psi'| \, du \, dt = \int_{[-T,T]} \left(\int_{Y_t} f \, d\sigma \right) dt$$

(d) Consider the 'generalised mollifier' $\chi_\varepsilon : O \to \mathbb{R}$ defined by $\chi_\varepsilon(x) = \phi_\varepsilon(t(x))$ where ϕ_ε is a mollifier on $\mathbb{R}$. Complete the argument to show that

$$\lim_{\varepsilon\to 0} \int_O f \, \chi_\varepsilon \, d\mu = \int_{Y_0} f \, d\sigma.$$

This tempts us to define the integral of F on Y_0 to be $\lim_{\varepsilon\to0} F(\chi_\varepsilon)$ when this exists. This is similar to how we cannot always find the 'value' of a distribution at a point, but when we can it is $\lim_{\varepsilon\to0} F(\phi_\varepsilon)$.

(e) Let us denote the functions f and g_x in the weak mean value property and Weyl's lemma by a common notation

$$g_{x,\psi}(y) := \frac{\psi(|x-y|)}{n\omega_n |x-y|^{n-1}}.$$

Use the above results to interpret the expression $U(g_{x,\psi})$. What is the significance of $\int \psi = 0$ compared with $\int_\psi = 1$?

13. A detail in the proof of the Poisson Representation Formula (Poissonschen Darstellungsformel).

We denote by $K(x, y)$ the Poisson kernel as in Section 2.3 of the lecture notes. This has the following properties (do *not* prove these properties again, refer to the lecture notes):

(i) $K(x, y) > 0$ for $x \in B(0, 1)$, $y \in \partial B(0, 1)$.

(ii) $\int_{\partial B(0,1)} K(x,y)\mathrm{d}\sigma(y) = 1$ for $x \in B(0,1)$.

(iii) For all $x_0 \in \partial B(0,1)$, in the limit $x \to x_0$, $x \in B(0,1)$ the map $y \mapsto K(x,y)$ converges uniformly to 0 with respect to y on compact subsets of $\partial B(0,1) \setminus \{x_0\}$.

Let a continuous function $u \in C(\partial B(0,1))$ be given. We define

$$\tilde{u} : B(0,1) \to \mathbb{R},\ x \mapsto \int_{\partial B(0,1)} K(x,y)u(y)\mathrm{d}\sigma(y)\ . \qquad (*)$$

Show that the function $\tilde{u}$ can be extended continuously to the boundary $\partial B(0,1)$ and that the extension on $\partial B(0,1)$ agrees with u.

[Hint: For any given $x_0 \in \partial B(0,1)$ consider $x \in B(0,1)$ in a neighbourhood of x_0 and break the integral $(*)$ into a piece close to x_0 and the "rest". Use the properties of K given above to show that the "rest" is well behaved with respect to the limit and goes to zero. For the part close to x_0 use the continuity of u to approximate the function values of $u(y)$ and $u(x_0)$.]

14. A detail in the proof of the Weak Maximum Principle.

Let H be a real $n \times n$-matrix with

$$H = H^t \quad \text{and} \quad x^t H x \leq 0 \ \ \forall x.$$

We will show that there is a matrix D such that $H = -D \cdot D^t$.

(a) Optional: Show that the eigenvalues of a real symmetric $n \times n$ matrix are real.

(b) Consider the map $f : \partial B(0,1) \to \mathbb{R}$ defined by $x \mapsto x^T H x$. Let v be a maximum point of f. Show that $Hv = f(v)v$. [Hint. Consider a path $\alpha : (-\varepsilon, \varepsilon) \to \partial B(0,1)$ with $\alpha(0) = v$.]

(c) Suppose that v is an eigenvector of H. Let $v^\perp := \{x \in \mathbb{R}^n \mid x \cdot v = 0\}$ be the orthogonal complement. Show that $Hv^\perp \subset v^\perp$.

(d) Prove inductively that there exists a matrix O and real numbers λ_i such that $H = O\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,O^T$.

(e) Finally, show there is a matrix D with $H = -D\,D^T$.