

6. Extension of Continuous Linear Operators. Let X be a normed vector space and \bar{X} its completion. Let Y be a complete normed vector space Y . Suppose that $L : X \rightarrow Y$ is a continuous linear operator. This means that there is a constant C such that $\|Lx\| \leq C\|x\|$ for all $x \in X$. Show that there is a unique continuous linear operator $\bar{L} : \bar{X} \rightarrow Y$ extending L (ie $\bar{L}x = Lx$ for all $x \in X$). (3 Points)

Solution. Every element of $x \in \bar{X}$ is the limit of a sequence $x_n \in X$. Since we want \bar{L} to be continuous, we are forced to define $\bar{L}x := \lim Lx_n$. We see that Lx_n is a Cauchy sequence

$$\|Lx_n - Lx_m\| = \|L(x_n - x_m)\| \leq C\|x_n - x_m\| \rightarrow 0$$

and therefore it converges in Y , so this definition produces a value. If x'_n is another sequence converging to x , then $L(x_n - x'_n)$ is a null sequence, so this other sequence gives the same value. Hence this definition of \bar{L} is well-defined.

It's easy to see it's linear. It's trivial that it's an extension of L . Since the norm is continuous

$$\|\bar{L}x\| = \lim \|Lx_n\| \leq \lim C\|x_n\| = C\|x\|.$$

7. Distributions I.

(a) Show directly from Definition 2.6 that the Heaviside distribution

$$H : C_0^\infty(\mathbb{R}) \rightarrow \mathbb{R}, \phi \mapsto \int_0^\infty \phi(x) dx$$

is a distribution on \mathbb{R} .

(2 Points)

(b) By the definition of the derivative of a distribution

$$\partial H(\phi) = -H(\partial\phi) = -\int_0^\infty \phi'(x) dx.$$

Simplify this expression in order to give a description of ∂H . (∂ here is the derivative in one-dimension. It seems weird to use an index.) (3 Points)

(c) What is the support of ∂H (in the sense of distributions)? Why does this show that there is no function $f \in L^1_{loc}(\mathbb{R})$ with $\partial H = F_f$? (3 Points)

(d) Consider a function $f \in C_0^\infty(\mathbb{R}^n)$. Show that $\partial_i(F_f) = F_{\partial_i f}$. What is the connection to Exercise 6?

(2 Points)

Solution.

- (a) The integral is always finite because ϕ has compact support. The integral is a linear operator. It only remains to show that H is continuous with respect to the semi-norms. Choose any compact set K and test function ϕ with support in K . Then

$$|H(\phi)| \leq \int_0^\infty |\phi| \leq \int_{\mathbb{R}} |\phi| \leq \mu(K) \|\phi\|_\infty = \mu(K) \|\phi\|_{K,0}$$

- (b) If we apply integration by parts we get $-\phi(\infty) + \phi(0)$. Since ϕ has compact support, the first term is zero. Therefore $H(\phi) = \phi(0)$. This is the delta distribution.
- (c) By definition, $(\text{supp } \delta)^c$ is the union of all open sets U such that $\delta(\phi) = 0$ for every test function ϕ with support in U . If ϕ is supported in $\mathbb{R} \setminus \{0\}$ then $\delta(\phi) = \phi(0) = 0$. This shows that $\mathbb{R} \setminus \{0\} \subseteq (\text{supp } \delta)^c$. Conversely, if $0 \in U$ then there is an $\varepsilon > 0$ such that the bump function $\phi_{B(0,\varepsilon)}$ has support in U . Because $\phi_{B(0,\varepsilon)}(0) \neq 0$ we conclude that this U is not part of the union that comprises $(\text{supp } \delta)^c$ and thus $0 \notin (\text{supp } \delta)^c$. We have shown that $(\text{supp } \delta)^c = \mathbb{R} \setminus \{0\}$.

$\text{supp } \delta = \{0\}$. Therefore, if $\delta = F_f$ then $\text{supp } f = \{0\}$. But if a function is supported only at one point, then it is equivalent to 0 in the sense of L^1 functions. Since the delta distribution is not the zero distribution, it can not be from a function.

- (d)

$$\partial_i(F_f)(\phi) = -F_f(\partial_i\phi) = -\int_{\mathbb{R}^n} f\partial_i\phi = \int_{\mathbb{R}^n} \partial_i f\phi = F_{\partial_i f}(\phi).$$

8. On Convolutions.

- (a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. (2 Points)
- (b) Show that the convolution of C_0^∞ -functions on \mathbb{R}^n is a bilinear, commutative, and associative operation. (1+2 Points + 2 Bonus Points)
- (c) Denote a constant function on \mathbb{R} by 1. The Heaviside function $H : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $H(x) := 1$ for $x \geq 0$ and $H(x) := 0$ for $x < 0$. The derivative of the Dirac distribution δ' acts by $\delta'(\phi) = -\phi'(0)$. Let $\phi \in C_0^\infty(\mathbb{R})$ be a test function.
- (i) Consider the distribution $\phi * P\delta'$. Which result from the script tells us that this distribution comes from a smooth function, even though δ' does not? (1 Point)
- (ii) Prove that $\phi * P\delta' = F_{-\phi}$. (3 Points)
- (iii) Thereby show that $\delta' * H = \delta$ and $1 * \delta' = 0$. (2 Points)
- (iv) Complete the calculation of both $1 * (\delta' * H)$ and $(1 * \delta') * H$ in the sense of distributions and see that they are not equal. This shows that the convolution of distributions with non-compact support (on \mathbb{R}) is not necessarily associative, even when it is well-defined. (1 Point)

Solution.

(a)

$$\begin{aligned} f * f(x) &= \int_{\mathbb{R}} f(x-y)f(y) dy = \int_{-1}^1 f(x-y) dy = \int_{x+1}^{x-1} f(z) (-dz) = \int_{x-1}^{x+1} f(z) dz \\ &= \int_{[x-1, x+1] \cap [-1, 1]} 1 dz = \text{len}([x-1, x+1] \cap [-1, 1]). \end{aligned}$$

(b) Commutativity follows from the substitution $z = x - y$. So it is enough to check bilinearity in the first argument, which is clear. The tricky thing is associativity. The easiest way is to try to reduce both integrals to a common form:

$$\begin{aligned} (f * g) * h(x) &= \int (f * g)(x-z) h(z) dz = \int \left(\int f(x-z-y)g(y) dy \right) h(z) dz \\ &= \iint f(x-y-z)g(y)h(z) dy dz, \\ f * (g * h)(x) &= \int f(x-y) g * h(y) dy = \int f(x-y) \left(\int g(y-z)h(z) dz \right) dy \\ &= \iint f(x-y)g(y-z)h(z) dz dy \quad \text{let } y = w + z, z = z \\ &= \iint f(x-w-z)g(w)h(z) dz dw. \end{aligned}$$

Ask yourself, where have we used the assumption that the functions are compactly supported?

- (c) (i) Lemma 1.11: The convolution of a distribution and a smooth function with compact support is a smooth function.
- (ii) Let ψ be a test function. By definition of convolution of distributions $\phi * P\delta'(\psi) = P\delta'(P\phi * \psi)$. We should therefore investigate the argument.

$$P\phi * \psi(x) = \int P\phi(x-y)\psi(y) dy = \int \phi(y-x)\psi(y) dy.$$

Call this function $j(x)$ for clarity. Now we must act $P\delta'$ on this. We first unwind the definition of P : $P\delta'(j) = \delta'(Pj)$. We see

$$\begin{aligned} Pj(x) &= j(-x) = \int \phi(y+x)\psi(y) dy \\ \delta'(Pj) &= - \left. \frac{d}{dx} \right|_{x=0} \int \phi(y+x)\psi(y) dy = \int [-\phi'(y)]\psi(y) dy. \end{aligned}$$

In other words

$$\phi * P\delta' : \psi \mapsto \int [-\phi'(y)]\psi(y) dy.$$

Therefore we see that $\phi * P\delta'$ acts on a test function ψ by integrating it along with $-\phi'$. Hence this distribution comes from $-\phi'$.

(iii)

$$\begin{aligned}\delta' * H(\phi) &= H(\phi * P\delta') = H(-\phi') = \int_0^\infty -\phi'(y) dy \\ &= \phi(0).\end{aligned}$$

$$\begin{aligned}1 * \delta'(\phi) &= 1(\phi * P\delta') = 1(-\phi') = \int_{-\infty}^\infty -\phi'(y) dy \\ &= 0.\end{aligned}$$

(iv) $1 * (\delta' * H) = 1 * \delta = 1$ and $(1 * \delta') * H = 0 * H = 0$.

9. Distributions II.

(a) Show that

$$V(\phi) = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx$$

is a distribution on \mathbb{R} . Hint: Split the integral into $[0, 1]$ and $[1, \infty]$ and use the mean value theorem. (2 Bonus Points)

(b) What is the relation of V to x^{-1} ? (1 Bonus Point)

(c) Show that the function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto \|x\|^k$ is a locally integrable function for $k > -n$. (2 Bonus Points)

(d) Let $n = 3$ and $k = -1$. Let $U = F_u$ be the distribution associated to u . It follows from 5(b) that $\partial_i u = -x_i \|x\|^{-3}$, which is also locally integrable, so expect $\partial_i U$ to correspond to $\partial_i u$. However we only know this correspondence holds in situations like Exercise 7(d). Using careful manipulation of the integrals (in particular, cut-out a ball $B(0, \varepsilon)$) show that our expectation holds. (4 Bonus Points)

Solution.

(a) V is clearly linear. One can use L'Hopital's rule to see that the integrand is continuous at 0, and therefore integrable. Hence V defines a linear functional, and it remains to show that it is continuous. Let K be a compact set and ϕ a test function on K . We can bound K in the interval $[-R, R]$. Let us therefore split the integral into two parts

$$V(\phi) = \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx + \int_1^R \frac{\phi(x) - \phi(-x)}{x} dx.$$

The second integral is easily bound

$$\left| \int_1^R \frac{\phi(x) - \phi(-x)}{x} dx \right| \leq 2\|\phi\|_{K,0} \int_1^R dx \leq 2R\|\phi\|_{K,0}$$

By the mean value theorem, for all x there is a $y \in [-x, x]$ such that

$$\phi'(y) = \frac{\phi(x) - \phi(-x)}{2x}.$$

This allows us to bound the first integral

$$\left| \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx \right| \leq 2\|\phi\|_{K,1}$$

Together this gives $|V(\phi)| \leq 2R\|\phi\|_{K,0} + 2\|\phi\|_{K,1}$.

- (b) x^{-1} is not a local integrable function on \mathbb{R} so does not correspond to a distribution. However, if $0 \notin \text{supp } \phi$ then $V(\phi) = F_{x^{-1}}(\phi)$. So we can think of V as one possible extension of x^{-1} .
- (c) Away from the origin u is continuous, so of course it is locally integrable. So we just need to consider an open set containing the origin:

$$\int_{B(0,1)} |u(x)| dx = \int_0^1 \int_{B(0,r)} r^k d\sigma dr = \int_0^1 r^k n\omega_n r^{n-1} dr = n\omega_n \int_0^1 r^{k+n-1} dr$$

is finite if $k > -n$.

- (d) By definition

$$\begin{aligned} \partial_i U(\phi) &= -U(\partial_i \phi) = - \int_{\mathbb{R}^3} u \partial_i \phi \\ &= - \int_{B(0,\varepsilon)} u \partial_i \phi - \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} u \partial_i \phi \\ &= - \int_{B(0,\varepsilon)} u \partial_i \phi - \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \partial_i(u\phi) + \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \partial_i u \phi \\ &= - \int_{B(0,\varepsilon)} u \partial_i \phi + \int_{\partial B(0,\varepsilon)} u\phi x_i \|x\|^{-1} d\sigma + \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \partial_i u \phi \end{aligned}$$

We examine the three integrals in the limit as $\varepsilon \rightarrow 0$. The first integral we know goes to zero, because we can pull out $\|\partial_i \phi\|_\infty$ and then we know that u is integrable. The idea for the second integral is similar

$$\left| \int_{\partial B(0,\varepsilon)} u\phi x_i \|x\|^{-1} d\sigma \right| \leq \|\phi\|_\infty \varepsilon^{-2} \int_{\partial B(0,\varepsilon)} |x_i| d\sigma \leq \|\phi\|_\infty \varepsilon^{-2} \times \varepsilon 4\pi\varepsilon^2 \rightarrow 0.$$

The third term is exactly what we want. So taking limits

$$\partial_i U(\phi) = 0 + 0 + \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \partial_i u \phi = \int_{\mathbb{R}^3} \partial_i u \phi = F_{\partial_i u}(\phi)$$