

6. Extension of Continuous Linear Operators. Let X be a normed vector space and \bar{X} its completion. Let Y be a complete normed vector space Y . Suppose that $L : X \rightarrow Y$ is a continuous linear operator. This means that there is a constant C such that $\|Lx\| \leq C\|x\|$ for all $x \in X$. Show that there is a unique continuous linear operator $\bar{L} : \bar{X} \rightarrow Y$ extending L (ie $\bar{L}x = Lx$ for all $x \in X$). (3 Points)

7. Distributions I.

(a) Show directly from Definition 2.6 that the Heaviside distribution

$$H : C_0^\infty(\mathbb{R}) \rightarrow \mathbb{R}, \phi \mapsto \int_0^\infty \phi(x) dx$$

is a distribution on \mathbb{R} .

(2 Points)

(b) By the definition of the derivative of a distribution

$$\partial H(\phi) = -H(\partial\phi) = -\int_0^\infty \phi'(x) dx.$$

Simplify this expression in order to give a description of ∂H . (∂ here is the derivative in one-dimension. It seems weird to use an index.) (3 Points)

(c) What is the support of ∂H (in the sense of distributions)? Why does this show that there is no function $f \in L^1_{loc}(\mathbb{R})$ with $\partial H = F_f$? (3 Points)

(d) Consider a function $f \in C_0^\infty(\mathbb{R}^n)$. Show that $\partial_i(F_f) = F_{\partial_i f}$. What is the connection to Exercise 6? (2 Points)

8. On Convolutions.

(a) Let $f(x) = 1$ for $-1 \leq x \leq 1$ and 0 otherwise. Compute $f * f$. (2 Points)

(b) Show that the convolution of C_0^∞ -functions on \mathbb{R}^n is a bilinear, commutative, and associative operation. (1+2 Points + 2 Bonus Points)

(c) Denote a constant function on \mathbb{R} by 1. The Heaviside function $H : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $H(x) := 1$ for $x \geq 0$ and $H(x) := 0$ for $x < 0$. The derivative of the Dirac distribution δ' acts by $\delta'(\phi) = -\phi'(0)$. Let $\phi \in C_0^\infty(\mathbb{R})$ be a test function.

(i) Consider the distribution $\phi * P\delta'$. Which result from the script tells us that this distribution comes from a smooth function, even though δ' does not? (1 Point)

(ii) Prove that $\phi * P\delta' = F_{-\phi}$. (3 Points)

(iii) Thereby show that $\delta' * H = \delta$ and $1 * \delta' = 0$. (2 Points)

(iv) Complete the calculation of both $1 * (\delta' * H)$ and $(1 * \delta') * H$ in the sense of distributions and see that they are not equal. This shows that the convolution of distributions with non-compact support (on \mathbb{R}) is not necessarily associative, even when it is well-defined. (1 Point)

9. Distributions II.

(a) Show that

$$V(\phi) = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx$$

is a distribution on \mathbb{R} . Hint: Split the integral into $[0, 1]$ and $[1, \infty]$ and use the mean value theorem. *(2 Bonus Points)*

(b) What is the relation of V to x^{-1} ? *(1 Bonus Point)*

(c) Show that the function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto \|x\|^k$ is a locally integrable function for $k > -n$. *(2 Bonus Points)*

(d) Let $n = 3$ and $k = -1$. Let $U = F_u$ be the distribution associated to u . It follows from 5(b) that $\partial_i u = -x_i \|x\|^{-3}$, which is also locally integrable, so expect $\partial_i U$ to correspond to $\partial_i u$. However we only know this correspondence holds in situations like Exercise 7(d). Using careful manipulation of the integrals (in particular, cut-out a ball $B(0, \varepsilon)$) show that our expectation holds. *(4 Bonus Points)*