Tutorial 2

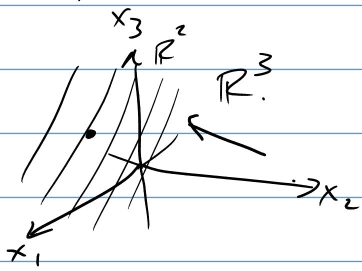
"Projection" a map onto a subspace. It fixes the subspace (idempotent)

Stereographic proj From $S^n \setminus \{e_0\}$ to $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$

coordinate proj $\pi_2 : \mathbb{R}^3 \to \mathbb{R}^2$

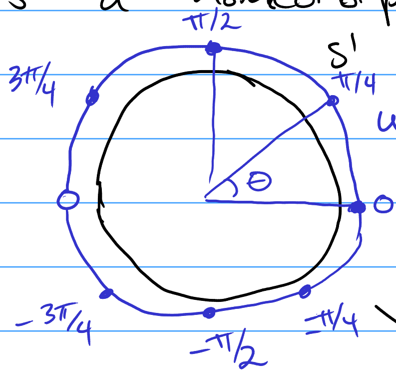
$$(x_1, x_2, x_3) \mapsto (x_1, x_3)$$

$$\pi_2(x_1, 0, x_3) = (x_1, x_3)$$



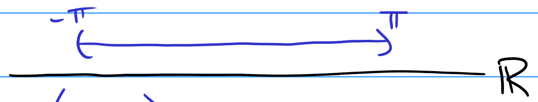
Lets do the questions 5, 6 for S^1 the circle.

Chart. A Chart is a homeomorphism $\phi : U \subset M \to \phi[U] \subseteq \mathbb{R}^n$



$S^1 = \{x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$

$\phi_{-\pi}$ = angle in range $(-\pi, \pi)$



$\phi_{-\pi}(1,0) = 0$

$\phi_{-\pi}(0,1) = \pi/2$

$\phi_{-\pi}(0,-1) = -\pi/2$

Justify the definition of a chart.

- Two points should not have the same label. $\phi_{-\pi}$ should be <u>injective</u>. Every injective function is bijective onto its image.

- Nearby points of U should have nearby labels. $\phi_{-\pi}$ should be <u>continuous / strictly</u>

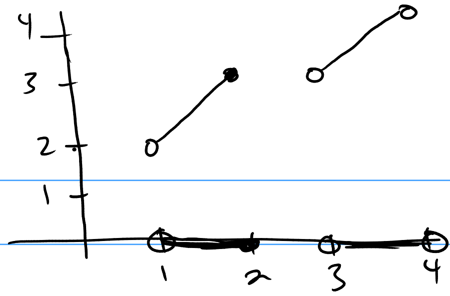
- f is bijective and continuous $\not\Rightarrow$ f^{-1} is continuous? No.

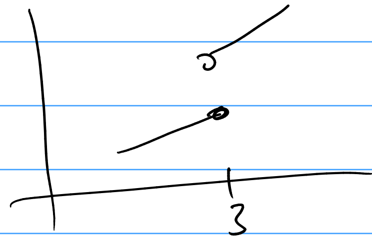
- Points not near each other should not have nearby labels. $\Rightarrow \phi_{-\pi}^{-1}$ should be continuous.

Example f^{-1} is not continuous.

$$f : (1,2] \cup (3,4) \longrightarrow (2,4)$$

$$f(x) = \begin{cases} x+1 & , x \in (1,2] \\ x & , x \in (3,4) \end{cases}$$





φ = varphi
ϕ = phi
ψ = psi

Go through steps to show ϕ_α is a chart.

$\phi_\alpha = \psi_\alpha^{-1}$

$$\psi_\alpha : (\alpha, \alpha + 2\pi) \longrightarrow \underbrace{\{(x,y) \in S^1 \mid (x,y) = (\cos\alpha, \sin\alpha)\}}_{\shortparallel}$$

$\psi_\alpha(t) = (\cos t, \sin t)$

$$S^1 \setminus \{(\cos\alpha, \sin\alpha)\} = U_\alpha$$

1. Show ψ_α is injective + surjective $\longrightarrow$ 1. find all points $t \in \mathbb{R}$ with $(\cos t, \sin t) = (x_0, y_0)$ see only one is in $(\alpha, \alpha + 2\pi)$
2. Show ψ_α is continuous
3. Show ψ_α^{-1} is continuous.

2. cos and sin are continuous functions $\mathbb{R} \to \mathbb{R}$. So ψ_α is cont.

3. Quick way using inverse function theorem.

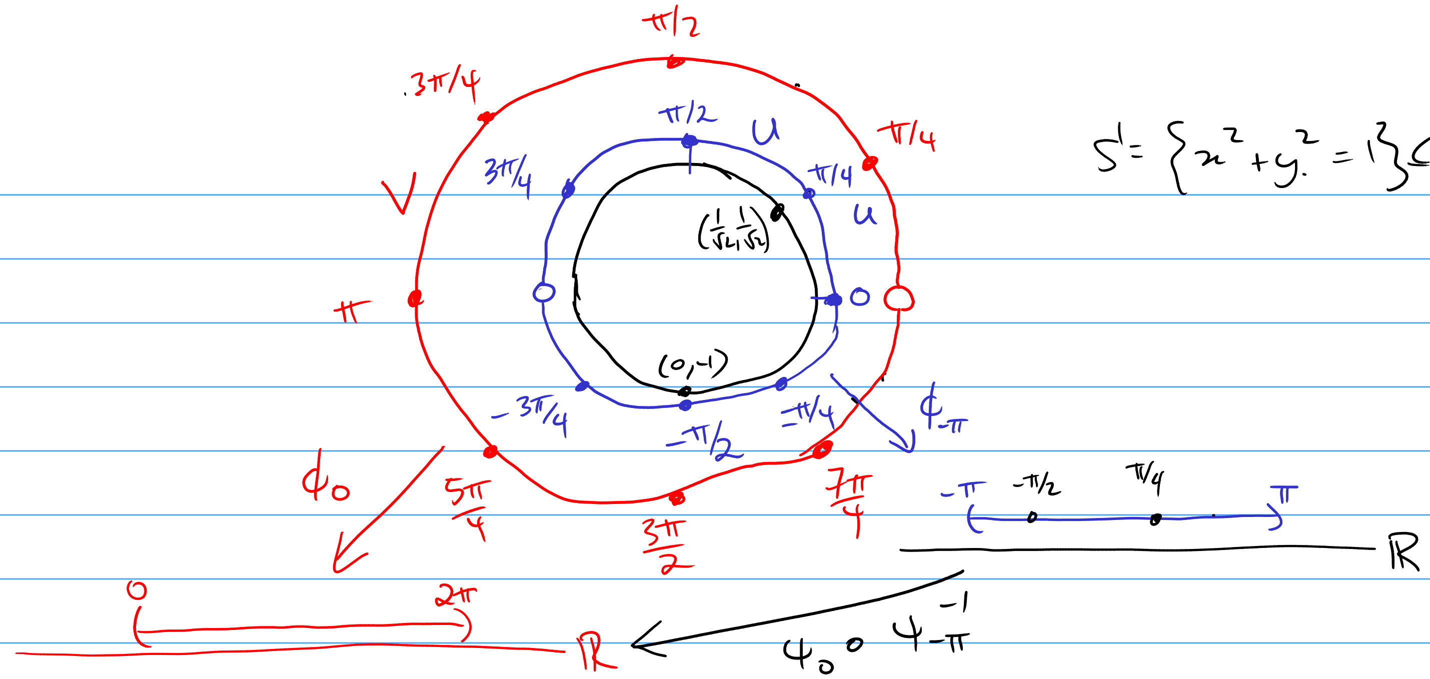
$n \geq m$ $F : \mathbb{R}^m \to \mathbb{R}^n$ and Jacobian matrix of F / matrix of derivatives, F smooth
is max rank, then there is a smooth inverse locally

$$J\psi_\alpha = \begin{bmatrix} \frac{\partial \cos t}{\partial t} \\ \frac{\partial \sin t}{\partial t} \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \quad \text{is rank 1.}$$

Rang = rank.

ψ_α^{-1} is smooth

$S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2$



"Change of coordinates" = "transition function"

$$\phi_0 \circ \phi_{-\pi}^{-1}\left(\frac{\pi}{4}\right) = \phi_0\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right) = \frac{\pi}{4}$$

$$\phi_0 \circ \phi_{-\pi}^{-1}(-\pi/2) = \phi_0(0,-1) = 3\pi/2$$

Both charts are defined on $U \cap V$

$$\phi_0 \circ \phi_{-\pi}^{-1} : \phi_{-\pi}(U\cap V) \longrightarrow \phi_0(U \cap V)$$

- Work out $\phi_0 \circ \phi_{-\pi}^{-1}(x)$ the domain and codomain the formula.

$U \cap V = S^1 \setminus \{(1,0), (-1,0)\} \subset S^1$

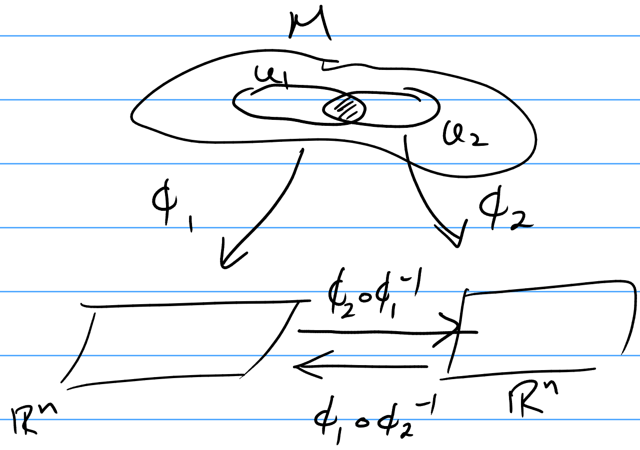
$\phi_{-\pi}(U\cap V) = (-\pi, 0) \cup (0, \pi) = (-\pi,\pi) \setminus \{0\} \subseteq \mathbb{R}$

$\phi_0(U \cap V) = (0, 2\pi) \setminus \{\pi\} \subseteq \mathbb{R}$

$$\phi_0 \circ \phi_{-\pi}^{-1}(x) = \begin{cases} x + 2\pi & \text{if } x \in (-\pi, 0) \\ x & \text{if } x \in (0, \pi) \end{cases}$$

go from $\phi_{-\pi}$ coordinate to ϕ_0 coordinate

Two charts are "compatible" verträglich if the transition functions both ways are smooth / glatt.



An <u>atlas</u> is a collection of charts

- the domains of the charts cover M
- all charts are compatible.

Manifold "=" atlas.

S^1 $\qquad \phi_0 \circ \phi_{-\pi}^{-1}(t) = \begin{cases} t + 2\pi & t \in (-\pi, 0) \\ t & t \in (0, \pi) \end{cases}$ smooth

$\updownarrow$ inverse

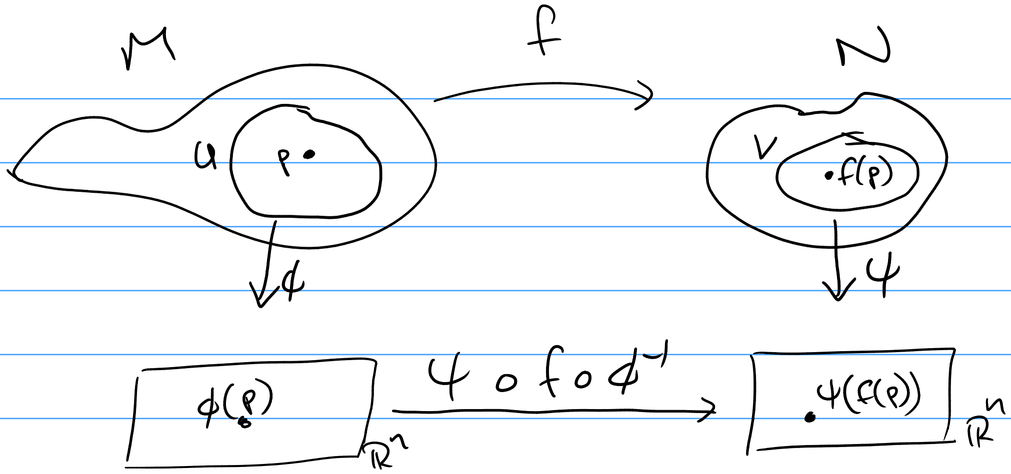
$$\phi_{-\pi} \circ \phi_0^{-1}(s) = \begin{cases} s - 2\pi & s \in (\pi, 2\pi) \\ s & s \in (0, \pi) \end{cases}$$ smooth

$\mathcal{A} = \{\phi_0, \phi_{-\pi}\}$ is an atlas for S^1.

$(S^1, \mathcal{A})$ is a manifold.

Check f is smooth at p

Q8.



f is smooth at p $\iff$ $\underbrace{\psi \circ f \circ \phi^{-1}}_{\text{function in coordinates}}$ is smooth at $\phi(p)$