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Analysis III
14. Exercises

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44. A differential form which is closed but not exact.

Consider on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ the 1-form

$$\omega := -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y.$$

(a) Show that ω is closed.

(b) Compute $\int_{\mathbb{S}^1} \omega$.

(c) Why does it follow from that ω is not exact?

Remark. Due to $\mathrm{d}(\mathrm{d}\eta) = 0$ we see that every exact form is closed. *Poincaré's Lemma* says that on *star-shaped* regions in $\mathbb{R}^n$ that the converse is also true: every closed form is exact. The example in this exercise shows that such a converse result cannot hold for general regions.

Solution.

(a)

$$d\omega = -\left(\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right) dy \wedge dx + \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right) dx \wedge dy = 0$$

(b) We use the parametrisation f and result from Exercise 43 on the last tutorial sheet:

$$\int_{\mathbb{S}^1} \omega = \int_0^{2\pi} -\frac{\sin t}{1}\, d(\cos t) + \frac{\cos t}{1}\, d(\sin t) = \int_0^{2\pi} \sin^2 t\, dt + \cos^2 t\, dt = 2\pi.$$

(c) By Stokes' theorem if ω were exact then this integral would be zero.

45. An integration.

Let $\omega = y\,\mathrm{d}x + z\,\mathrm{d}y$ be a 1-form on $\mathbb{R}^3$. Consider the restriction of ω to the 2-sphere $\mathbb{S}^2$, with the parametrisation

$$S^2 = \{\, (\sin(\varphi)\sin(\vartheta), \cos(\varphi)\sin(\vartheta), \cos(\vartheta)) \in \mathbb{R}^3 \mid \varphi \in [0, 2\pi), \vartheta \in [0, \pi] \,\}.$$

Verify through direct computation that Stokes' theorem holds for this case:

$$\int_{S^2} \mathrm{d}\omega = 0.$$

Solution. First,

$$d\omega = dy \wedge dx + dz \wedge dy.$$

We will also need to calculate the pullback by the parametrisation $f(\varphi, \vartheta) = (\sin(\varphi)\sin(\vartheta), \cos(\varphi)\sin($

$$\begin{aligned}
f^*dx &= d(\sin(\varphi)\sin(\vartheta)) = \cos(\varphi)\sin(\vartheta)d\varphi + \sin(\varphi)\cos(\vartheta)d\vartheta \\
f^*dy &= -\sin(\varphi)\sin(\vartheta)d\varphi + \cos(\varphi)\cos(\vartheta)d\vartheta \\
f^*dz &= -\sin(\vartheta)d\vartheta \\
f^*(dy \wedge dx) &= -\sin^2(\varphi)\sin(\vartheta)\cos(\vartheta)d\varphi \wedge d\vartheta + \cos^2(\varphi)\sin(\vartheta)\cos\vartheta d\vartheta \wedge d\varphi \\
&= -\sin(\vartheta)\cos(\vartheta)d\varphi \wedge d\vartheta \\
f^*(dz \wedge dy) &= -\cos(\varphi)\sin^2(\vartheta)d\vartheta \wedge d\varphi \\
&= \cos(\varphi)\sin^2(\vartheta)d\varphi \wedge d\vartheta
\end{aligned}$$

Similar to Exercise 43, we can ignore sets of measure zero when pulling back using the parametrisation.

$$\begin{aligned}
\int_{S^2} \mathrm{d}\omega &= \int_{[0,2\pi]\times[0,\pi]} f^*\mathrm{d}\omega \\
&= \int_{[0,2\pi]\times[0,\pi]} \Big[-\sin(\vartheta)\cos(\vartheta) + \cos(\varphi)\sin^2(\vartheta)\Big] d\varphi \wedge d\vartheta \\
&= \int_0^{\pi} \left[\int_0^{2\pi} -\sin(\vartheta)\cos(\vartheta) + \cos(\varphi)\sin^2(\vartheta)\ d\varphi\right] d\vartheta \\
&= \int_0^{\pi} -2\pi\sin(\vartheta)\cos(\vartheta)\ d\vartheta = \int_0^{\pi} -\pi\sin(2\vartheta)\ d\vartheta = 0.
\end{aligned}$$

46. The Divergence Theorem (aka Gauss' Theorem).

Let $X \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ with $\overline{X^0} = X$ that is an n-dimensional manifold with boundary. It is know that X must be orientable and that $\omega := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a volume form on X. Further, let a smooth $(n-1)$-form η on X be given.

(a) Show that there is a unique vector field $F \in \mathrm{Vec}^\infty(X)$ with $\eta = i_F\omega$.

(b) Write $F = (F_1, \ldots, F_n)$ for functions $F_1, \ldots, F_n \in C^\infty(X, \mathbb{R})$. Define the divergence operator $\mathrm{div}(F) \in C^\infty(X, \mathbb{R})$ as

$$\mathrm{div}(F) := \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k}.$$

Prove the following connection between the divergence operator and the exterior derivative:

$$\mathrm{d}(i_F\omega) = \mathrm{div}(F) \cdot \omega.$$

(c) Prove Gauss' divergence theorem:

$$\int_{\partial X} \eta = \int_X \operatorname{div}(F) \cdot \omega \ .$$

Solution.

(a) We know that we can write $\eta = \sum \eta_i dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n$. Consider then the vector field $F := (\eta_1, \ldots, (-1)^{n-1}\eta_n)$ and in particular how $\iota_F \omega$ acts on $E_1 \otimes \widehat{E}_i \otimes E_n$:

$$\begin{aligned}
\langle \iota_F \omega, E_1 \otimes \widehat{E}_i \otimes E_n \rangle &= \langle \omega, F \otimes E_1 \otimes \widehat{E}_i \otimes E_n \rangle \\
&= \sum_j \langle \omega, (-1)^{j-1} \eta_j E_j \otimes E_1 \otimes \widehat{E}_i \otimes E_n \rangle \\
&= \sum_j (-1)^{j-1} \eta_j \det(dx_k(v_l))_{k,l} \\
&= (-1)^{i-1} \eta_i \det(dx_k(v_l))_{k,l} \\
&= (-1)^{i-1} \eta_i \cdot (-1)^{i-1},
\end{aligned}$$

because the only determinant that does not have a repeated column is the one where $j = i$. For that matrix, you then have to do $j-1$ column swaps to make it the identity matrix. This shows that $\iota_F \omega$ acts identically to η.

(b) We can apply part (a) in reverse, so that $\iota_F \omega = \sum_i (-1)^{i-1} F_i dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n$. Now we apply the exterior derivative

$$\begin{aligned}
d(\iota_F \omega) &= \sum_i d\left((-1)^{i-1} F_i\right) \wedge dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n \\
&= \sum_i \left[\sum_j \frac{\partial}{\partial x_j} (-1)^{i-1} F_i dx_j \right] \wedge dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n \\
&= \sum_i (-1)^{i-1} \frac{\partial F_i}{\partial x_i} dx_i \wedge dx_1 \wedge \ldots \widehat{dx_i} \cdots \wedge dx_n \\
&= \sum_i \frac{\partial F_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \\
&= \operatorname{div}(F) \cdot dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

(c)

$$\int_{\partial X} \eta = \int_X d\eta = \int_X d(\iota_F \omega) = \int_X \operatorname{div}(F) \cdot \omega$$

47. Volume forms on compact connected manifolds.

Let X be a compact connected orientable n-dimensional manifold without boundary, and suppose that ω is a non-vanishing n-form. Show that ω is not exact.

Hint. Calculate $\int_X \omega$ in two ways: with Stokes' theorem and with Definition 3.21.

Solution. Suppose that ω was exact: $\omega = d\eta$. Then by Stokes' theorem

$$0 = \int_{\emptyset} \eta = \int_{\partial X} \eta = \int_X d\eta = \int_X \omega.$$

On the other hand, from the definition of integration on manifolds, let $\{(U_k, \phi_k)\}$ be an oriented atlas of X and f_k the corresponding partition of unity. Without loss of generality, assume all the sets U_k are connected. Write $\omega = g_k d\phi_{k,1} \wedge \cdots \wedge d\phi_{k,n}$. Because g_k is non-vanishing, it has a definite sign on U_k. Because we are using an orientable atlas, all of the functions g_k have the same sign. Assume this sign is positive. Then

$$\int_X \omega = \sum_k \int_{\phi_k[U_k]} f_k(\phi_k^{-1}(x)) g_k(\phi_k^{-1}(x))\, dx_1 \dots dx_n \geq \int_{\phi_0[U_0]} f_0(\phi_0^{-1}(x)) g_0(\phi_0^{-1}(x))\, dx_1 \dots dx_n > 0$$

since the integral of a non-negative continuous function that is positive at a point must be positive. This is contradiction. Hence ω is not exact.