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Analysis III
14. Exercises

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44. A differential form which is closed but not exact.

Consider on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ the 1-form

$$\omega := -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y.$$

(a) Show that ω is closed.

(b) Compute $\int_{\mathbb{S}^1} \omega$.

(c) Why does it follow from that ω is not exact?

Remark. Due to $\mathrm{d}(\mathrm{d}\eta) = 0$ we see that every exact form is closed. *Poincaré's Lemma* says that on *star-shaped* regions in $\mathbb{R}^n$ that the converse is also true: every closed form is exact. The example in this exercise shows that such a converse result cannot hold for general regions.

45. An integration.

Let $\omega = y\,\mathrm{d}x + z\,\mathrm{d}y$ be a 1-form on $\mathbb{R}^3$. Consider the restriction of ω to the 2-sphere $\mathbb{S}^2$, with the parametrisation

$$S^2 = \{\, (\sin(\varphi)\sin(\vartheta), \cos(\varphi)\sin(\vartheta), \cos(\vartheta)) \in \mathbb{R}^3 \mid \varphi \in [0, 2\pi), \vartheta \in [0, \pi] \,\}.$$

Verify through direct computation that Stokes' theorem holds for this case:

$$\int_{S^2} \mathrm{d}\omega = 0.$$

46. The Divergence Theorem (aka Gauss' Theorem).

Let $X \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ with $\overline{X^0} = X$ that is an n-dimensional manifold with boundary. It is know that X must be orientable and that $\omega := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a volume form on X. Further, let a smooth $(n-1)$-form η on X be given.

(a) Show that there is a unique vector field $F \in \mathrm{Vec}^\infty(X)$ with $\eta = i_F\omega$.

(b) Write $F = (F_1, \ldots, F_n)$ for functions $F_1, \ldots, F_n \in C^\infty(X, \mathbb{R})$. Define the divergence operator $\mathrm{div}(F) \in C^\infty(X, \mathbb{R})$ as

$$\mathrm{div}(F) := \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k}.$$

Prove the following connection between the divergence operator and the exterior derivative:

$$\mathrm{d}(i_F\omega) = \mathrm{div}(F) \cdot \omega.$$

(c) Prove Gauss' divergence theorem:

$$\int_{\partial X} \eta = \int_X \operatorname{div}(F) \cdot \omega \ .$$

47. Volume forms on compact connected manifolds.

Let X be a compact connected orientable n-dimensional manifold without boundary, and suppose that ω is a non-vanishing n-form. Show that ω is not exact.

Hint. Calculate $\int_X \omega$ in two ways: with Stokes' theorem and with Definition 3.21.