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Analysis III
13. Exercises

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40. The pullback of differential forms.

(a) Let X, Y be manifolds of dimension n and $f : X \to Y$ a smooth map. Further take the standard local set-up of charts $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$ and $\psi = (\psi_1, \ldots, \psi_n) : V \to \mathbb{R}^n$ on open sets $U \subset X$ and $V \subset Y$ with $f(U) \subset V$.

Show the following local formula for the pullback holds for every smooth function $g \in \mathcal{C}^\infty(V, \mathbb{R})$:

$$f^*(g\,\mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_n) = (g \circ f) \cdot \det\left(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\right) \cdot \mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_n\,.$$

Hint. Make use of the determinant formula for the evaluation of forms $\langle A_1 \wedge \cdots \wedge A_p\,,\, v_1 \otimes \ldots \otimes v_p\rangle = \det\big(A_i(v_j)\big)_{i,j}$, from page 71 of the script.

(b) Consider the *canonical volume form* on $\mathbb{R}^3$, namely $\omega := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and *spherical coordinates*

$$f : \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3,\ (r, \vartheta, \varphi) \mapsto \big(r\ \cos(\vartheta)\ \cos(\varphi)\,,\ r\ \cos(\vartheta)\ \sin(\varphi)\,,\ r\ \sin(\vartheta)\big).$$

Compute "ω in spherical coordinates", by which we mean the pullback $f^*\omega$.

Solution.

(a) A special case of this is used in the proof of Theorem 3.17 to change between n-forms coming from two sets of coordinates on the same manifold. The operation itself is defined in Theorem 3.12(iii) and Theorem 3.5.

In particular, because we know from Theorem 3.12(iv) that pulling back is an exterior algebra homomorphism, we can compute the effect on each part and then recombine them. The pullback of a smooth function is just pre-composition: $f^*g = g \circ f$. Moreover, by Theorem 3.15(ii) pullbacks commute with exterior derivatives, so

$$f^* d\psi_j = d(f^*\psi_i) = d(\psi_j \circ f) = \sum_i \frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\, d\phi_i$$

using the formula for exterior derivative in terms of coordinate charts on page 75. We have one such 1-form for each $d\psi_j$, and clearly it would be a pain to try to expand out the exterior product directly. This is where we follow the hint. Let E_k

be coordinate vector fields on U, that is $d\phi_i(E_k) = \delta_{i,k}$. Then

$$\begin{aligned}
\langle f^* d\psi_1 \wedge \cdots \wedge f^* d\psi_n, E_1 \otimes \cdots \otimes E_n \rangle &= \det \Big(\sum_i \frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i} \, d\phi_i(E_k) \Big)_{j,k} \\
&= \det \Big(\sum_i \frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i} \, \delta_{i,k} \Big)_{j,k} \\
&= \det \Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i} \Big)_{j,i}
\end{aligned}$$

$$\begin{aligned}
\Big\langle \det \Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i} \Big)_{i,j} d\phi_1 \wedge \cdots \wedge d\phi_n, E_1 \otimes \cdots \otimes E_n \Big\rangle \\
&= \det \Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i} \Big)_{i,j} \det \Big(d\phi_i(E_k) \Big)_{i,k} \\
&= \det \Big(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i} \Big)_{i,j}
\end{aligned}$$

Any other pure n-form in increasing order must repeat vectors, so for the first form leads to a repeated column (and so the determinant is zero) and for the second form leads to a column of zeroes. This shows that the two differential forms act identically, and so are equal.

(b) Here we interpret the interior of the domain of f as a manifold X and let $Y = \mathbb{R}^3$. In particular $X = (0, \infty) \times (0, 2\pi) \times (0, \pi)$ is an open subset of $\mathbb{R}^3$ so we use the identity function as the chart ϕ, but label the components r, ϑ, φ, whereas on $Y = \mathbb{R}^3$ we also use the identity function as a chart but label the coordinates with the usual x, y, z. Therefore we have

$$(x, y, z) = \psi \circ f \circ \phi^{-1}(r, \vartheta, \varphi) = f(r, \vartheta, \varphi).$$

The determinant that we need to compute is nothing other than the determinant of the Jacobian matrix:

$$\begin{aligned}
&\begin{vmatrix} \cos(\vartheta)\cos(\varphi) & -r\sin(\vartheta)\cos(\varphi) & -r\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\sin(\varphi) & -r\sin(\vartheta)\sin(\varphi) & r\cos(\vartheta)\cos(\varphi) \\ \sin(\vartheta) & r\cos(\vartheta) & 0 \end{vmatrix} \\
&= \sin(\vartheta) \begin{vmatrix} -r\sin(\vartheta)\cos(\varphi) & -r\cos(\vartheta)\sin(\varphi) \\ -r\sin(\vartheta)\sin(\varphi) & r\cos(\vartheta)\cos(\varphi) \end{vmatrix} - r\cos(\vartheta) \begin{vmatrix} \cos(\vartheta)\cos(\varphi) & -r\cos(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\sin(\varphi) & r\cos(\vartheta)\cos(\varphi) \end{vmatrix} \\
&= r^2 \sin^2(\vartheta)\cos(\vartheta) \begin{vmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{vmatrix} - r^2\cos^3(\vartheta) \begin{vmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{vmatrix} \\
&= -r^2\cos(\vartheta)
\end{aligned}$$

41. Orientable manifolds.

(a) Show that the n-dimensional sphere $\mathbb{S}^n$ is orientable by finding an oriented atlas.

Hint. For the sphere, consider the atlas that uses stereographic projection. An extra trick is also needed.

(b) Show that the Möbius band is not orientable.

Hint. This is difficult. Good luck.

(c) Let X and Y be orientable manifolds. Show that the Cartesian product $X \times Y$ is also orientable.

(d) Let X be a manifold. Show that every coordinate neighbourhood of X is orientable. More precisely, let (U, ϕ) be a chart of X with $\phi = (\phi_1, \dots, \phi_n) : U \to \mathbb{R}^n$, and show that $\mathrm{d}\phi_1 \wedge \ldots \wedge \mathrm{d}\phi_n$ is a non-vanishing n-form on U.

(e) Prove that the tangent bundle of any manifold is orientable.

Solution.

(a) As we have seen many times, we can cover the sphere with two charts using stereographic projection. The transistion function between these two charts was already computed in Example 1.18(iii) as $y \mapsto z := \|y\|^{-2} y$ for $y \in \mathbb{R}^n \setminus \{0\}$. We have that

$$\frac{\partial z_j}{\partial y_i} = \begin{cases} \frac{\|y\|^2 - 2y_i^2}{\|y\|^4} & \text{for } i = j \\ \frac{-2y_i y_j}{\|y\|^4} & \text{for } i \neq j. \end{cases}$$

Thus we need to find the sign of the matrix $\|y\|^2 I - (2y_i y_j)_{i,j}$. Recall that the determinant is the product of the eigenvalues of a matrix. The eigenvalues of the matrix $bI - A$ are related to those of A because

$$(bI - A)v = \lambda v \Leftrightarrow Av = (b - \lambda)v.$$

Thus we need to calculate the eigenvalues of $(2y_i y_j)_{i,j}$. But we recognise that this is the product $2yy^T$ for y a column vector (notice this is not the familiar order used to write the dot product). We see immediately that y is itself an eigenvector, since

$$(2yy^T)y = 2y(y^T y) = 2\|y\|^2 y.$$

On the other hand, if v is perpendicular to y, then the same computation shows that v is a null vector of A. This gives us a basis of eigenvectors of A and all of the eigenvalues. Therefore the eigenvalues of $\|y\|^2 I - (2y_i y_j)_{i,j}$ are $\|y\|^2 - 2\|y\|^2 = -\|y\|^2$ and $n - 1$ copies of $\|y\|^2$. This shows that transition matrix has everywhere *negative* determinant and the atlas of the two stereographic projections is not an oriented atlas.

However, we can easily make an oriented atlas now. Let N, S be stereographic projection from the north and south pole respectively. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be map $A(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$ which reflects in the first coordinate. Consider the atlas $\{N, A \circ S\}$. There is only one transition to consider here, and the determinant of the derivative of the transition is, by the chain rule,

$$\det(A \circ S \circ N^{-1})' = \det A' \det(S \circ N^{-1}) = -\det(S \circ N^{-1}) > 0.$$

Thus this is an oriented atlas, and shows $\mathbb{S}^n$ is oriented.

There is also a more abstract way to make this argument that avoids the computation of the determinant: Note that the Jacobian is everywhere full-rank because it has no kernel (or that the transition function is diffeomorphism). Therefore its determinant is non-zero and so must be a single sign on $\mathbb{R}^n \setminus \{0\}$ (this set is path connected. If it had two points with opposite sign, connected them with a path and apply the intermediate value theorem along this path). If the sign is positive, we have an oriented atlas. If the sign is everywhere negative, compose one of the charts with the reflection A of $\mathbb{R}^n$. The result is an oriented atlas. In either case, $\mathbb{S}^n$ must be orientable.

(b) To prove something is non-orientable, we must show that there does not exist any oriented atlas for the manifold; it is not enough to say that the usual atlas is not oriented. For this reason, will be prove that the Möbius band is non-orientable by contradiction. Suppose then that there is an oriented atlas $\{(U_i, \phi_i)\}$ of the Möbius band M. Let the standard atlas for M be $\{(U_N, \psi_N), (U_s, \psi_S)\}$. There is some coordinate chart U_0 that contains $(0, x_S)$, so it contains also (ε, x_S) and $(-\varepsilon, x_S)$ for some small $\varepsilon > 0$. Without loss of generality, suppose that U_0 is connected and so $\operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'$ is constant on U_0

Consider the continuous path

$$\alpha(t) = \begin{cases} (\varepsilon, x_S)_{U_N} & \text{for } t = -\pi/2 \\ (\varepsilon, (\cos t, \sin t))_{U_S} & \text{for } -\pi/2 < t < 3\pi/2 \\ (-\varepsilon, x_S)_{U_N} & \text{for } t = 3\pi/2 \end{cases}$$

At any point $\alpha(t)$ with $-\pi/2 < t < 3\pi/2$ consider the sign of $\det(\phi_i \circ \psi_S^{-1})$. This is independent of the chart ϕ_i because

$$\begin{aligned} \operatorname{sign}\det(\phi_i \circ \psi_S^{-1})' &= \operatorname{sign}\Big(\det(\phi_i \circ \phi_j^{-1} \circ \phi_j \circ \psi_S^{-1})'\Big) \\ &= \operatorname{sign}\det(\phi_i \circ \phi_j^{-1})' \operatorname{sign}\det(\phi_j \circ \psi_S^{-1})' \\ &= \operatorname{sign}\det(\phi_j \circ \psi_S^{-1})'. \end{aligned}$$

Extend this sign s to the end points of the path. We see now however that near $t = -\pi/2$

$$\begin{aligned}
s(-\pi/2) &= \lim_{t\to-\pi/2^+} \operatorname{sign}\det(\phi_0 \circ \psi_S^{-1})'(\psi_S(\alpha(t))) \\
&= \lim_{t\to-\pi/2^+} \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1} \circ \psi_N \circ \psi_S^{-1})'(\psi_S(\alpha(t))) \\
&= \lim_{t\to-\pi/2^+} \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'(\psi_N(\alpha(t))) \cdot \operatorname{sign}\det(\psi_N \circ \psi_S^{-1})'(\psi_S(\alpha(t))) \\
&= \lim_{t\to-\pi/2^+} \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'(\psi_N(\alpha(t))) \cdot 1 \\
&= \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'(\psi_N(\alpha(-\pi/2)))
\end{aligned}$$

where the last step follows because by the definition of U_0 we know that $\operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'$ is constant on it. On the other hand

$$\begin{aligned}
s(3\pi/2) &= \lim_{t\to3\pi/2^-} \operatorname{sign}\det(\phi_0 \circ \psi_S^{-1})'(\psi_S(\alpha(t))) \\
&= \lim_{t\to3\pi/2^-} \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'(\psi_N(\alpha(t))) \cdot \operatorname{sign}\det(\psi_N \circ \psi_S^{-1})'(\psi_S(\alpha(t))) \\
&= \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'(\psi_N(\alpha(3\pi/2))) \cdot (-1) \\
&= \operatorname{sign}\det(\phi_0 \circ \psi_N^{-1})'(\psi_N(\alpha(-\pi/2))) \cdot (-1).
\end{aligned}$$

This contradicts the fact that s is constant along the path. Therefore there cannot be an oriented atlas on M: it is non-orientable.

(c) Choose oriented atlases $\{\phi_i\}$ for X and $\{\psi_j\}$ for Y. Then the standard atlas $\{\phi_i \times \psi_j\}$ for $X \times Y$ is oriented because

$$\operatorname{sign}\det((\phi_i \times \psi_j) \circ (\phi_k \times \psi_l)^{-1})' = \operatorname{sign}\begin{vmatrix} (\phi_i \circ \phi_k^{-1})' & 0 \\ 0 & (\psi_j \circ \psi_l^{-1})' \end{vmatrix} = 1 \cdot 1 = 1.$$

(d) Let E_k be the coordinate vector fields on U. We have already seen that this n-form is 1 at every point when applied to $E_1 \otimes \cdots \otimes E_n$. Therefore it is non-vanishing.

(e) Again, there is a standard atlas on the tangent bundle TX that arises from any atlas $\{(U_i, \phi_i)\}$ on X given by the tangent map, namely

$$T(\phi_i) : \pi^{-1}[U_i] \to \mathbb{R}^n \times \phi[U_i], \quad (v, x) \mapsto (T_x(\phi_i)v, \phi_i(x)).$$

We can compute the Jacobian of the transition function $T(\phi_i \circ \phi_j^{-1})$ in block form, for $w \in \mathbb{R}^n$ and $y = \phi_j(x)$:

$$\Big(T(\phi_i) \circ T(\phi_j)\Big)'(w, y) = \begin{pmatrix} T_y(\phi_i \circ \phi_j^{-1})w & \frac{\partial}{\partial y_k} T_y(\phi_i \circ \phi_j^{-1})(w, y) \\ 0 & (\phi_i \circ \phi_j^{-1})'(w, y) \end{pmatrix}.$$

The first block is so because the derivative of a linear map is the same linear map. The off diagonal block is difficult to compute, but not needed to find the determinant. Observe finally that the two diagonal blocks are actually two different notations for the same thing. Therefore the determinant is a square, and hence always positive.

42. Orientable hypersurfaces defined by an equation.

Let $f : \mathbb{R}^n \to \mathbb{R}$ a smooth map, $q \in \mathbb{R}$ a point in its range so $X := f^{-1}(\{q\}) \neq \varnothing$, and f is submersive at all points $x \in X$. Show that X is an $(n-1)$-dimensional orientable submanifold.

Hint. Let ω be the volume form on $\mathbb{R}^n$ and F the gradient field of f (i.e $T_x(f)(v) = F(x) \cdot v$). Investigate $i_F\omega|X$, defined in Definition 3.11.

Solution. Because f is a submersion, we know that X is an $(n-1)$-dimensional submanifold. The gradient field $F = \nabla f$ is a vector field on $\mathbb{R}^n$, so it makes sense to contract the volume form with it. $\iota_F\omega$ is a $(n-1)$-form, so we should try show that this form is non-vanishing on X, so that Theorem 3.17(iv) applies.

Choose any point $x \in X$ and take a basis v_i of T_xX. Then $\{F(x), v_1, \ldots, v_{n-1}\}$ is a basis of $T_x\mathbb{R}^n$ because F is perpendicular to the vectors v_i and does not vanishing at x because f is a submersion. Hence

$$\langle \iota_F\omega, v_1 \otimes \cdots \otimes v_{n-1} \rangle = \langle \omega, F \otimes v_1 \otimes \cdots \otimes v_{n-1} \rangle \neq 0$$

shows that $\iota_F\omega|X$ is non-vanishing.

43. Integration on the unit circle.

Let ω be a 1-form on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and

$$f : \mathbb{R} \to \mathbb{S}^1,\ t \mapsto (\cos t, \sin t)$$

the standard paramterisation.

(a) Show that $\displaystyle\int_{\mathbb{S}^1} \omega = \int_{[0,2\pi]} f^*\omega.$

Hint. Obviously we want to use Corollary 3.22, but we can not so do immediately, because $f|[0, 2\pi]$ is not injective. However ....

(b) Prove from (a) Stokes' theorem for $\mathbb{S}^1$. Actually, show the stronger result that ω is exact if and only if

$$\int_{\mathbb{S}^1} \omega = 0.$$

($\mathbb{S}^1$ is a manifold whose boundary is empty, so the right side of Stokes' theorem is zero.)

Solution.

(a) We know that an integral is not changed by the exclusion of a null set. Thus

$$\int_{\mathbb{S}^1} \omega = \int_{\mathbb{S}^1 \setminus \{(1,0)\}} \omega = \int_{(0,2\pi)} f^*\omega = \int_{[0,2\pi]} f^*\omega$$

using Corollary 3.22 in the middle step, because f is a diffeomorphisms between $\mathbb{S}^1 \setminus \{(1,0)\}$ and $(0, 2\pi)$.

(b) Suppose that $\omega = dg$ for a smooth function $g : \mathbb{S}^1 \to \mathbb{R}$. Then we have that

$$\int_{\mathbb{S}^1} dg = \int_{[0,2\pi]} d(g \circ f) = g(f(2\pi)) - g(f(0)) = g(1,0) - g(1,0) = 0.$$

This shows Stokes' theorem in this case.

For the stronger statement, which also has the converse, suppose that the integral of ω is 0 over the circle. Define the real function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(t) = \int_0^t f^*\omega.$$

This function is in fact periodic with period 2π because f is and

$$G(t + 2\pi) = \int_0^{2\pi} f^*\omega + \int_{2\pi}^{t+2\pi} f^*\omega = 0 + \int_0^t f^*\omega = G(t).$$

Thus G defines a function g on the circle. Every point of the circle has a neighbourhood where f restricts to give a coordinate chart. In this coordinate chart, we see that $G = g \circ f = f^*g$. Since $dG = G'\,dt = f^*\omega$, it follows that $dg = \omega$. This shows that ω is exact as required.

Terminology

zurückziehen = pullback.