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Analysis III
13. Exercises

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40. The pullback of differential forms.

(a) Let X, Y be manifolds of dimension n and $f : X \to Y$ a smooth map. Further take the standard local set-up of charts $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$ and $\psi = (\psi_1, \ldots, \psi_n) : V \to \mathbb{R}^n$ on open sets $U \subset X$ and $V \subset Y$ with $f(U) \subset V$.

Show the following local formula for the pullback holds for every smooth function $g \in \mathcal{C}^\infty(V, \mathbb{R})$:

$$f^*(g \,\mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_n) = (g \circ f) \cdot \det\left(\frac{\partial(\psi_j \circ f \circ \phi^{-1})}{\partial x_i}\right) \cdot \mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_n \,.$$

Hint. Make use of the determinant formula for the evaluation of forms $\langle A_1 \wedge \cdots \wedge A_p \,,\, v_1 \otimes \ldots \otimes v_p \rangle = \det\big(A_i(v_j)\big)_{i,j}$, from page 71 of the script.

(b) Consider the *canonical volume form* on $\mathbb{R}^3$, namely $\omega := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and *spherical coordinates*

$$f : \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3,\ (r, \vartheta, \varphi) \mapsto \big(r\, \cos(\vartheta)\, \cos(\varphi)\,,\, r\, \cos(\vartheta)\, \sin(\varphi)\,,\, r\, \sin(\vartheta)\big).$$

Compute "ω in spherical coordinates", by which we mean the pullback $f^*\omega$.

41. Orientable manifolds.

(a) Show that the n-dimensional sphere $\mathbb{S}^n$ is orientable by finding an oriented atlas.

Hint. For the sphere, consider the atlas that uses stereographic projection. An extra trick is also needed.

(b) Show that the Möbius band is not orientable.

Hint. This is difficult. Good luck.

(c) Let X and Y be orientable manifolds. Show that the Cartesian product $X \times Y$ is also orientable.

(d) Let X be a manifold. Show that every coordinate neighbourhood of X is orientable. More precisely, let (U, ϕ) be a chart of X with $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{R}^n$, and show that $\mathrm{d}\phi_1 \wedge \ldots \wedge \mathrm{d}\phi_n$ is a non-vanishing n-form on U.

(e) Prove that the tangent bundle of any manifold is orientable.

42. Orientable hypersurfaces defined by an equation.

Let $f : \mathbb{R}^n \to \mathbb{R}$ a smooth map, $q \in \mathbb{R}$ a point in its range so $X := f^{-1}(\{q\}) \neq \varnothing$, and f is submersive at all points $x \in X$. Show that X is an $(n-1)$-dimensional orientable submanifold.

Hint. Let ω be the volume form on $\mathbb{R}^n$ and F the gradient field of f (i.e $T_x(f)(v) = F(x) \cdot v$). Investigate $i_F\omega|X$, defined in Definition 3.11.

43. Integration on the unit circle.

Let ω be a 1-form on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and

$$f : \mathbb{R} \to \mathbb{S}^1,\ t \mapsto (\cos t, \sin t)$$

the standard paramterisation.

(a) Show that $\int_{\mathbb{S}^1} \omega = \int_{[0,2\pi]} f^*\omega$.
Hint. Obviously we want to use Corollary 3.22, but we can not so do immediately, because $f|[0, 2\pi]$ is not injective. However ....

(b) Prove from (a) Stokes' theorem for $\mathbb{S}^1$. Actually, show the stronger result that ω is exact if and only if

$$\int_{\mathbb{S}^1} \omega = 0.$$

($\mathbb{S}^1$ is a manifold whose boundary is empty, so the right side of Stokes' theorem is zero.)

Terminology
zurückziehen = pullback.