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Analysis III
12. Exercises

1 June 2021

36. The wedge product

In this exercise we will do some calculations with the antisymmetric algebras of $V = \mathbb{R}^3$. Let e_i be a basis of V and α_i the corresponding dual basis of V'.

(a) Show that $\beta = \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1$ belongs to $\bigwedge^2 V'$.

(b) Consider the antisymmeterising operation $\mathcal{A}$ defined in the proof of Theorem 3.4. Compute $\mathcal{A}^1(\alpha_3) \in \bigwedge^1 V'$ and $\mathcal{A}^2(\alpha_1 \otimes \alpha_2) \in \bigwedge^2 V'$.

(c) Compute $\langle \beta, e_1 \otimes e_3 \rangle = \beta(e_1, e_3)$ using the definition of tensors as linear maps and also using the formula on page 71 of the script (notice that $\beta = \alpha_1 \wedge \alpha_2$).

(d) Compute the wedge product $\beta \wedge \alpha_3$.

(e) In the proof of Theorem 3.4, it is proved that the wedge products span the space of antisymmetric tensors by a dimension count argument. Given an antisymmetric tensor, can you find an algorithm to write it as a sum of wedge products of basis elements?

Solution.

(a) The only non-trivial element of the symmetric group S_2 is the transposition (12). More generally, the symmetric group is generated by the $n-1$ adjacent transpositions and it is sufficient to check the relation holds for these. The sign of transpositions is by definition -1.

$$(12).\beta = \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_2 = -\beta$$

(b) The only element of S_1 is the identity:

$$\mathcal{A}^1(\alpha_3) = \sum_{\sigma \in S_1} \operatorname{sign}(\sigma)\sigma.\alpha_3 = 1\,\mathrm{id}.\alpha_3 = \alpha_3.$$

The two elements of S_2 are the identity and (12)

$$\begin{aligned}\mathcal{A}^2(\alpha_1 \otimes \alpha_2) &= \sum_{\sigma \in S_2} \operatorname{sign}(\sigma)\sigma.(\alpha_1 \otimes \alpha_2) \\ &= \mathrm{id}.(\alpha_1 \otimes \alpha_2) - (12).(\alpha_1 \otimes \alpha_2) \\ &= \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1 = \beta.\end{aligned}$$

(c) Using the definition as linear maps

$$\beta(e_1, e_3) = \alpha_1(e_1) \cdot \alpha_2(e_3) - \alpha_2(e_1) \cdot \alpha_1(e_3) = 1 \cdot 0 - 0 \cdot 0 = 0.$$

On the other hand, we have that $\beta = \frac{1}{1!1!}\mathcal{A}(\alpha_1 \otimes \alpha_2) = \alpha_1 \wedge \alpha_2$ Using the determinant formula

$$\langle \beta, e_1 \otimes e_3 \rangle = \begin{vmatrix} \alpha_1(e_1) & \alpha_1(e_3) \\ \alpha_2(e_1) & \alpha_2(e_3) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

The number is not so interesting, but notice if we expanded the determinant as a formula we would get the linear maps version.

(d)

$$\begin{aligned}
\beta \wedge \alpha_3 &= \frac{1}{2!1!}\mathcal{A}^3(\beta \otimes \alpha_3) \\
&= \frac{1}{2}\mathcal{A}^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3) \\
\mathcal{A}^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) &= (1 - (12) - (13) - (23) + (123) + (132)).(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \\
&= \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_3 \otimes \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_3 \otimes \alpha_2 \\
&\quad + \alpha_2 \otimes \alpha_3 \otimes \alpha_1 + \alpha_3 \otimes \alpha_1 \otimes \alpha_2 \\
\mathcal{A}^3(\alpha_2 \otimes \alpha_1 \otimes \alpha_3) &= (1 - (12) - (13) - (23) + (123) + (132)).(\alpha_2 \otimes \alpha_1 \otimes \alpha_3) \\
&= \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_3 \otimes \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_3 \otimes \alpha_1 \\
&\quad + \alpha_1 \otimes \alpha_3 \otimes \alpha_2 + \alpha_3 \otimes \alpha_2 \otimes \alpha_1 \\
&= -\mathcal{A}^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \\
\beta \wedge \alpha_3 &= \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_3 \otimes \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_3 \otimes \alpha_2 \\
&\quad + \alpha_2 \otimes \alpha_3 \otimes \alpha_1 + \alpha_3 \otimes \alpha_1 \otimes \alpha_2
\end{aligned}$$

(e) First we give an overview of the idea. Suppose we are given an antisymmetric tensor as a sum of pure tensors of basis elements. Our procedure is to choose one pure tensor from this, apply $\mathcal{A}$ to it and then subtract the result. This must reduce the number of pure tensors in the sum, so after a finite number of iterations we have written our antisymmetric tensor as a sum of $\mathcal{A}$ applied to pure tensors. But this is exactly a sum of wedge products.

More precisely, suppose that $\beta \in \bigwedge^p V'$. We can write it as

$$\beta = \sum \beta_{i_1,\dots,i_p} \alpha_{i_1} \otimes \cdots \otimes \alpha_{i_p}.$$

Choose a non-zero term of this sum $\beta_{j_1,\dots,j_p}\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$ and apply $\mathcal{A}^p$ to it to get the antisymmetric tensor $A_1 := \beta_{j_1,\dots,j_p}\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_p}$. Consider $\beta - A_1$. I claim that all terms whose indices are a permutation of $\{j_1, \dots, j_p\}$ are zero. Suppose for contradiction that its $\sigma.\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$ term was non-zero. But $\sigma.(\beta - A_1) = \operatorname{sign}\sigma(\beta - A_1)$ because they are antisymmetric tensors. This means the coefficient of the $\sigma.\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$ term is equal to $\operatorname{sign}\sigma$ times the coefficient of the $\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_p}$ term, which is $\beta_{j_1,\dots,j_p} - \beta_{j_1,\dots,j_p} = 0$ by construction.

Thus $\beta - A_1$ is again an antisymmetric tensor, with fewer non-zero terms. Iterating this procedure we get that

$$\beta = A_1 + \cdots + A_k = \sum \beta_{j_{1,k},\ldots,j_{p,k}} \alpha_{j_{1,k}} \wedge \cdots \wedge \alpha_{j_{p,k}}$$

is a sum of wedge products of basis elements.

Let us give an example of this procedure. Consider the antisymmetric tensor

$$\beta = 3\alpha_1 \otimes \alpha_2 - 2\alpha_1 \otimes \alpha_3 - 3\alpha_2 \otimes \alpha_1 + 2\alpha_2 \otimes \alpha_3 + 2\alpha_3 \otimes \alpha_1 - 2\alpha_3 \otimes \alpha_2.$$

Then we choose the $\alpha_1 \otimes \alpha_2$ term

$$A_1 = 3\alpha_1 \wedge \alpha_2 = \mathcal{A}(3\alpha_1 \otimes \alpha_2) = 3\alpha_1 \otimes \alpha_2 - 3\alpha_2 \otimes \alpha_1$$

and so

$$\beta - A_1 = -2\alpha_1 \otimes \alpha_3 + 2\alpha_2 \otimes \alpha_3 + 2\alpha_3 \otimes \alpha_1 - 2\alpha_3 \otimes \alpha_2$$

does indeed have fewer terms. Now we choose the $\alpha_1 \otimes \alpha_3$ term

$$A_2 = -2\alpha_1 \wedge \alpha_3 = -2\alpha_1 \otimes \alpha_3 + 2\alpha_3 \otimes \alpha_1$$

so

$$\beta - A_1 - A_2 = 2\alpha_2 \otimes \alpha_3 - 2\alpha_3 \otimes \alpha_2.$$

We can repeat this one more time with $A_3 = 2\alpha_2 \wedge \alpha_3$ to exhaust the non-zero terms. This gives at the end

$$\beta = A_1 + A_2 + A_3 = 3\alpha_1 \wedge \alpha_2 - 2\alpha_1 \wedge \alpha_3 + 2\alpha_2 \wedge \alpha_3.$$

37. Dual 1-forms to a vector field (using dot product).

Let $F \in \mathrm{Vec}^\infty(\mathbb{R}^3)$ be a smooth vector field on $\mathbb{R}^3$ that is nowhere vanishing. Find a tensor field α of $T'\mathbb{R}^3$ (a 1-differential form on $\mathbb{R}^3$), so that the kernel of α at every point is orthogonal to F. Orthogonal means using the dot product of $\mathbb{R}^3$

Solution. Let e_i be the standard basis of $\mathbb{R}^3$ which gives a non-vanishing vector field. Thus we can write $F(x) = \sum F_i(x)e_i$. This induces dual basis fields α_j on $T'\mathbb{R}^3$, which act as $\alpha_j(F) = F_j(x)$. Any 1-form α can be written as $\alpha(x) = \sum a_j(x)\alpha_j$ for smooth functions a_j.

What we require in this question is to find a form, such that $G \cdot F = 0$ at every point if and only if $\alpha(G) \equiv 0$. Observe

$$\alpha(G)(x) = \sum a_j(x)G_j(x) = a(x) \cdot G(x)$$

where we treat the coefficients of α as the coefficients of a vector field. Clearly then we should take $\alpha(x) = \sum F_j(x)\alpha_j$ as the 1-form.

One point to see here is that for a vector field F in Euclidean space $\langle F, \cdot \rangle$ is a 1-form. This explains why we also use inner product notation for the pairing between forms and fields.

38. Local representations of tensor fields.

Let X be an n-dimensional smooth manifold and f a tensor field in $T^q_p X$. To simplify the notation, we will only consider the case $p = 0, q = 2$, but for other tensor spaces everything holds completely analogously. Further, let (U, ϕ) be a chart of X and denote the components of ϕ by $\phi_1, \dots, \phi_n : U \to \mathbb{R}$. These induce 1-forms $\alpha_k := \mathrm{d}\phi_k$ in $T^1_0 U \subset T^1_0 X$. By definitions these 1-forms act as $\alpha_k(x)(v) = T_x(\phi_k)(v)$ for every $x \in U$ und $v \in T_x X$.

General Hint. It will be useful to consider the (local) vector fields E_k on U

$$E_k(x) = T_x(\phi)^{-1}(e_k).$$

These are often called the coordinate vector fields.

(a) Show there exist functions $f_{k,l} : U \to \mathbb{R}$, so that

$$f|U = \sum_{k,l=1}^{n} f_{k,l} \cdot \alpha_k \otimes \alpha_l.$$

More precisely, we mean that for all $x \in U$ and $v, w \in T_x X$

$$f(x)(v, w) = \sum_{k,l=1}^{n} f_{k,l}(x)\, \alpha_k(x)(v)\, \alpha_l(x)(w).$$

Show further that these functions are unique. This way of describing f is called representing f in local coordinates. The functions $f_{k,l}$ are called the coefficient functions.

(b) Show that $f|U$ is smooth exactly when the functions $f_{k,l}$ are all smooth.

(c) Show that f is a 2-form exactly when the coefficients are antisymmetric: $f_{k,l} = -f_{l,k}$.

Solution.

(a) Note that the 1-forms form a local trivialisation of $T^1_0 U$: Due to the relation

$$d\phi_i(x)(E_j(x)) = T_x(\phi_i)(T_x(\phi)^{-1}(e_j)) = T_x(\pi_i \circ \phi)(T_x(\phi)^{-1}(e_j)) = \pi_i(e_j) = \delta_{i,j}$$

it follows that the are non-vanishing at every point and linearly independent. Therefore $\alpha_i(x) \otimes \alpha_j(x)$ for a basis for $T^2_{0,x} X$ and hence every $(0, 2)$ tensor $f(x)$ is a unique linear combination of them.

(b) If the coefficient functions are all smooth, so is the sum, because the basis sections α_j are smooth since they are the exterior derivative of smooth functions (Theorem 3.14). In the converse direction, the coefficient functions may be realised as the projections of f with respect to the coordinate chart, but this is the composition of smooth functions.

(c) f is a 2-form when it is antisymmetric in every fibre. So the question is really: when is a 2-tensor $\sum a_{i,j} e_i \otimes e_j$ antisymmetric? There is only one non-trivial element of S_2 to consider:

$$-\sum a_{i,j} e_i \otimes e_j = (12).(\sum a_{i,j} e_i \otimes e_j) = \sum a_{i,j} e_j \otimes e_i = \sum a_{j,i} e_i \otimes e_j$$

from which we require $a_{i,j} = -a_{j,i}$.

39. Closed and exact differential forms.

A p-form ω on a manifold X is called closed if $\mathrm{d}\omega = 0$, and it is called exact if there is a $(p-1)$-form θ on X with $\omega = \mathrm{d}\theta$.

Consider $X = \mathbb{R}^3$ and let $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ be the usual coordinate functions. Investigate whether the following forms are closed and or exact.

(a) $\omega = yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z$

(b) $\omega = x\,\mathrm{d}x + x^2y^2\,\mathrm{d}y + yz\,\mathrm{d}z$

(c) $\omega = 2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$

Solution. Before we begin, recall Theorem 3.14(iii): $d(df) = 0$ for a function f, and for part (c) the more general result of Theorem 3.15(i): $d(d\alpha) = 0$ for any differential form α. This implies that any exact form is closed, or that a form cannot be exact if it is not closed. This can be an effective test.

(a)

$$\begin{aligned}
&d(yz\,dx + xz\,dy + xy\,dz) \\
&\quad = d(yz) \wedge dx + d(xz) \wedge dy + d(xy) \wedge dz \\
&\quad = (ydz + zdy) \wedge dx + (xdz + zdx) \wedge dy + (xdy + ydx) \wedge dz \\
&\quad = ydz \wedge dx + zdy \wedge dx + xdz \wedge dy + zdx \wedge dy + xdy \wedge dz + ydx \wedge dz \\
&\quad = -ydx \wedge dz - zdx \wedge dy - xdy \wedge dz + zdx \wedge dy + xdy \wedge dz + ydx \wedge dz \\
&\quad = 0.
\end{aligned}$$

So ω is closed. That means it might be exact (but in general it doesn't have to be). In this case it's relatively easy to see that $\omega = d(xyz)$.

(b)

$$\begin{aligned} d(x\,\mathrm{d}x + x^2y^2\,\mathrm{d}y + yz\,\mathrm{d}z) &= dx \wedge dx + d(x^2y^2) \wedge dy + d(yz) \wedge dz \\ &= 0 + (2xy^2dx + 2x^2ydy) \wedge dy + (zdy + ydz) \wedge dz \\ &= 2xy^2dx \wedge dy + 0 + zdy \wedge dz + 0. \end{aligned}$$

So ω is not closed, and therefore cannot be exact.

(c)

$$\begin{aligned} d(2xy^2\,\mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z) &= d(2xy^2) \wedge dx \wedge dy + dz \wedge dy \wedge dz \\ &= (2y^2dx + 4xydy) \wedge dx \wedge dy + 0 \\ &= 0. \end{aligned}$$

So ω is closed. Let $\theta = adx + bdy + cdz$ for functions a, b, c. Then we must try to solve $d\theta = \omega$. I will use subscripts for the partial derivatives.

$$\begin{aligned} d\theta &= (a_xdx + a_ydy + a_zdz) \wedge dx + (b_xdx + b_ydy + b_zdz) \wedge dy + (c_xdx + c_ydy + c_zdz) \wedge dz \\ &= -a_ydx \wedge dy - a_zdx \wedge dz + b_xdx \wedge dy - b_zdy \wedge dz + c_xdx \wedge dz + c_ydy \wedge dz \\ &= (b_x - a_y)dx \wedge dy + (c_x - a_z)dx \wedge dz + (c_y - b_z)dy \wedge dz. \end{aligned}$$

Hence we must find functions that satisfy $b_x - a_y = 2xy^2$, $c_x - a_z = 0$ and $c_y - b_z = z$. Assume that $c = 0$. That means that a is a function of x, y only and $b(x, y, z) = -\frac{1}{2}z^2 + \tilde{b}(x, y)$. Then we only need to further satisfy the first equation

$$\tilde{b}_x - a_y = 2xy^2.$$

This can be done by choosing $b = -\frac{1}{2}z^2 + x^2y^2$ and $a = 0$. So $\theta = (-\frac{1}{2}z^2 + x^2y^2)dy$ shows that ω is exact.

Note, this is not the only solution: if f is any function then $d(\theta + df) = d\theta = \omega$.