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Analysis III
11. Exercises

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In the exercises below, let $V, V_1, \dots, V_n, W$ be finite dimensional normed vector spaces over $\mathbb{K}$.

33. (a) **Dimension of $\mathcal{L}(V_1, \dots, V_n; W)$.**

Show that

$$\dim \mathcal{L}(V_1, \dots, V_n; W) = \dim(V_1) \cdot \ldots \cdot \dim(V_n) \cdot \dim(W)$$

Solution. We give two approaches.

1. Note that $\mathcal{L}(V_1; W)$ is the familiar space of linear maps from V_1 to W, each of which can be written as a $\dim W$ rows by $\dim V_1$ columns matrix. Thus it is a vector space of $\dim V_1 \cdot \dim W$. Next, notice that for $A \in \mathcal{L}(V_1, \dots, V_n; W)$ the map $x_1 \mapsto A(x_1, \cdot)$ is a linear map from V_1 to $\mathcal{L}(V_2, \dots, V_n; W)$, and conversely every such linear map give an element of $\mathcal{L}(V_1, \dots, V_n; W)$. The formula follows by induction.

2. We can also give an explicit basis to this space. Let $e_{i,j}$ for $i = 1, \dots, \dim V_j$ be a basis of V_j. Then every vector in $V_1 \times \ldots \times V_n$ can be written as $v = (\sum_{i=1}^{\dim V_j} c_{i,j} e_{i,j})_j$. Choose a linear map $A \in \mathcal{L}(V_1, \dots, V_n; W)$. It acts on v as

$$\begin{aligned}
A(\sum_{i=1}^{\dim V_j} c_{i,j} e_{i,j}) &= A(c_{1,1} e_{1,1} + \sum_{i_1=2}^{\dim V_1} c_{i_1,1} e_{i_1,1}, \sum_{i_2=1}^{\dim V_2} c_{i_2,2} e_{i_2,2}, \dots) \\
&= A(c_{1,1} e_{1,1}, \sum_{i_2=1}^{\dim V_2} c_{i_2,2} e_{i_2,2}, \dots) + A(\sum_{i_1=2}^{\dim V_1} c_{i_1,1} e_{i_1,1}, \sum_{i_2=1}^{\dim V_2} c_{i_2,2} e_{i_2,2}, \dots) \\
&= \sum_{i_1=1}^{\dim V_1} A(c_{i_1,1} e_{i_1,1}, \sum_{i_2=1}^{\dim V_2} c_{i_2,2} e_{i_2,2}, \dots) \\
&= \sum_{i_1=1}^{\dim V_1} \sum_{i_2=1}^{\dim V_2} A(c_{i_!,1} e_{i_1,1}, c_{i_2,2} e_{i_2,2}, \sum_{i_3=1}^{\dim V_3} c_{i_3,3} e_{i_3,3}, \dots) \\
&= \sum_{i_1=1}^{\dim V_1} \sum_{i_2=1}^{\dim V_2} \cdots \sum_{i_n=1}^{\dim V_n} A(c_{i_1,1} e_{i_1,1}, c_{i_2,2} e_{i_2,2}, \dots, c_{i_n,n} e_{i_n,n}) \\
&= \sum_{i_1=1}^{\dim V_1} \sum_{i_2=1}^{\dim V_2} \cdots \sum_{i_n=1}^{\dim V_n} c_{i_1,1} c_{i_2,2} \ldots c_{i_n,n} A(e_{i_1,1}, e_{i_2,2}, \dots, e_{i_n,n})
\end{aligned}$$

What this shows is that the effect of A on a vector is determined by the values $A(e_{i_1,1}, e_{i_2,2}, \dots, e_{i_n,n}) \in W$. Conversely, if you specify these values then there is a unique multi-linear map A. Since there are $\dim(V_1) \cdot \ldots \cdot \dim(V_n)$ different values to be chosen from a $\dim W$ dimensional space, this shows the formula.

It might be a useful extra exercise to compare how the linear maps $\mathcal{L}(V_1 \times V_2; W)$ are different from the multilinear maps $\mathcal{L}(V_1, V_2; W)$

(b) Alternative description of the norm on $\mathcal{L}(V_1, \ldots, V_n; W)$.

In the lectures we saw that the norm on $\mathcal{L}(V_1, \ldots, V_n; W)$ was defined as

$$\|A\| := \sup\{ \|A(x_1, \ldots, x_n)\| \,|\, x_k \in V_k,\ \|x_k\| \leq 1 \}.$$

Prove the following alternative characterisations are equivalent:

$$\begin{aligned}\|A\| &= \sup\{ \|A(x_1, \ldots, x_n)\| \,|\, x_k \in V_k,\ \|x_k\| = 1 \} \\ &= \sup\left\{ \left\|A\left(\tfrac{x_1}{\|x_1\|}, \ldots, \tfrac{x_n}{\|x_n\|}\right)\right\| \,\middle|\, x_k \in V_k \setminus \{0\} \right\} .\end{aligned}$$

Solution. Note that $A(x_1, \ldots, x_n)$ is zero if any of the input vectors are zero. So we may consider only non-zero vectors. Consider for $\|x_k\| \leq 1$

$$\|A(x_1, \ldots, x_n)\| = \|x_1\| \ldots \|x_n\| \, \|A(\hat{x}_1, \ldots, \hat{x}_n)\| \leq \|A(\hat{x}_1, \ldots, \hat{x}_n)\|$$

With equality if and only if $\|x_k\| = 1$ for all k. This shows the equality of (1) and (2). (2) and (3) are equal for the reason that they consider precisely the same set of values.

(c) An isomorphism between $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$.

Show that the map

$$\Phi : \mathcal{L}(V; W) \to \mathcal{L}(V, W'; \mathbb{K}),\ A \mapsto \Phi(A)$$

with

$$\Phi(A) : V \times W' \to \mathbb{K},\ (v, B) \mapsto (B \circ A)(v)$$

is an isomorphism between the normed vector spaces $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$, i.e. Φ is a vector space isomorphism and it preserves the respective norms: for all $A \in \mathcal{L}(V; W)$ we have $\|\Phi(A)\| = \|A\|$.

Solution. First note that Φ is a linear map

$$\begin{aligned}\Phi(A + \lambda\tilde{A})(v, B) &= B\Big((A + \lambda\tilde{A})(v)\Big) = B\Big(A(v) + \lambda\tilde{A}(v)\Big) = B(A(v)) + \lambda B(\tilde{A}(v)) \\ &= \Phi(A)(v, B) + \lambda\Phi(\tilde{A})(v, B).\end{aligned}$$

Next suppose that $\Phi(A)$ is the zero map for some A. Let v_i, w_j and w'_k be (unit length) bases of V, W and W', with $w'_k(w_j) = \delta_{i,j}$. Write $A(v_i) = \sum a_{i,j} w_j$ in matrix form. Then

$$0 = \Phi(A)(v_i, w'_k) = w'_k \left(\sum a_{i,j} w_j \right) = a_{i,k},$$

which shows $A = 0$. Thus Φ is injective. On the other hand, we know the dimensions of the two spaces are equal, so Φ must be surjective. Therefore we have shown that Φ is a vector space isomorphism.

Finally we must show that it preserves the norms. Observe $|B(w)| \leq \|B\| \, \|w\|$ by the properties of norms of linear operators. Choose $A \in \mathcal{L}(V; W)$. We proceed

$$\|\Phi(A)\|_{\mathcal{L}(V,W';\mathbb{K})} = \sup\{|B(A(v))| \mid \|B\| = 1, \|v\| = 1\} \leq \sup\{\|A(v)\| \mid \|v\| = 1\} = \|A\|_{\mathcal{L}(V;W)}.$$

On the other hand, these are finite dimensional spaces, so $\{\|A(v)\| \mid \|v\| = 1\}$ actually has a maximum at say $\tilde{v}$, and if A is not the zero operator then $A(\tilde{v}) \in W$ is nonzero. Let w_1 be the unit length normalisation of $A(\tilde{v})$ and extend this to a basis $w_2, \ldots, w_{\dim W}$ of W. Now let $\tilde{B}$ be the dual element. That is $\tilde{B}(w_1) = 1$ and $\tilde{B}(w_j) = 0$ for $j \geq 2$. Both $\tilde{B}$ and $\tilde{v}$ are unit length, so

$$\|\Phi(A)\|_{\mathcal{L}(V,W';\mathbb{K})} = \sup\{|B(A(v))| \mid \|B\| = 1, \|v\| = 1\} \geq |\tilde{B}(A(\tilde{v}))| = |A(\tilde{v})| = \|A\|_{\mathcal{L}(V;W)}.$$

This shows that Φ is norm preserving.

34. The tensor product.

(a) Prove or disprove:

(i) the tensor product of vectors

$$V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n, \; (v_1, \ldots, v_n) \mapsto v_1 \otimes \ldots \otimes v_n$$

is not commutative in the case $V_1 = \ldots = V_n$.

(ii) every vector in $V_1 \otimes \ldots \otimes V_n$ is *pure (coherent).*

Solution.

(i) This is true, the tensor product is not commutative even when the vector spaces are all the same. Consider $n = 2$ and $V = \mathbb{R}^2$ with the standard basis e_1, e_2. Let the dual space V' have the dual basis α_1, α_2. By the construction of the double dual, V acts on V' by $v(\alpha) := \alpha(v)$. Let's apply Definition 3.2 to the following two tensors

$$\begin{aligned}
&e_1 \otimes e_2, e_2 \otimes e_1 : V' \times V' \to \mathbb{R} \\
&e_1 \otimes e_2(\alpha_1, \alpha_2) := e_1(\alpha_1) \cdot e_2(\alpha_2) = \alpha_1(e_1) \cdot \alpha_2(e_2) = 1, \\
&e_2 \otimes e_1(\alpha_1, \alpha_2) := e_2(\alpha_1) \cdot e_1(\alpha_2) = \alpha_1(e_2) \cdot \alpha_2(e_1) = 0,
\end{aligned}$$

so clearly they are different tensors.

(ii) This is false. Let's continue the example from the previous part. I claim that $t = e_1 \otimes e_2 - e_2 \otimes e_1$ is not a pure tensor. Let it act on two arbitrary vectors of V', namely $\alpha = a_1\alpha_1 + a_2\alpha_2$ and $\beta = b_1\alpha_1 + b_2\alpha_2$:

$$t(\alpha, \beta) = e_1(\alpha)e_2(\beta) - e_2(\alpha)e_1(\beta) = a_1b_2 - a_2b_1.$$

However, a pure tensor would produce

$$(c_1e_1 + c_2e_2) \otimes (d_1e_1 + d_2e_2)(\alpha, \beta) = (c_1a_1 + c_2a_2)(d_1b_1 + d_2b_2)$$
$$= c_1d_1a_1b_1 + c_2d_1a_2b_1 + c_1d_2a_1b_2 + c_2d_2a_2b_2$$

So we would need for $c_1d_2 = 1$ and $c_2d_1 = -1$ but also $c_1d_1 = 0$. This is not possible.

(b) Show that in $V_1 \otimes \ldots \otimes V_n$ the linear span of the pure tensors is $V_1 \otimes \ldots \otimes V_n$, ie. every element of $V_1 \otimes \ldots \otimes V_n$ is a finite linear combination of the pure tensors.

Solution. We have already seen in 33(a) that an element A of $\mathcal{L}(V_1, \ldots, V_n; \mathbb{K})$ is exactly determined by its values $A(e_{i_1,1}, e_{i_2,2}, \ldots, e_{i_n,n})$. But

$$\alpha_{j_1,1} \otimes \cdots \otimes \alpha_{j_n,n}(e_{i_1,1}, e_{i_2,2}, \ldots, e_{i_n,n}) = \delta_{i_1,j_1} \cdot \cdots \cdot \delta_{i_n,j_n}.$$

This allows us to write

$$A = \sum_{i_1=1}^{\dim V_1} \sum_{i_2=1}^{\dim V_2} \cdots \sum_{i_n=1}^{\dim V_n} A(e_{i_1,1}, e_{i_2,2}, \ldots, e_{i_n,n}) \alpha_{i_1,1} \otimes \cdots \otimes \alpha_{i_n,n}.$$

Thus we have written every element of $\mathcal{L}(V_1, \ldots, V_n; \mathbb{K}) = V_1' \otimes \ldots V_n'$ as a sum of pure tensors. Conversely, pure tensors are multilinear maps, and so so are their linear combinations.

(c) Construct the following isomorphisms between normed vector spaces:

(i) $\mathcal{L}(V_1, \ldots, V_n; W) \cong \mathcal{L}(V_1 \otimes \ldots \otimes V_n; W)$

(ii) $V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$

(iii) $\mathcal{L}(V; W) \cong V' \otimes W$.

Solution. In each case we will name the isomorphism from left to right as Φ. The question says only to construct the isomorphism and I am feeling a little lazy, so I will omit the proof that it is in fact an isomorphism.

(ii) We will prove the first isomorphism. By definition $V_1 \otimes V_2 \otimes V_3 = \mathcal{L}(V_1', V_2', V_3'; \mathbb{K})$. On the other hand, we have seen that the space of multi-linear maps can be understood inductively as linear maps into the space of multi-linear maps. Hence

$$\mathcal{L}(V_1', V_2', V_3'; \mathbb{K}) \cong \mathcal{L}(V_1'; \mathcal{L}(V_2', V_3'; \mathbb{K})) = \mathcal{L}(V_1'; V_2 \otimes V_3) \cong \mathcal{L}(V_1', (V_2 \otimes V_3)'; \mathbb{K})$$
$$= V_1 \otimes (V_2 \otimes V_3).$$

(iii) This is probably the most useful isomorphism, because it enables us to reduce spaces of linear maps to tensor products, and I find tensor products easier. Simply $V' \otimes W = \mathcal{L}(V, W'; \mathbb{K}) \cong \mathcal{L}(V; W)$.

(i) First, let us show that dualising distributes over the tensor product: $(V \otimes W)' = V' \otimes W'$. This follows since

$$(V \otimes W)' \cong \mathcal{L}(V'; W)' = \mathcal{L}(W; V') \cong \mathcal{L}(V, W; \mathbb{K}) = V' \otimes W'.$$

(Perhaps it is also a good exercise as to why the dual of the linear maps from V to W is the linear maps from W to V)

We can now prove (i) using (ii) and (iii). I'll show only the proof in the case $n = 2$, higher n follow similarly by induction.

$$\begin{aligned}
\mathcal{L}(V_1, V_2; W) &\cong \mathcal{L}(V_1; \mathcal{L}(V_2; W)) = \mathcal{L}(V_1; V_2' \otimes W) = V_1' \otimes (V_2' \otimes W) \\
&\cong V_1' \otimes V_2' \otimes W \\
\mathcal{L}(V_1 \otimes V_2; W) &\cong \mathcal{L}(V_1 \otimes V_2, W'; \mathbb{K}) = (V_1 \otimes V_2)' \otimes W \cong (V_1' \otimes V_2') \otimes W \\
&\cong V_1' \otimes V_2' \otimes W
\end{aligned}$$

35. Riemannian metric.

Let X be a manifold. Let $L(TX, TX; \mathbb{R})$ denote the vector bundle whose fibre over $x \in X$ is the $\mathbb{R}$-vector space of bilinear forms $T_xX \times T_xX \to \mathbb{R}$. A Riemannian metric (or simply a metric) on X is a global smooth section G of this vector bundle, such that $g(x)$ is a scalar product on T_xX for ever $x \in X$ (it is symmetric and positive definite).

Show that every manifold has a Riemannian metric.

Hint. Choose a cover of X by coordinate charts. Construct a Riemannian metric in each coordinate chart. 'Glue' them all together using a partition of unity.

Solution. Let us first do this in a single coordinate chart $\phi : U \to \mathbb{R}^n$. Then we know that $T(\phi)$ is a diffeomorphism between TU and TR^n. This has an obvious Riemannian metric, namely the dot product. Explicitly, if $v, w \in T_xU$, then we define

$$g(x)(v, w) = T_x(\phi)v \cdot T_x(\phi)w.$$

We see that this very much depends on the choice of chart.

Now let X be covered by an atlas $\mathcal{A}$ and let (α_α) be a subordinate partition of unity. In each coordinate neighbourhood U_α we have a Riemannian metric g_α. Let $g(x) = \sum \alpha_\alpha(x) g_\alpha(x)$. This is a well-defined global section, because α_α vanishes outside U_α and at any point at most finitely many of the terms are non-zero. Bilinearity and symmetry are also immediate, because the sum of symmetric bilinear forms is again a symmetric bilinear form. It remains to show positive definiteness. But the partition of unity is non-negative, so $g(x)$ must be non-negative. Suppose that $v \in T_xX$ is a non-zero vector. There must be at least one α_α that does not vanish at x because they sum to 1, so it follows that

$$g(x)(v,v) = \sum \alpha_\alpha(x) g_\alpha(x)(v,v) \geq \alpha_0(x) g_0(x)(v,v) > 0.$$

This shows positive definiteness.

Terminology
köherent = coherent. In English, we called these tensors pure, simple, or elementary.