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Analysis III
11. Exercises

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In the exercises below, let $V, V_1, \ldots, V_n, W$ be finite dimensional normed vector spaces over $\mathbb{K}$.

33. (a) **Dimension of $\mathcal{L}(V_1, \ldots, V_n; W)$.**
Show that

$$\dim \mathcal{L}(V_1, \ldots, V_n; W) = \dim(V_1) \cdot \ldots \cdot \dim(V_n) \cdot \dim(W)$$

(b) **Alternative description of the norm on $\mathcal{L}(V_1, \ldots, V_n; W)$.**
In the lectures we saw that the norm on $\mathcal{L}(V_1, \ldots, V_n; W)$ was defined as

$$\|A\| := \sup\{ \|A(x_1, \ldots, x_n)\| \mid x_k \in V_k,\ \|x_k\| \leq 1 \}.$$

Prove the following alternative characterisations are equivalent:

$$\begin{aligned} \|A\| &= \sup\{ \|A(x_1, \ldots, x_n)\| \mid x_k \in V_k,\ \|x_k\| = 1 \} \\ &= \sup \left\{ \left\| A\left(\tfrac{x_1}{\|x_1\|}, \ldots, \tfrac{x_n}{\|x_n\|}\right) \right\| \,\middle|\, x_k \in V_k \setminus \{0\} \right\} . \end{aligned} \tag{1}$$

(c) **An isomorphism between $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$.**
Show that the map

$$\Phi : \mathcal{L}(V; W) \to \mathcal{L}(V, W'; \mathbb{K}),\ A \mapsto \Phi(A)$$

with

$$\Phi(A) : V \times W' \to \mathbb{K},\ (v, B) \mapsto (B \circ A)(v)$$

is an isomorphism between the normed vector spaces $\mathcal{L}(V; W)$ and $\mathcal{L}(V, W'; \mathbb{K})$, i.e. Φ is a vector space isomorphism and it preserves the respective norms: for all $A \in \mathcal{L}(V; W)$ we have $\|\Phi(A)\| = \|A\|$.

34. The tensor product.

(a) Prove or disprove:

(i) the tensor product of vectors

$$V_1 \times \ldots \times V_n \to V_1 \otimes \ldots \otimes V_n,\ (v_1, \ldots, v_n) \mapsto v_1 \otimes \ldots \otimes v_n$$

is not commutative in the case $V_1 = \ldots = V_n$.

(ii) every vector in $V_1 \otimes \ldots \otimes V_n$ is *pure (coherent).*

(b) Show that in $V_1 \otimes \ldots \otimes V_n$ the linear span of the pure tensors is $V_1 \otimes \ldots \otimes V_n$, ie. every element of $V_1 \otimes \ldots \otimes V_n$ is a finite linear combination of the pure tensors.

(c) Construct the follow isomorphisms between normed vector spaces:

(i) $\mathcal{L}(V_1, \ldots, V_n; W) \cong \mathcal{L}(V_1 \otimes \ldots \otimes V_n; W)$

(ii) $V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$

(iii) $\mathcal{L}(V; W) \cong V' \otimes W$.

35. Riemannian metric.

Let X be a manifold. Let $L(TX, TX; \mathbb{R})$ denote the vector bundle whose fibre over $x \in X$ is the $\mathbb{R}$-vector space of bilinear forms $T_xX \times T_xX \to \mathbb{R}$. A Riemannian metric (or simply a metric) on X is a global smooth section G of this vector bundle, such that $g(x)$ is a scalar product on T_xX for ever $x \in X$ (it is symmetric and positive definite).

Show that every manifold has a Riemannian metric.

Hint. Choose a cover of X by coordinate charts. Construct a Riemannian metric in each coordinate chart. 'Glue' them all together using a partition of unity.

Terminology
köherent = coherent. In English, we called these tensors pure, simple, or elementary.