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Analysis III
10. Exercises

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29. The computation of the Lie Bracket for submanifolds of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and $F, G \in \mathrm{Vec}^\infty(X)$. With the help of Theorem 2.22(iii),(iv) devise a formula to compute $[F,G]$ similar to Exercise 22. Prove your formula.

Solution. Using Theorem 2.22(iii), extend F and G to vector fields on $\mathbb{R}^n$ called $\tilde{F}, \tilde{G}$. Then by Theorem 2.22(iii) and Exercise 22 we have that

$$[F,G]_X = [\tilde{F},\tilde{G}]_{\mathbb{R}^n} = \tilde{G}'\tilde{F} - \tilde{F}'\tilde{G}.$$

So here we already have a formula that avoids using coordinate charts. There is the practical question of how to find extensions of vector fields on X. Many times, the formulas for the vector fields will still be valid. For example, in the solution of Exercise 23(b) we computed the Lie bracket of fields on $\mathbb{S}^3$, but these fields were already coming from $\mathbb{R}^4$. This exercise explains why our calculation in 23(b) agreed with the one in 22.

If you are in the situation where there is not an easy extension, here is a practical way to construction one. Choose a point $x \in X$. Because X is a submanifold, we know that locally X is the graph of a function $h : U \to \mathbb{R}^{n-k}$. For simplicity, assume it is a graph over the coordinates $y = (x_1, \dots, x_k)$. In other words, $y \mapsto (y, h(y))$ is the inverse of the chart $\phi(x) = (x_1, \dots, x_k)$ of X. Thus we can write the vector fields $F(y,h(y)), G(y,h(y))$ in this neighbourhood as functions of y alone. Then $\tilde{F}(x) := F(y,h(y))$ is an extension of F to $U \times \mathbb{R}^{n-k}$, and likewise for $\tilde{G}$. The advantage of this choice of extension is that they are constant in the variables $x_{k+1}, \dots, x_n$, so for example

$$\tilde{G}'\tilde{F} = \begin{pmatrix} \frac{\partial \tilde{G}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{G}_1}{\partial x_k} & 0\dots0 \\ \vdots & & \vdots & \\ \frac{\partial \tilde{G}_n}{\partial x_1} & \cdots & \frac{\partial \tilde{G}_n}{\partial x_k} & 0\dots0 \end{pmatrix} \begin{pmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{G}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{G}_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial \tilde{G}_n}{\partial x_1} & \cdots & \frac{\partial \tilde{G}_n}{\partial x_k} \end{pmatrix} \begin{pmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_k \end{pmatrix}$$

$$(\tilde{G}'\tilde{F})_j(x) = \sum_{i=0}^{k} \frac{\partial \tilde{G}_j}{\partial x_i} F_i(x) = \sum_{i=0}^{k} \left(\frac{\partial G_j}{\partial x_i} + \sum_{l=1}^{n-k} \frac{\partial G_j}{\partial x_{k+l}} \frac{\partial h_l}{\partial x_i} \right) F_i(x)$$

The derivatives of h can also be found relatively easily by solving the linear system

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial(x_{k+1}, \dots, x_n)} Jh = 0,$$

where $f(y,h(y)) = c$ describes X in this neighbourhood as a level set. Hence we can compute the Lie bracket at this point x using just the vector fields F, G defined on X and a level set describing X locally. If your submanifold is not defined using level sets, well then it probably has nice charts and you should probably just compute the Lie bracket using them.

30. My hat it has three corners, three corners has my hat.

Let X be a manifold, F a smooth vector field on X, $x_0 \in X$, and $\gamma : J \to X$ the maximal integral curve of F with $\gamma(0) = x_0$.

(a) Show there is a trichotomy: either γ is constant, or γ is injective, or γ is periodic, and these are mutually exclusive. Periodic means that $J = \mathbb{R}$, γ is non-constant, and there is a number $p > 0$ so that

$$\gamma(t + p) = \gamma(t) \quad \text{for all } \ t \in \mathbb{R}.$$

This number p is called a *period* of γ. It is not unique; for example if p is a period, so is $2p$.

Hint: Assume that γ is not constant or injective, and try to show that it is periodic.

(b) Show γ is constant exactly when $F(x_0) = 0$.

(c) Suppose that γ is periodic. Show that there is a *minimal period* $p_0 > 0$: that means p_0 is a period of γ and there are no other periods in the interval $0 < p < p_0$.

Hint: Prove this by contradiction.

(d) Suppose that γ is periodic. Show that any period is a multiple of the minimal period.

(e) Suppose that γ is periodic. Show that $\gamma|_{[0,p_0)}$ is injective and the map $f : \mathbb{S}^1 \to X$ defined by

$$f(\cos(\theta), \sin(\theta)) = \gamma\left(\frac{p_0}{2\pi} \cdot \theta\right) \quad \text{for all } \ \theta \in \mathbb{R}$$

is an embedding with $f[\mathbb{S}^1] = \gamma[\mathbb{R}]$. It follows that that the image $\gamma[\mathbb{R}]$ is a submanifold of X.

Hint: Exercise 14.

(f) Suppose that γ is injective and X is compact. We know then that $J = \mathbb{R}$. Prove that if $\gamma[\mathbb{R}]$ has an accumulation point in $X \setminus \gamma[\mathbb{R}]$ then γ is *not* an embedding.

Solution.

(a) Clearly if γ is constant or periodic then it is not injective, and conversely if γ is injective then it is not constant or periodic. Periodic functions are by definition not constant. Therefore the three types are mutually exclusive.

Suppose now that γ is not constant or injective. Then there exists times such that $\gamma(t_0) = \gamma(t_1) = x_1$. Suppose that $t_0 < t_1$ without loss of generality. Now we apply the uniqueness of integral curves, Theorem 2.5(ii). Let $p = t_1 - t_0$ and $\alpha(t) = \gamma(t+p)$, which is still an integral curve of F and has $\alpha(t_0) = \gamma(t_1)$. Then $\alpha(t) = \gamma(t)$ for all t for which they are both defined.

In particular, because J is an open interval γ is defined for at least $[t_0, t_1] = [t_0, t_0+p]$ and α for at least $[t_0 - p, t_0]$. But then

$$\tilde{\gamma} : t \mapsto \begin{cases} \gamma(t) & \text{for } t \in [t_0, t_0 + p] \\ \alpha(t) & \text{for } t \in [t_0 - p, t_0] \end{cases}$$

is an integral curve of F with $\tilde{\gamma}(t_0) = \gamma(t_0)$. Since γ is maximal, it must be that in fact it is defined on at least $[t_0 - p, t_0 + p]$. On the other hand, α must also be a maximal integral curve, and $\tilde{\gamma}$ shows it is also defined on at least $[t_0 - p, t_0 + p]$. From the definition of α, γ must be defined on at least $[t_0 - p, t_0 + 2p]$. Every time that we iterate this argument, we show that the domain of γ extends $-p$ and $+p$ further than we had assumed. The only possibility is that $J = \mathbb{R}$. Finally then we have shown that $\gamma(t + p) = \gamma(t)$ for all $t \in \mathbb{R}$; it is periodic.

(b) If γ is constant, then $[\gamma]$ is the zero vector and so from the integral curve equation $[\gamma(t)] = F(\gamma(t))$ we have that $F(x_0) = 0$.

Conversely, if $F(x_0) = 0$ then the curve $\gamma(t) = x_0$ solves the integral curve equation. The solutions are unique.

(c) Let P be the set of positive periods. Suppose there were no minimal period. Because P is bounded from below by 0, it has an infimum $p = \inf P$. If $p > 0$, choose a sequence $p_k \in P$ converging to the infimum. Then by the continuity of γ

$$\gamma(t + p) = \lim_{k\to\infty} \gamma(t + p_k) = \lim_{k\to\infty} \gamma(t) = \gamma(t).$$

This contradicts the fact that P has no minimum. It must be that if there is no minimal period then $\inf P = 0$.

Now we continue the argument in local coordinates. Choose a chart ϕ containing $x_0 = \gamma(0)$ and consider the curve $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^n$, $\tilde{\gamma} = \phi \circ \gamma$. Again, take a sequence of periods p_k, this time which converge to zero. We compute the derivative of $\tilde{\gamma}$ at $t = 0$, using the fact that we know it exists (γ is smooth) and the equivalence between limits of functions and limits of sequences of function values:

$$\tilde{\gamma}'(0) = \lim_{k\to\infty} \frac{\tilde{\gamma}(0 + p_k) - \tilde{\gamma}(0)}{p_k - 0} = \lim_{k\to\infty} \frac{\tilde{\gamma}(0) - \tilde{\gamma}(0)}{p_k} = 0.$$

Thus $F(x_0) = 0$ and it follows from the previous question that γ is constant. But this contradicts the definition of periodic. Therefore there must exist a minimal period.

(d) Let p_0 be the minimal period and p be any other period. Then so is $p + kp_0$ for any integer $k \in \mathbb{Z}$. Thus there is a unique period $p + kp_0 \in [0, p_0)$. But the only period in this interval is 0. Therefore $p = -kp_0$.

(e) If $\gamma|_{[0,p_0)}$ were not injective, then there would be points $t_0, t_1 \in [0, p_0)$ with $\gamma(t_0) = \gamma(t_1)$. We have seen in the proof of (a) that if $\gamma(t_0) = \gamma(t_1)$ then $t_1 - t_0$ is a period. So we would have a period $0 < |t_1 - t_0| < p_0$, which is a contradiction.

Consider the function $f : \mathbb{S}^1 \to X$. It is well defined because γ is periodic. Again from (b) we know that γ has non-vanishing derivative, so f is an immersion. And we have just seen that $\gamma|_{[0,p_0)}$ is injective, so f must be too. It remains to show that f is a homeomorphism, specifically that the inverse is continuous.

Choose any point x_1 of $\gamma[\mathbb{R}]$. Since f has constant rank, we know from Exercise 14 that there are charts (ϕ, U) of $\mathbb{S}^1$ and (ψ, V) of X with $x_1 \in U$ so that

$$\psi \circ f \circ \phi^{-1}(\theta) = (\theta, 0, \ldots, 0).$$

Let $\Pi(x_1, x_2, \ldots, x_n) = x_1$. Then $\phi^{-1} \circ \Pi \circ \psi$ is continuous, and is equal to f^{-1} on $\gamma[\mathbb{R}] \cap V$.

(f) We know that γ is a smooth injective immersion. The only way that it is not an embedding is if the inverse is not continuous. $\gamma^{-1} : \gamma[\mathbb{R}] \to \mathbb{R}$ is not continuous exactly when there is a closed set $A \subset \mathbb{R}$ such that $(\gamma^{-1})^{-1}[A] = \gamma[A]$ is not closed.

Let $x \in \overline{\gamma[\mathbb{R}]} \setminus \gamma[\mathbb{R}]$ be the accumulation point of $\gamma[\mathbb{R}]$ outside itself and $x_k \in \gamma[\mathbb{R}]$ be a sequence that converges to x. Define t_k by $\gamma(t_k) = x_k$. This is well-defined because γ is injective. The set $A := \{t_k\} \subset \mathbb{R}$ has no accumulation points. If it had an accumulation point t, then (x_k) would have to converge to $\gamma(t) \in \gamma[\mathbb{R}]$ by the continuity of γ. Thus the set A is closed. But $\gamma[A] = \{x_k\}$ is not closed, because it has an accumulation point x. This argument shows that γ^{-1} is not continuous.

31. Integral curves on the circle and torus.

Let $\mathbb{S}^1 = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1 \,\}$ be the unit circle, as we have seen many times before. We know from Exercise 15 that its tangent bundle is trivial. Specifically

$$\psi : \mathbb{R} \times \mathbb{S}^1 \to T\mathbb{S}^1, \; \big(s, (x, y)\big) \mapsto s \cdot (-y, x)$$

is a global trivialisation of $T\mathbb{S}^1$ (using the identification of $T_{(x,y)}\mathbb{S}^1$ with $\{\, v \in \mathbb{R}^2 \mid \langle v, (x, y)\rangle = 0 \,\} = \mathbb{R} \cdot (-y, x)$ also from this exercise).

For each $\alpha > 0$ consider the non-vanishing smooth vector field

$$F_\alpha : \mathbb{S}^1 \to T\mathbb{S}^1, \; (x, y) \mapsto \psi\big(\alpha, (x, y)\big)$$

and the maximal integral curve $\gamma_\alpha : \mathbb{R} \to \mathbb{S}^1$ of F_α with $\gamma_\alpha(0) = (1, 0)$.

(a) Show that
$$\gamma_\alpha(t) = \big(\cos(\alpha \cdot t), \sin(\alpha \cdot t)\big)$$
is the maximal integral curve. Determine the minimal period.

Next we consider the 2-dimensional manifold $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. This subset of $\mathbb{R}^2 \times \mathbb{R}^2$ is a torus, a doughnut (donut). For constants $\alpha, \beta > 0$ we define the vector field
$$G_{\alpha,\beta} : \mathbb{T}^2 \to T\mathbb{T}^2,\ \big((x_1, y_1), (x_2, y_2)\big) \mapsto \big(F_\alpha(x_1, y_1)\,,\, F_\beta(x_2, y_2)\big)\ .$$

(b) I don't think we've had an exercise about this, so take a moment to think about why $T(X \times Y) = TX \times TY$. Consult Definition 1.41 and try to write $T_{(x,y)}(X \times Y)$ as a product.

(c) Prove that the curve
$$\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{T}^2,\ t \mapsto \big(\gamma_\alpha(t), \gamma_\beta(t)\big)$$
is the maximal integral curve of $G_{\alpha,\beta}$ with $\eta_{\alpha,\beta}(0) = \big((1, 0), (1, 0)\big) \in \mathbb{T}^2$.

(d) Suppose $\frac{\alpha}{\beta} \in \mathbb{Q}$. Show that $\eta_{\alpha,\beta}$ is periodic and determine the minimal period.

From Exercise 30 we knot that the image is a submanifold. This is call a *torus knot.*

(e) Suppose $\frac{\alpha}{\beta} \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\eta_{\alpha,\beta}$ is injective, but that it is not an embedding.

Remark. In this case, the image $\eta_{\alpha,\beta}[\mathbb{R}]$ is in fact dense in $\mathbb{T}^2$.

Solution.

(a) $\gamma'_\alpha(t) = \alpha(-\sin(\alpha t), \cos(\alpha t)) = F_\alpha(\gamma_\alpha(t))$ shows that it is an integral curve of F_α. It is defined for all $t \in \mathbb{R}$ so it is maximal. The minimal period is $2\pi/\alpha$.

(b) Let (ϕ, U) be a chart of X and (ψ, V) be a chart of Y. By Definition 1.41 we have that $(\phi \times \psi, U \times V)$ is a chart of $X \times Y$. Consider the tangent space of the product. Choose any vector in $T_{(x,y)}(X \times Y)$ represented by $\alpha = (\alpha_X, \alpha_Y) : (-\varepsilon, \varepsilon) \to X \times Y$. Then we get vectors $[\alpha_X] \in T_x X$ and $[\alpha_Y] \in T_y Y$ given by the two components of α. Conversely, given vectors in $T_x X$ and $T_y Y$ we can make a path in $X \times Y$ and get a vector in $T_{(x,y)}(X \times Y)$. Moreover, if $\beta = (\beta_X, \beta_Y)$ is another path, then $[\alpha] = [\beta]$ if and only if
$$((\phi \times \psi) \circ \alpha)'(0) = ((\phi \times \psi) \circ \beta)'(0)$$
$$(\phi \circ \alpha_X, \psi \circ \alpha_Y)'(0) = (\phi \circ \beta_X, \psi \circ \beta_Y)'(0)$$
$$((\phi \circ \alpha_X)'(0), (\psi \circ \alpha_Y)'(0)) = ((\phi \circ \beta_X)'(0), (\psi \circ \beta_Y)'(0)),$$
which is exactly the condition that $[\alpha_X] = [\beta_X]$ and $[\alpha_Y] = [\beta_Y]$. Hence we see that $T_{(x,y)}(X \times Y) = T_x X \times T_y Y$.

This splitting of the tangent space of $X \times Y$ allows us to write projection maps $p_X : T(X \times Y) \to TX$ and $p_Y : T(X \times Y) \to TY$. It is a short exercise to check that this makes $T(X \times Y)$ into a product manifold.

(c) From (b), $[\eta_{\alpha,\beta}] = ([\gamma_\alpha], [\gamma_\beta]) \in TS^1 \times TS^1$. We have already seen that γ_α is the integral curve of F_α on $\mathbb{S}^1$ in part (a), so $[\eta_{\alpha,\beta}] = (F_\alpha, F_\beta) = G_{\alpha,\beta}$. It is maximal because it is defined for all $\mathbb{R}$, and it starts at the given point $\eta(0) = (\gamma_\alpha(0), \gamma_\beta(0)) = ((1,0),(1,0))$.

(d) Suppose that $\alpha/\beta = r/s \in \mathbb{Q}$ for $r, s \in \mathbb{N}$ with no common factors. Let $p_0 = 2\pi r/\alpha = 2\pi s/\beta$. This is a period, because

$$\eta_{\alpha,\beta}(t + p_0) = (\gamma_\alpha(t + 2\pi r/\alpha), \gamma_\beta(t + 2\pi s/\beta)) = (\gamma_\alpha(t), \gamma_\beta(t)) = \eta_{\alpha,\beta}(t).$$

Since γ_α is not constant, neither is $\eta_{\alpha,\beta}$ and thus it must be periodic.

If p is a period of $\eta_{\alpha,\beta}$ then it must be a period of both components. But we know the minimal period of γ_α is $2\pi/\alpha$ and any other period is a multiple of this. Therefore $p = 2\pi k/\alpha$. Likewise $p = 2\pi l/\beta$. So $\alpha/\beta = k/l$. Because we assumed r, s had no common factors, it follows that $k = nr$ and $l = ns$. Therefore p is a multiple of p_0. Since this applies to any period, p_0 must be minimal.

(e) Suppose that α/β is irrational but η is not injective, $\eta(t_0) = \eta(t_1)$. It follows that $p = t_1 - t_0$ is a period of γ_α and γ_β and so therefore a multiple $p = 2\pi k/\alpha = 2\pi l/\beta$ for integers k, l. But then $\alpha/\beta = k/l$ is rational. By contradiction, if α/β is irrational, then η is injective.

To see that it is not an embedding, we use Exercise 30(f). Consider the sequence $t_k = 2\pi k/\alpha$. This gives a sequence of distinct points $((1,0), s_k) = \eta_{\alpha,\beta}(t_k)$, where $s_k = \gamma_\beta(t_k)$. Every infinite collection of points in a compact space must have an accumulation point, so $\{s_k\} \subset S^1$ must have an accumulation point $s \in \mathbb{S}^1 \setminus \{s_k\}$. But then $((1,0), s)$ is an accumulation point of $\{\eta_{\alpha,\beta}(t_k)\}$ that does not lie in $\eta_{\alpha,\beta}[\mathbb{R}]$. Exercise 30(f) now tells us that this is not an embedding.