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Analysis III
10. Exercises

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29. The computation of the Lie Bracket for submanifolds of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and $F, G \in \mathrm{Vec}^\infty(X)$. With the help of Theorem 2.22(iii),(iv) devise a formula to compute $[F, G]$ similar to Exercise 22. Prove your formula.

30. My hat it has three corners, three corners has my hat.

Let X be a manifold, F a smooth vector field on X, $x_0 \in X$, and $\gamma : J \to X$ the maximal integral curve of F with $\gamma(0) = x_0$.

(a) Show there is a trichotomy: either γ is constant, or γ is injective, or γ is periodic, and these are mutually exclusive. Periodic means that $J = \mathbb{R}$, γ is non-constant, and there is a number $p > 0$ so that

$$\gamma(t + p) = \gamma(t) \quad \text{for all } \ t \in \mathbb{R}.$$

This number p is called a *period* of γ. It is not unique; for example if p is a period, so is $2p$.

Hint: Assume that γ is not constant or injective, and try to show that it is periodic.

(b) Show γ is constant exactly when $F(x_0) = 0$.

(c) Suppose that γ is periodic. Show that there is a *minimal period* $p_0 > 0$: that means p_0 is a period of γ and there are no other periods in the interval $0 < p < p_0$.

Hint: Prove this by contradiction.

(d) Suppose that γ is periodic. Show that any period is a multiple of the minimal period.

(e) Suppose that γ is periodic. Show that $\gamma|_{[0,p_0)}$ is injective and the map $f : \mathbb{S}^1 \to X$ defined by

$$f(\cos(\theta), \sin(\theta)) = \gamma\left(\frac{p_0}{2\pi} \cdot \theta\right) \quad \text{for all } \ \theta \in \mathbb{R}$$

is an embedding with $f[\mathbb{S}^1] = \gamma[\mathbb{R}]$. It follows that that the image $\gamma[\mathbb{R}]$ is a submanifold of X.

Hint: Exercise 14.

(f) Suppose that γ is injective and X is compact. We know then that $J = \mathbb{R}$. Prove that if $\gamma[\mathbb{R}]$ has an accumulation point in $X \setminus \gamma[\mathbb{R}]$ then γ is *not* an embedding.

31. Integral curves on the circle and torus.

Let $\mathbb{S}^1 = \{ (x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1 \}$ be the unit circle, as we have seen many times before. We know from Exercise 15 that its tangent bundle is trivial. Specifically

$$\psi : \mathbb{R} \times \mathbb{S}^1 \to T\mathbb{S}^1, \ \big(s, (x,y)\big) \mapsto s \cdot (-y, x)$$

is a global trivialisation of $T\mathbb{S}^1$ (using the identification of $T_{(x,y)}\mathbb{S}^1$ with $\{ v \in \mathbb{R}^2 \mid \langle v, (x,y) \rangle = 0 \} = \mathbb{R} \cdot (-y, x)$ also from this exercise).

For each $\alpha > 0$ consider the non-vanishing smooth vector field

$$F_\alpha : \mathbb{S}^1 \to T\mathbb{S}^1, \ (x,y) \mapsto \psi\big(\alpha, (x,y)\big)$$

and the maximal integral curve $\gamma_\alpha : \mathbb{R} \to \mathbb{S}^1$ of F_α with $\gamma_\alpha(0) = (1,0)$.

(a) Show that

$$\gamma_\alpha(t) = \big(\cos(\alpha \cdot t), \sin(\alpha \cdot t) \big)$$

is the maximal integral curve. Determine the minimal period.

Next we consider the 2-dimensional manifold $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. This subset of $\mathbb{R}^2 \times \mathbb{R}^2$ is a torus, a doughnut (donut). For constants $\alpha, \beta > 0$ we define the vector field

$$G_{\alpha,\beta} : \mathbb{T}^2 \to T\mathbb{T}^2, \ \big((x_1,y_1),(x_2,y_2)\big) \mapsto \big(F_\alpha(x_1,y_1)\,,\, F_\beta(x_2,y_2)\big)\ .$$

(b) I don't think we've had an exercise about this, so take a moment to think about why $T(X \times Y) = TX \times TY$. Consult Definition 1.41 and try to write $T_{(x,y)}(X \times Y)$ as a product.

(c) Prove that the curve

$$\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{T}^2, \ t \mapsto \big(\gamma_\alpha(t), \gamma_\beta(t)\big)$$

is the maximal integral curve of $G_{\alpha,\beta}$ with $\eta_{\alpha,\beta}(0) = \big((1,0),(1,0)\big) \in \mathbb{T}^2$.

(d) Suppose $\frac{\alpha}{\beta} \in \mathbb{Q}$. Show that $\eta_{\alpha,\beta}$ is periodic and determine the minimal period.

From Exercise 30 we knot that the image is a submanifold. This is call a *torus knot.*

(e) Suppose $\frac{\alpha}{\beta} \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\eta_{\alpha,\beta}$ is injective, but that it is not an embedding.

Remark. In this case, the image $\eta_{\alpha,\beta}[\mathbb{R}]$ is in fact dense in $\mathbb{T}^2$.