Martin Schmidt
Ross Ogilvie

Analysis III
9. Exercises

10 May 2021

25. Flows of vector fields.

(a) Let F be a smooth vector field on $\mathbb{R}^2$ given by

$$F(x,y) = (y, -x)$$

Determine the maximal flow of F.

(b) Let $\mathbb{S}^2 \subset \mathbb{R}^3$ and $a \in \mathbb{R}$. Define $\mathbb{F} : \mathbb{S}^2 \to \mathbb{R}^3$ by

$$F(x,y,z) = (a\,y, -a\,x, 0).$$

(i) Show that F is a vector field on S^2 (using the identification that comes from the inclusion map $\iota : \mathbb{S}^2 \to \mathbb{R}^3$).

(ii) Determine the maximal flow ψ_F of F.

(iii) Let $M := \mathbb{S}^2 \setminus \{(1,0,0)\}$. Find an open neighbourhood W of $\{0\} \times M$ in $\mathbb{R} \times M$ so that $\psi_F|W$ is a flow on M. Is $\psi_F|W$ a global flow on M?

Solution.

(a) If we draw F, we see that it is a circular vector field. We can immediately write down the solution to the flow equation:

$$\gamma'(t) = F(\gamma(t)), \text{ with } \gamma(0) = (x_0, y_0),$$

namely $\gamma(t) = (r\cos(t - t_0), r\sin(t - t_0))$ with $r = \sqrt{x_0^2 + y_0^2}$ and $(x_0, y_0) = \gamma(0)$. We could also obtain this by solving the differential equation. The flow is then $\phi(t,(x_0,y_0)) := (r(x_0,y_0)\cos(t - t_0(x_0,y_0)), r(x_0,y_0)\sin(t - t_0(x_0,y_0)))$, just the solution of the differential equation with the dependence of the solution on the initial condition as part of the function.

But this is ugly. The flow looks nicer when you write it using rotation matrices, because this separates the initial conditions in a clear way:

$$\gamma(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

So then we can write

$$\phi(t,(x_0,y_0)) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

In either case, this flow is defined for all $t \in \mathbb{R}$ so it is maximal.

(b) **(i)** Because $F \cdot (x, y, z) = ayx - axy = 0$, we understand that it corresponds to a vector field on $\mathbb{S}^2$ using the map $T\iota$ (see Exercise 15).

(ii) Firstly, we can find the flow for all of $\mathbb{R}^3$ and then because F is a vector field of $\mathbb{S}^2$ the restriction will give a flow on $\mathbb{S}^2$. Because the z-component of F is zero, the z-component of a point must be constant during its flow. Thus the question reduces the part (a). We therefore have the flow

$$\phi(t, (x, y, z)) = \begin{pmatrix} \cos at & \sin at & 0 \\ -\sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Notice that if $p \in \mathbb{S}^2$ then $\psi(t, p) \in \mathbb{S}^2$ for all time, so it does give a flow on $\mathbb{S}^2$ as we claimed it would. It is defined for all times, so maximal.

(iii) The issue with removing a point of $\mathbb{S}^2$ is that the flow may want to move to this point. Therefore let us understand which points p flow into $(1, 0, 0)$, ie $\psi(t, p) = (1, 0, 0)$ for some t. Due to the time-homomorphism property of flows (Definition 2.8(ii)), these points are just

$$p = \phi(-t, (1, 0, 0)) = \begin{pmatrix} \cos at & -\sin at & 0 \\ \sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos at \\ \sin at \\ 0 \end{pmatrix}.$$

Thus, all points of the equator want to flow through $(1, 0, 0)$ at some time. This means that there cannot be a global flow of this field on M, no matter how we choose W.

To find W we need to know an amount of time a point can flow before it tries to go to through the removed point $(1, 0, 0)$, not necessarily the maximum amount of time. There are many ways to do this. Observe that the difference in longitude between any point $(x, y, 0)$ on the equator and $(1, 0, 0)$ is greater than $1 - x$. Thus for any point on the equator $\psi(t, (x, y, z)) \neq (1, 0, 0)$ for all $t \in (-(1 - x)/a, (1 - x)/a)$. In fact, notice that this is true for all points of M and $(1 - x)/a$ is a positive function on M. Thus define

$$W = \left\{ (t, (x, y, z)) \in \mathbb{R} \times M \,\middle|\, t \in (-(1 - x)/a, (1 - x)/a) \right\},$$

and $\psi|_W$ is a local flow of M.

26. An example of an non-complete vector field.

Let

$$W := \{\, (t,(x,y)) \in \mathbb{R} \times \mathbb{R}^2 \,|\, 2\,(x^2+y^2)\cdot t < 1 \,\}$$

and

$$\psi : W \to \mathbb{R}^2,\ (t,(x,y)) \mapsto \frac{1}{\sqrt{1-2\,(x^2+y^2)\cdot t}} \cdot (x,y).$$

(a) Show that ψ is a flow on $\mathbb{R}^2$.

(b) Determine the corresponding vector field $F \in \mathrm{Vec}^\infty(\mathbb{R}^2)$.

(c) Explain why ψ is the maximal flow of F, and why F is not a complete vector field.

Solution.

(a) Let us check the properties in Defintion 2.8. For each point of $\mathbb{R}^2$ we see that $2(x^2+y^2)\cdot t < 1$ is an open interval containing zero. Notice that we can also write the set W as those points (t,p) with $2\|p\|^2\, t < 1$. So if $(s,p) \in W$ and $(t, \phi(s,p)) \in W$ then

$$\begin{aligned} 1 > 2\|\psi(s,p)\|^2\, t &= 2 \cdot \frac{1}{1-2\|p\|^2\cdot s}\|p\|^2 \cdot t \\ 1-2\|p\|^2\, s > 2\|p\|^2\, t& \\ 1 &> 2\|p\|^2(t+s), \end{aligned}$$

which shows $(t+s,p)$ belongs to W also. Thus $\psi(t,\psi(s,p))$ is well-defined and we can compute

$$\begin{aligned} \psi(t,\psi(s,p)) &= \left(1-2\|\psi(s,p)\|^2\, t\right)^{-0.5} \psi(s,p) \\ &= \left(1-\frac{2t\|p\|^2}{1-2s\|p\|^2}\right)^{-0.5} \frac{1}{\sqrt{1-2s\|p\|^2}} p \\ &= \left(1-2s\|p\|^2-2t\|p\|^2\right)^{-0.5} p \\ &= \psi(t+s,p). \end{aligned}$$

Finally, $\psi(0,p) = (1-0)^{-0.5}p = p$.

(b) Recall the relationship between a flow and a vector field. The flow is the set of integral curves of the vector field, and conversely given a point p we get a curve $\gamma(t) = \psi(t,p)$ with $\psi(0,p) = p$, so the tangent vector at p is $[\gamma]$. When we are in Euclidean space, we identify $[\gamma]$ with $\gamma'(0)$. So in this situation $F(p) = \partial_t\psi(t,p)|_{t=0}$.

$$\begin{aligned} F(p) &= \frac{\partial}{\partial}\bigg|_{t=0} \psi(t,p) = -0.5(1-2\|p\|^2\, t)^{-1.5}(-2\|p\|^2)p\bigg|_{t=0} \\ &= \|p\|^2\, p. \end{aligned}$$

The flow ψ is a maximal flow because if we add any additional point (s, p) to W with $2\|p\|^2 s > 1$, then necessarily $(t, p) \in W$ for all $(-\infty, s)$. But then this includes the point where $2\|p\|^2 t = 1$ and ψ is cannot be extended to such points. So the maximal flow of F is not defined on $\mathbb{R} \times \mathbb{R}^2$, it is not a global flow, and by Definition 2.13 we say that F is not complete.

A way to think about the completeness of a vector field without talking about its flow, is that a vector field is complete when every point has an integral curve that exists for all time. If we consider the vector field $F(p) = \|p\|^2\, p$, we see immediately that the direction of the integral curve γ is constant:

$$(\hat{\gamma})' = -\|\gamma\|^{-3}(\gamma' \cdot \gamma)\gamma + \|\gamma\|^{-1}\gamma' = -\|\gamma\|^{-3}(\|\gamma\|^4)\gamma + \|\gamma\|^{-1}(\|\gamma\|^2\gamma) = 0.$$

Therefore, for the initial point $(1, 0)$ we have $\gamma'(0) = F(1, 0) = (1, 0)$, so $\gamma(t) = (x(t), 0)$. The differential equation reduces to $x'(t) = x(t)^3$ with $x(0) = 1$. This has the solution $x(t) = (1 - 2t)^{-0.5}$, which only exists up until time $t = 0.5$. This demonstrates that there is an integral curve that does not exist for all time, so F cannot be complete.

Think why this phrasing in terms of integral curves is exactly equivalent to the statement with flows.

27. The integral curves of vector fields with the form $\boldsymbol{\lambda F}$.

Let X be a manifold, $F \in \mathrm{Vec}^\infty(X)$ a vector field, $\lambda \in C^\infty(X, \mathbb{R})$ a smooth function, $G := \lambda F \in \mathrm{Vec}^\infty(X)$ the rescaling of F, and $p_0 \in X$ a point.

Suppose that $\alpha : I \to X$ is an integral curve of F with $\alpha(0) = p_0$ and that $f : J \to I$ is a solution to the initial value problem

$$f'(t) = \lambda(\alpha(f(t))) \quad \text{with} \quad f(0) = 0.$$

Show then that $\beta := \alpha \circ f : J \to X$ is an integral curve of G with $0 \in J$ and $\beta(0) = p_0$.

Moreover, show that every integral curve of G can be obtained in this way.

Solution. First, $0 \in J$ because the initial condition $f(0) = 0$ means that f is defined at 0, and $\beta(0) = \alpha(f(0)) = \alpha(0) = p_0$. This leaves the main property, that β is an integral curve of G. Choose any chart ϕ containing p_0. We must show that $[\beta(t)] = G(\beta(t))$, or in other words

$$(\phi \circ \beta)'(t) = T_{\beta(t)}(\phi)\, G(\beta(t)),$$

because this is the meaning of tangent vectors in a manifold being equal. We compute both sides

$$(\phi \circ \beta)'(t) = (\phi \circ \alpha \circ f)'(t) = (\phi \circ \alpha)'(f(t)) \cdot f'(t) = (\phi \circ \alpha)'(f(t)) \cdot \lambda(\alpha(f(t)))$$

$$\begin{aligned} T_{\beta(t)}(\phi)\, G(\beta(t)) &= T_{\beta(t)}(\phi) \left[\lambda(\beta(t)) \cdot F(\beta(t))\right] = \lambda(\beta(t)) \cdot T_{\beta(t)}(\phi)\, F(\beta(t)) \\ &= \lambda(\beta(t)) \cdot T_{\alpha(f(t))}(\phi)\, F(\alpha(f(t))) = \lambda(\beta(t)) \cdot (\phi \circ \alpha)'(f(t)). \end{aligned}$$

The last equality follows because α is an integral curve for F, so $(\phi \circ \alpha)'(s) = T_{\alpha(s)}(\phi)\, F(\alpha(s))$ for all s. This shows that the two sides are in fact equal, and thus β is an integral curve of G.

Conversely, suppose that we have integral curves α of F and β of G with $\alpha(0) = \beta(0) = p_0$. If $F(p_0) = 0$, then so is G and the integral curves are simply $\alpha(t) = \beta(t) = p_0$ for all t. In this case we have $f(t) = \lambda(p_0)t$, which solves the DE.

Let us assume then that $F(p_0) \neq 0$, and so there is a neighbourhood of p_0 where F is non-zero. Thus there exists an interval $I = (-\varepsilon, \varepsilon)$ so that $\alpha : I \to \alpha[I] \subset X$ is a diffeomorphism by the inverse function theorem. Likewise, we can restrict the domain of β to J so that $f := \alpha^{-1} \circ \beta : J \to I$ is a well-defined smooth map. It remains to show that f satisfies the DE. But we have already calculated that

$$(\phi \circ \beta)'(t) = (\phi \circ \alpha \circ f)'(t) = (\phi \circ \alpha)'(f(t)) \cdot f'(t)$$

and

$$[\beta(t)] = G(\beta(t)) = \lambda(\alpha(f(t))) \cdot F(\alpha(f(t))) = \lambda(\alpha(f(t))) \cdot [\alpha(f(t))],$$

so f must satisfy this DE.

28. Aligning coordinates with a vector field.

Again let X be a manifold. Let $n := \dim(X)$ be its dimension, $x_0 \in X$ a point, and $F \in \mathrm{Vec}^\infty(X)$ a vector field with $F(x_0) \neq 0$. Show that there is a chart (U, ϕ) containing $x_0 \in U$ such that

$$T_x(\phi)^{-1}(e_1) = F(x) \quad \text{for all } x \in U.$$

Hint: Let ψ be the maximal flow of F. Then we know that ψ is defined on $(-\varepsilon, \varepsilon) \times U'$ for some $\varepsilon > 0$ and neighbourhood $U' \ni x_0$. Next choose an $(n-1)$-dimensional submanifold S of U' with $x_0 \in S$ and $F(x_0) \notin T_{x_0}S$ (explain why there must exists such an S). Finally, apply the inverse function theorem to ψ.

Solution. We follow the advice of the hint. Let ψ be the maximal flow of F. Then we know that ψ is defined on $(-\varepsilon, \varepsilon) \times U'$ for some $\varepsilon > 0$ and neighbourhood $U' \ni x_0$.

Choose some chart $\phi_1 : U'' \to \mathbb{R}^n$ with $\phi_1(x_0) = 0$ and $U'' \subseteq U'$, and let us write $F_1(y) := T_{\phi_1^{-1}(y)}(\phi_1)\, F(\phi_1^{-1}(y)) : \mathbb{R}^n \to \mathbb{R}^n$ for the vector field F in these local coordinates.

Consider the hyperplane $H \subset \mathbb{R}^n$ perpendicular to $F_1(0)$. There must be some neighbourhood $V \subset H$ of the origin such that $F_1(y)$ is not parallel to H for all $y \in V$. One way to see this is to consider the 'vertical component' $F_1(y) \cdot F_1(0)$ which is non-zero at the origin and therefore is non-zero on a neighbourhood of the origin.

We now push this back up to the manifold. Let $S = \phi(V)$ be the $(n-1)$-dimensional manifold. The key idea here is that S is *transverse* to the vector field, ie the vector field is not tangent to this submanifold, so when we flow this submanifold with ψ, S_t sweeps out points of X and does not just slide along itself. We want to use this motion of S_t to make a coordinate system.

Consider the function $h : \mathbb{R}\times V \to U''$, $(t, y) \mapsto \psi(t, \phi_1^{-1}(y))$ where we identify $V \subseteq H$ with subsets $\mathbb{R}^{n-1}$. This map is full rank at $(0, 0)$, because the derivatives in the y direction are tangent to S and the derivative in the t-direction is $F(x_0)$ which is not tangent to S. By the inverse function theorem there is a neighbourhood U and a smooth function $\phi : U \to \mathbb{R}^n$ which is its inverse.

In fact ϕ is a chart compatible with the atlas of X. Clearly ϕ is bijection onto its image because it is the inverse function of h. This also explains why it is a homeomorphism. Compatibility with the atlas in this follows from the fact that h and ϕ are smooth as maps between manifolds.

To explain this coordinate chart a little further, if a point $x \in X$ can be written as $\psi(t, x_1)$ with x_1 in S, then $\phi(x) = (t, \phi_1(x_1))$. You could say that we pick a point $x_1 \in S$ and then flow it with ψ for some time. The points it flows through then have the coordinates of the 'starting point' x_1 and the 'arrival time' t. Of course, x_1 is a point in the manifold, not a point in Euclidean space, so we can't use it as a coordinate directly; instead we push it down into Euclidean space with ϕ_1, where it lies in H by definition.

The desired result now follows easily, because

$$T_x(\phi)^{-1}(e_1) = T_x(h)(e_1) = \partial_t\psi(t, x) = F(x).$$