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Analysis III
9. Exercises

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25. Flows of vector fields.

(a) Let F be a smooth vector field on $\mathbb{R}^2$ given by

$$F(x, y) = (y, -x)$$

Determine the maximal flow of F.

(b) Let $\mathbb{S}^2 \subset \mathbb{R}^3$ and $a \in \mathbb{R}$. Define $\mathbb{F} : \mathbb{S}^2 \to \mathbb{R}^3$ by

$$F(x, y, z) = (a\,y, -a\,x, 0).$$

(i) Show that F is a vector field on S^2 (using the identification that comes from the inclusion map $\iota : \mathbb{S}^2 \to \mathbb{R}^3$).

(ii) Determine the maximal flow ψ_F of F.

(iii) Let $M := \mathbb{S}^2 \setminus \{(1, 0, 0)\}$. Find an open neighbourhood W of $\{0\} \times M$ in $\mathbb{R} \times M$ so that $\psi_F|W$ is a flow on M. Is $\psi_F|W$ a global flow on M?

26. An example of an non-complete vector field.

Let

$$W := \{\, (t, (x, y)) \in \mathbb{R} \times \mathbb{R}^2 \,|\, 2\,(x^2 + y^2) \cdot t < 1 \,\}$$

and

$$\psi : W \to \mathbb{R}^2,\ (t, (x, y)) \mapsto \frac{1}{\sqrt{1 - 2\,(x^2 + y^2) \cdot t}} \cdot (x, y).$$

(a) Show that ψ is a flow on $\mathbb{R}^2$.

(b) Determine the corresponding vector field $F \in \mathrm{Vec}^\infty(\mathbb{R}^2)$.

(c) Explain why ψ is the maximal flow of F, and why F is not a complete vector field.

27. The integral curves of vector fields with the form $\boldsymbol{\lambda F}$.

Let X be a manifold, $F \in \mathrm{Vec}^\infty(X)$ a vector field, $\lambda \in C^\infty(X, \mathbb{R})$ a smooth function, $G := \lambda F \in \mathrm{Vec}^\infty(X)$ the rescaling of F, and $p_0 \in X$ a point.

Suppose that $\alpha : I \to X$ is an integral curve of F with $\alpha(0) = p$ and that $f : J \to I$ is a solution to the initial value problem

$$f'(t) = \lambda(\alpha(f(t))) \quad \text{with} \quad f(0) = 0.$$

Show then that $\beta := \alpha \circ f : J \to X$ is an integral curve of G with $0 \in J$ and $\beta(0) = p_0$.

Moreover, show that every integral curve of G can be obtained in this way.

28. Aligning coordinates with a vector field.

Again let X be a manifold. Let $n := \dim(X)$ be its dimension, $x_0 \in X$ a point, and $F \in \mathrm{Vec}^\infty(X)$ a vector field with $F(x_0) \neq 0$. Show that there is a chart (U, ϕ) containing $x_0 \in U$ such that

$$T_x(\phi)^{-1}(e_1) = F(x) \quad \text{for all } x \in U.$$

Hint: Let ψ be the maximal flow of F. Then we know that ψ is defined on $(-\varepsilon, \varepsilon) \times \widehat{U}$ for some $\varepsilon > 0$ and neighbourhood $\widehat{U} \ni x_0$. Next choose an $(n-1)$-dimensional submanifold S of $\widehat{U}$ with $x_0 \in S$ and $F(x_0) \notin T_{x_0}S$ (explain why there must exists such an S). Finally, apply the inverse function theorem to ψ.