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Analysis III
8. Exercises

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21. A little bit more about submanifolds.

Let X , Y be manifolds and $f : X \to Y$ be a smooth map with constant rank. Then we know that for every $y \in f[X]$ the preimage $M := f^{-1}[\{y\}]$ is a submanifold of X. Show the following holds for $x \in M$:

$$T_x M = \ker T_x(f) \ .$$

Hint. Take $v \in T_x M$, so a smooth path $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Then consider the path $f \circ \gamma$ in Y.

Remark. This is the 'complement' of the idea that for an embedding $\iota : M \to X$ the tangent vectors to $i[M]$ considered as a subset of X are $\operatorname{img} T_x(\iota) \subset T_x X$

Solution. This is one part of Corollary 1.46, but we give a more explicit proof.

Following the hint, take a vector in the submanifold $[\gamma] \in T_x M$. Then we have $f \circ \gamma(t) = y$ for all t, because the submanifold is a level set of f. Thus $T_x(f)([\gamma]) = [f \circ \gamma] = 0 \in T_{f(x)} Y$ because the constant map of a point represents the zero tangent vector. This shows that $[\gamma]$ belongs to the kernel of $T_x(f)$. Therefore $T_x M \subseteq \ker T_x(f)$.

On the other hand, we have seen in Exercise 14 that there are charts ϕ on X and ψ on Y so that

$$\psi \circ f \circ \phi^{-1}(x_1, \dots, x_{\dim X}) = (x_1, \dots, x_r, 0, \dots) \in \mathbb{R}^r \times \mathbb{R}^{\dim Y - r}.$$

This shows that M has dimension $\dim X - r$ and so must $T_x M$. But by the rank-nullity theorem of linear algebra, $\dim T_x X = \dim \ker T_x(f) + r$, which shows that $\ker T_x(f)$ has the same dimension as $T_x M$. If a subspace has the same dimension as the space it lies in, it must be the whole space. Hence $T_x M = \ker T_x(f)$.

22. The Lie bracket in $\mathbb{R}^n$.

The Lie bracket is the name of the operation on vector fields defined in Corollary 2.3.

(a) For a vector field F on X, describe the difference and relationship between the derivation θ_F defined by Theorem 2.2 and D_v described by Theorem 1.40.

(b) Let us focus now on $X = \mathbb{R}^n$. We can write a vector field on X as $F : \mathbb{R}^n \to \mathbb{R}^n$. How can we calculate $\theta_F(f)$ for some function $f : \mathbb{R}^n \to \mathbb{R}$?

(c) Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields on $\mathbb{R}^n$. Show

$$[F, G](x) = G'(x) \cdot F(x) - F'(x) \cdot G(x) \ .$$

(d) Consider the three vector fields on $\mathbb{R}^4$ (we have seen these already in Exercise 15(c)):

$$\begin{aligned} F(x_1,x_2,x_3,x_4) &:= (-x_2,x_1,x_4,-x_3)\ , \\ G(x_1,x_2,x_3,x_4) &:= (-x_3,-x_4,x_1,x_2) \\ \text{and}\quad H(x_1,x_2,x_3,x_4) &:= (-x_4,x_3,-x_2,x_1)\ . \end{aligned}$$

(i) Calculate $[F,G]\,,\ [G,H]$ und $[F,H]\,.$

(ii) For these three fields, check that the *Jacobi identity* holds (compare with the next exercise):

$$[F,[G,H]] = [[F,G],H] + [G,[F,H]]\ .$$

Solution.

(a) Let $f : X \to \mathbb{R}$ be a function. Just before Theorem 1.40 in the script, for every vector $v \in T_xX$ we define a derivation $D_v : C^\infty(X,\mathbb{R}) \to \mathbb{R}$. In essence, given a function and a vector at a point, we get a single number. If we have a vector at every point of X then we get a number at every point of X, ie a function $X \to \mathbb{R}$. This is the definition of the derivation $\theta_F(f)$:

$$\theta_F(f) = x \mapsto D_{F(x)}(f).$$

(b) If we have a tangent vector $v \in \mathbb{R}^n$ at a point $x \in R^n$, then $D_v(f)$ is the directional derivative:

$$D_v(f) = \frac{d}{dt}\bigg|_{t=0} f(x+vt) = \nabla f \cdot v,$$

because a path representing the tangent vector v is $y(t) = x + vt$. It follows then that $\theta_F(f)(x) = \nabla f(x) \cdot F(x)$.

(c) We firstly calculate $\theta_F \circ \theta_G - \theta_G \circ \theta_F$ and then try to see which vector field it could come from. Let $f : \mathbb{R}^n \to \mathbb{R}$ be any function. Note that

$$\frac{\partial}{\partial x_i}\Big(\nabla f \cdot G\Big) = \frac{\partial}{\partial x_i}\sum_j (\partial_j f)\, G_j = \sum_j (\partial_j \partial_i f)\, G_j + (\partial_j f)\,(\partial_i G_j)$$

Now we can compute half the expression.

$$\begin{aligned} \theta_F \circ \theta_G(f) = \theta_F\Big(\nabla f \cdot G\Big) &= \nabla\Big(\nabla f \cdot G\Big) \cdot F \\ &= \sum_{i,j} (\partial_j \partial_i f)\, G_j\, F_i + (\partial_j f)\,(\partial_i G_j)\, F_i \end{aligned}$$

Swapping F and G gives an expression for $\theta_G \circ \theta_F(f)$ too. The difference is

$$\begin{aligned}
\theta_F \circ \theta_G(f) - \theta_G \circ \theta_F(f) &= \sum_{i,j} (\partial_j f)\,(\partial_i G_j)\,F_i - (\partial_j f)\,(\partial_i F_j)\,G_i \\
&= \sum_j (\partial_j f) \sum_i (\partial_i G_j)\,F_i - (\partial_i F_j)\,G_i \\
&= \nabla f \cdot \left(\sum_i (\partial_i G_j)\,F_i - (\partial_i F_j)\,G_i \right)_j \\
&= \nabla f \cdot \Big(\nabla G_j \cdot F - \nabla F_j \cdot G \Big)_j
\end{aligned}$$

Thus we see that $[F, G]$ is the vector field whose j-th component has the formula in the bracket. But the Jacobian matrix F' of a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix whose j-th row is the gradient of F_j. Thus this formula is the same as the formula in the question.

(d) We can use the formula we just derived:

$$\begin{aligned}
[F, G](x) &= G'(x) \cdot F(x) - F'(x) \cdot G(x) \\
&= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \\ x_4 \\ -x_3 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -x_3 \\ -x_4 \\ x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} -x_4 \\ x_3 \\ -x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} -2x_4 \\ 2x_3 \\ -2x_2 \\ 2x_1 \end{pmatrix}
\end{aligned}$$

Similarly we have $[G, H](x) = (-2x_2, 2x_1, 2x_4, -2x_3)$ and $[F, H](x) = (2x_3, 2x_4, -2x_1, -2x_2)$.

(e) For this part, we could go ahead and calculate another three Lie brackets. But notice that in fact $[F, G] = 2H$, $[G, H] = 2F$ and $[F, H] = -2G$. It follows that

$$\begin{aligned}
&[F, [G, H]] = [F, 2F] = 2F' \cdot F - F' \cdot 2F = 0, \\
&[[F, G], H] + [G, [F, H]] = [2H, H] + [G, -2G] = 0.
\end{aligned}$$

If $[F, G] = 2H$ reminds you of the cross-product in $\mathbb{R}^3$, there's a good reason. Consider these vector fields at the point $x = (1, 0, 0, 0)$. Then

$$\begin{aligned}
F(1, 0, 0, 0) &:= (0, 1, 0, 0)\ , \\
G(1, 0, 0, 0) &:= (0, 0, 1, 0) \\
\text{and} \quad H(1, 0, 0, 0) &:= (0, 0, 0, 1)\ .
\end{aligned}$$

so we can see these vectors as the basis of $\mathbb{R}^3$ and the Lie bracket as twice the cross-product.

23. Properties of the Lie bracket. Let X be an n-dimensional manifold.

(a) Show: the Lie bracket has the following properties for all vector fields $F, G, H \in \mathrm{Vec}^\infty(X)$ and scalars $a \in \mathbb{R}$.

(i) $\mathbb{R}$-linear: $[aF, G] = a[F, G]$.

(ii) anti-symmetric: $[F, G] = -[G, F]$.

(iii) Jacobi identity: $[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0$.

Hint: The pairing $F \to \theta_F$ is injective (and for smooth vector fields and derivations it is bijective), so it is enough to show equality for the corresponding derivations. Eg. to show $[aF, G] = a[F, G]$ you can show $\theta_{[aF,G]} = \theta_{a[F,G]}$.

(b) Let $\phi : U \to \mathbb{R}^n$ be a chart of X for an open set $U \subset X$. Then consider the vector field $F_i \in \mathrm{Vec}^\infty(U)$ with

$$F_i(x) = T_x(\phi)^{-1}(e_i)\ ,$$

for $i \in \{1, \dots, n\}$ and where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ is the i-th standard unit vector of $\mathbb{R}^n$.

Show that these fields commute: $[F_i, F_j] = 0$ for every i, j.

Solution.

(a) The Lie bracket is a local construct, so choose any point $x \in X$ and a chart $\phi : U \to \mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be any smooth function. Applying the definitions of Theorems 1.40 and 2.2 to a general manifold gives

$$\begin{aligned}\theta_F(f) : x \mapsto D_{F(x)}(f) &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \text{ for } [\gamma] = F(x)\\ &= \left.\frac{d}{dt}\right|_{t=0} f(\phi^{-1}(\phi(x) + vt)) \text{ for } v = T_x(\phi)(F(x)).\end{aligned}$$

The formulas are equivalent, but depending on how the vectors of the vector field are described, whether as paths or in local coordinates, one formula might be easier than the other. Now the derivation $\theta_{[F,G]} = \theta_F \circ \theta_G - \theta_G \circ \theta_F$.

$\mathbb{R}$-linear: Notice that θ_F is $\mathbb{R}$-linear in F:

$$\begin{aligned}\theta_{aF}(f)(x) &= \left.\frac{d}{dt}\right|_{t=0} f(\phi^{-1}(\phi(x) + avt)) \text{ for } v = T_x(\phi)(F(x))\\ &= \left.\frac{d}{d(s/a)}\right|_{s=0} f(\phi^{-1}(\phi(x) + vs)) \text{ for } s = at\\ &= a\,\theta_F(f)(x).\end{aligned}$$

And linearity in f follows from the Leibniz rule. Thus

$$\begin{aligned}\theta_{[aF,G]}(f) &= \theta_{aF}\Big(\theta_G(f)\Big) - \theta_G\Big(\theta_{aF}(f)\Big)\\ &= a\,\theta_F\Big(\theta_G(f)\Big) - \theta_G\Big(a\,\theta_F(f)\Big)\\ &= a\,\theta_F\Big(\theta_G(f)\Big) - a\,\theta_G\Big(\theta_F(f)\Big)\\ &= a\,\theta_{[F,G]}(f) = \theta_{a[F,G]}(f)\end{aligned}$$

Anti-symmetry also follows from the linearity of θ_F in F:

$$\theta_{-[G,F]} = -\theta_{[G,F]} = -\theta_G \circ \theta_F + \theta_F \circ \theta_G = \theta_{[F,G]}$$

Finally we must show the Jacobi identity.

$$\begin{aligned}\theta_{[F,[G,H]]}(f) &= \theta_F\Big(\theta_G\theta_H(f) - \theta_H\theta_G(f)\Big) - (\theta_G\theta_H - \theta_H\theta_G)\Big(\theta_F(f)\Big)\\ &= \theta_F\theta_G\theta_H(f) - \theta_F\theta_H\theta_G(f) - \theta_G\theta_H\theta_F(f) + \theta_H\theta_G\theta_F(f).\end{aligned}$$

If you permute the F,G, and H to compute the other two terms, you see that every permutation of $\theta_F\theta_G\theta_H$ occurs twice, once with each sign. Therefore the sum is zero.

(b) Here the second version of the formula for θ_F is very useful, because $v = T_x(\phi)(F_i(x)) = T_x(\phi)T_x(\phi)^{-1}e_i = e_i$ for every point $x \in U$. Then

$$\begin{aligned}\theta_{F_i}\theta_{F_j}(f)(x) &= \theta_{F_i}\left(y \mapsto \frac{d}{dt}\bigg|_{t=0} f(\phi^{-1}(\phi(y) + e_j t))\right)(x)\\ &= \frac{d}{ds}\bigg|_{s=0}\frac{d}{dt}\bigg|_{t=0} f(\phi^{-1}(\phi(x) + e_i s + e_j t))\\ &= \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} f(\phi^{-1}(\phi(x) + e_i s + e_j t))\\ &= \theta_{F_j}\theta_{F_i}(f).\end{aligned}$$

This shows $\theta_{[F_i,F_j]} = 0$, and hence $[F_i, F_j] = 0$.

More explanation/another example: Perhaps it is useful to see how special this property is by doing the same computation for F and G from Exercise 22 considered as vector fields on $\mathbb{S}^3$. Choose the point $x_0 = (1, 0, 0, 0)$ and a small neighbourhood $U \subset \mathbb{S}^3$ of this point. Then we can use the chart $\phi(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4)$ which has inverse $\phi^{-1}(y_1, y_2, y_3) = (h(y), y_1, y_2, y_3)$ for $h(y) = +\sqrt{1 - \|y\|^2}$. Let $y = \phi(x)$ be the corresponding local coordinate for any point x. First we compute the vectors in local coordinates

$$\begin{aligned}v_F(y) &:= T_x(\phi)F(x(y)) = (x_1, x_4, -x_3) = (h(y), y_3, -y_2),\\ v_G(y) &:= T_x(\phi)G(x(y)) = (-x_4, x_1, x_2) = (-y_3, h(y), y_1).\end{aligned}$$

Take any smooth function $f : \mathbb{S}^3 \to \mathbb{R}$. The application of θ_F to f is standard:

$$\theta_F(f)(x) = \frac{d}{dt}\Big|_{t=0} f \circ \phi^{-1}(y + v_F(y)t).$$

Here is the important point. When we apply θ_G to this, we make the substitution $y + v_G(y)s$ for y, but the vector v_F also depends on y! This gives

$$\theta_G\theta_F(f)(x) = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f \circ \phi^{-1}\Big(y + v_G(y)s + v_F(y + v_G(y)s)t\Big).$$

Now I think you can see why the order of θ_F and θ_G is important. Let's complete this calculation now:

$$\begin{aligned}
y + v_G(y)s &= (y_1 - y_3 s, y_2 + h(y)s, y_3 + y_1 s)\\
v_F(y + v_G(y)s) &= v_F(y_1 - y_3 s, y_2 + h(y)s, y_3 + y_1 s)\\
&= (h(y + v_G(y)s), y_3 + y_1 s, -y_2 - h(y)s))\\
\theta_G\theta_F(f)(x) &= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f \circ \phi^{-1}\Big(y + v_G(y)s + v_F(y + v_G(y)s)t\Big)\\
&= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} f \circ \phi^{-1} \begin{pmatrix} y_1 - y_3 s + h(y + v_G(y)s)t \\ y_2 + h(y)s + (y_3 + y_1 s)t \\ y_3 + y_1 s + (-y_2 - h(y)s)t \end{pmatrix}
\end{aligned}$$

Let's assume that f is given as the restriction of a smooth function on $\mathbb{R}^4$, which is always possible, so that we can use vector calculus for the next steps. You can also do this with the chain rule for manifolds with the tangent map instead of the gradient and Jacobian, and of course it is basically the same thing, but I think it is a little clearer to write it this way. We continue

$$\begin{aligned}
\theta_G\theta_F(f)(x) &= \nabla f \cdot J(\phi^{-1}) \cdot \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} y_1 - y_3 s + h(y + v_G(y)s)t \\ y_2 + h(y)s + (y_3 + y_1 s)t \\ y_3 + y_1 s + (-y_2 - h(y)s)t \end{pmatrix}\\
&= \nabla f \cdot J(\phi^{-1}) \cdot \frac{d}{ds}\Big|_{s=0} \begin{pmatrix} h(y + v_G(y)s) \\ y_3 + y_1 s \\ -y_2 - h(y)s \end{pmatrix}\\
&= \nabla f \cdot J(\phi^{-1}) \cdot \begin{pmatrix} h'(y) \cdot v_G(y) \\ y_1 \\ -h(y) \end{pmatrix}.
\end{aligned}$$

In the same way

$$
\begin{aligned}
y+v_F(y)t &= (y_1+h(y)t, y_2+y_3t, y_3-y_2t) \\
v_G(y+v_F(y)t) &= (-y_3+y_2t, h(y+v_F(y)t), y_1+h(y)t) \\
\theta_F\theta_G(f)(x) &= \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} f\circ\phi^{-1}\Big(y+v_F(y)t+v_G(y+v_F(y)t)s\Big) \\
&= \nabla f\cdot J(\phi^{-1})\cdot\frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0}\begin{pmatrix} y_1+h(y)t+(-y_3+y_2t)s \\ y_2+y_3t+h(y+v_F(y)t)s \\ y_3-y_2t+(y_1+h(y)t)s \end{pmatrix} \\
&= \nabla f\cdot J(\phi^{-1})\cdot\frac{d}{dt}\bigg|_{t=0}\begin{pmatrix} -y_3+y_2t \\ h(y+v_F(y)t) \\ y_1+h(y)t \end{pmatrix} \\
&= \nabla f\cdot J(\phi^{-1})\cdot\begin{pmatrix} y_2 \\ h'(y)\cdot v_F(y) \\ h(y) \end{pmatrix}
\end{aligned}
$$

Finally we can say

$$
\begin{aligned}
\theta_{[F,G]}(f)(x) &= \nabla f\cdot J(\phi^{-1})\cdot\begin{pmatrix} y_2-h'(y)\cdot v_G(y) \\ h'(y)\cdot v_F(y)-y_1 \\ 2h(y) \end{pmatrix} \\
&= \nabla f\cdot J(\phi^{-1})\cdot\begin{pmatrix} x_3+x_1^{-1}(x_2,x_3,x_4)\cdot(-x_4,x_1,x_2) \\ -x_1^{-1}(x_2,x_3,x_4)\cdot(x_1,x_4,-x_3)-x_2 \\ 2x_1 \end{pmatrix} \\
&= \nabla f\cdot J(\phi^{-1})\cdot x_1^{-1}\begin{pmatrix} x_1x_3-x_2x_4+x_1x_3+x_2x_4 \\ -(x_1x_2+x_3x_4-x_3x_4)-x_1x_2 \\ 2x_1^2 \end{pmatrix} \\
&= \nabla f\cdot\begin{pmatrix} -x_1^{-1}x_2 & -x_1^{-1}x_3 & -x_1^{-1}x_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 2x_3 \\ -2x_2 \\ 2x_1 \end{pmatrix} = \nabla f\cdot\begin{pmatrix} -2x_4 \\ 2x_3 \\ -2x_2 \\ 2x_1 \end{pmatrix}
\end{aligned}
$$

and this is the same answer we found in 22(d).

24. Commuting flows.

Let $a,b,c\in\mathbb{R}$ be constants and the vector fields $F,G\in\mathrm{Vec}^\infty(\mathbb{R}^3)$ be given by

$$F(x_1,x_2,x_3)=(1,x_3,-x_2) \quad\text{and}\quad G(x_1,x_2,x_3)=(a,b,c)\,.$$

(a) Determine the flows ψ_F and ψ_G induced by F and G respectively, and determine for which values of a, b, c the flows commute with one another: i.e. for all $t, s \in \mathbb{R}$

$$\psi_F(t, \psi_G(s, x)) = \psi_G(s, \psi_F(t, x)).$$

(b) Calculate $[F, G]$, and determine for which values of a, b, c the Lie bracket is zero, $[F, G] = 0$.

Solution.

(a) There is a subtle difference between local and global flows, but here we will see that these vector fields are *complete* and so generate global flows. A global flow on X is a map $\psi : \mathbb{R} \times X \to X$ with the 'initial' property $\psi(0, x) = x$ and the 'continuation' property $\psi(t, \psi(s, x)) = \psi(s + t, x)$. If we fix a point x_0 and consider where this point moves as time t changes, we get a path $\alpha_{x_0}(t) := \psi(t, x_0)$. The vector field associated to a flow is $F(x) = [\alpha_x]$. Reversing this, finding the flow associated to a vector field, requires solving a differential equation.

For F, this is the differential equation

$$x_1'(t) = 1, \;\; x_2'(t) = x_3(t), \;\; x_3'(t) = -x_2(t),$$

with initial condition $x(0) = (x_{10}), x_{20}, x_{30})$. Immediately we have $x_1(t) = t + x_{10}$. The other two components are a well-known system with solution $x_2(t) = x_{20} \cos t + x_{30} \sin t$ and $x_3(t) = -x_{20} \sin t + x_{30} \cos t$. The flow is

$$\psi_F(t, x) = (t + x_1, \; x_2 \cos t + x_3 \sin t, \; -x_2 \sin t + x_3 \cos t).$$

The DE system for G is very easy, everything moves in a straight line with constant speed, giving the flow

$$\psi_G(t, x) = x + (a, b, c)t.$$

Now we can compute

$$\begin{aligned}
&\psi_F(t, \psi_G(s, x)) = \psi_F(t, (x_1 + as, x_2 + bs, x_3 + cs)) \\
&= (t + x_1 + as, (x_2 + bs) \cos t + (x_3 + cs) \sin t, -(x_2 + bs) \sin t + (x_3 + cs) \cos t) \\
&\psi_G(s, \psi_F(t, x)) = (t + x_1 + as, x_2 \cos t + x_3 \sin t + bs, -x_2 \sin t + x_3 \cos t + cs).
\end{aligned}$$

The first components are always equal. The second components are equal when $bs \cos t + cs \sin t = bs$ and the third when $-bs \sin t + cs \cos t = cs$. We can divide out the s, and due to the linear independence of trig functions, it must be that $b = c = 0$.

Geometrically, the flow ψ_F moves points in a circles around the axis $x_2 = x_3 = 0$, while increasing their x_1 at a constant rate. The two flows commute exactly when ψ_G moves parallel to this axis.

(b) This is in Euclidean space, so we use the formulas from Exercise 22.

$$[F,G] = G'F - F'G = 0 \cdot F - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}.$$

Thus the two fields commute when $b = c = 0$, exactly when the flows commute. This is a general truth: flows commute exactly when the vector fields commute (Corollary 2.21).

Terminology

Flüss = flow