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Analysis III
8. Exercises

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21. A little bit more about submanifolds.

Let X , Y be manifolds and $f : X \to Y$ be a smooth map with constant rank. Then we know that for every $y \in f[X]$ the preimage $M := f^{-1}[\{y\}]$ is a submanifold of X. Show the following holds for $x \in M$:

$$T_x M = \ker T_x(f) \ .$$

Hint. Take $v \in T_x M$, so a smooth path $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Then consider the path $f \circ \gamma$ in Y.

Remark. This is the 'complement' of the idea that for an embedding $\iota : M \to X$ the tangent vectors to $i[M]$ considered as a subset of X are $\operatorname{img} T_x(\iota) \subset T_x X$

22. The Lie bracket in $\mathbb{R}^n$.

The Lie bracket is the name of the operation on vector fields defined in Corollary 2.3.

(a) For a vector field F on X, describe the difference and relationship between the derivation θ_F defined by Theorem 2.2 and D_v described by Theorem 1.40.

(b) Let us focus now on $X = \mathbb{R}^n$. We can write a vector field on X as $F : \mathbb{R}^n \to \mathbb{R}^n$. How can we calculate $\theta_F(f)$ for some function $f : \mathbb{R}^n \to \mathbb{R}$?

(c) Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields on $\mathbb{R}^n$. Show

$$[F, G](x) = G'(x) \cdot F(x) - F'(x) \cdot G(x) \ .$$

(d) Consider the three vector fields on $\mathbb{R}^4$ (we have seen these already in Exercise 15(c)):

$$\begin{aligned} F(x_1, x_2, x_3, x_4) &:= (-x_2, x_1, x_4, -x_3) \ , \\ G(x_1, x_2, x_3, x_4) &:= (-x_3, -x_4, x_1, x_2) \\ \text{and} \quad H(x_1, x_2, x_3, x_4) &:= (-x_4, x_3, -x_2, x_1) \ . \end{aligned}$$

(i) Calculate $[F, G]$, $[G, H]$ und $[F, H]$.

(ii) For these three fields, check that the *Jacobi identity* holds (compare with the next exercise):

$$[F, [G, H]] = [[F, G], H] + [G, [F, H]] \ .$$

23. Properties of the Lie bracket. Let X be an n-dimensional manifold.

(a) Show: the Lie bracket has the following properties for all vector fields $F, G, H \in \mathrm{Vec}^\infty(X)$ and scalars $a \in \mathbb{R}$.

(i) $\mathbb{R}$-linear: $[aF, G] = a[F, G]$.

(ii) anti-symmetric: $[F, G] = -[G, F]$.

(iii) Jacobi identity: $[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0$.

Hint: The pairing $F \to \theta_F$ is injective (and for smooth vector fields and derivations it is bijective), so it is enough to show equality for the corresponding derivations. Eg. to show $[aF, G] = a[F, G]$ you can show $\theta_{[aF,G]} = \theta_{a[F,G]}$.

(b) Let $\phi : U \to \mathbb{R}^n$ be a chart of X for an open set $U \subset X$. Then consider the vector field $F_i \in \mathrm{Vec}^\infty(U)$ with

$$F_i(x) = T_x(\phi)^{-1}(e_i) \ ,$$

for $i \in \{1, \dots, n\}$ and where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ is the i-th standard unit vector of $\mathbb{R}^n$.

Show that these fields commute: $[F_i, F_j] = 0$ for every i, j.

24. Commuting flows.

Let $a, b, c \in \mathbb{R}$ be constants and the vector fields $F, G \in \mathrm{Vec}^\infty(\mathbb{R}^3)$ be given by

$$F(x_1, x_2, x_3) = (1, x_3, -x_2) \quad \text{and} \quad G(x_1, x_2, x_3) = (a, b, c)\,.$$

(a) Determine the flows ψ_F and ψ_G induced by F and G respectively, and determine for which values of a, b, c the flows commute with one another: i.e. for all $t, s \in \mathbb{R}$

$$\psi_F(t, \psi_G(s, x)) = \psi_G(s, \psi_F(t, x)).$$

(b) Calculate $[F, G]$, and determine for which values of a, b, c the Lie bracket is zero, $[F, G] = 0$.

Terminology

Flüss = flow