Martin Schmidt
Ross Ogilvie

Analysis III
7. Exercises

26 April 2022

17. Let (E, B, π) be a vector bundle.

(a) Cutting through a target. Let $b_0 \in B$ and $v_0 \in \pi^{-1}[\{b_0\}]$. Show: There exists a global section $f : B \to E$ with $f(b_0) = v_0$.

Hint: start with a local trivialisation of π in a neighbourhood U of b_0 and see if you can construct a local section whose support is contained in U.

(b) Yet another differentiability test. Show for every manifold Z and any arbitrary map $g : B \to Z$: if $g \circ \pi$ is smooth, so too is g.

(Compare this to Exercise 15.)

Solution.

(a) Let $\phi : \mathbb{R}^r \times U \to \pi^{-1}[U]$ be a local trivialisation of π in a neighbourhood U of b_0. Let $\varphi : U \to \mathbb{R}$ be a smooth function of whose support lies compactly within U and which takes the value $\varphi(b_0) = 1$. The idea is to make a 'constant' vector-valued function and then scale it by φ. Then we can defined a function on all of B by setting it to 0 outside U.

This is a little technical, because 'constant' sections do not exist (in general) on vector bundles, only sections which are constant with respect to a trivialisation. However this is enough. Let $(w_0, b_0) = \phi^{-1}(v_0)$ and set $g(b) = \phi(w_0, b)$ for $b \in U$. Clearly g is a local section because of the property $\pi \circ \phi(v, b) = b$. Hence we can define

$$f(b) = \begin{cases} \varphi(b)g(b) & \text{for } b \in U \\ 0_b & \text{for } b \notin U. \end{cases}$$

Remember that 0_b is a well-defined point of E regardless of the chart (Exercise 14(a)). This is also smooth because it is smooth at every point of U, and any point not in U has a neighbourhood where $\varphi(b)$ is identically zero. Finally $f(b_0) = 1 \cdot \phi(w_0, b_0) = v_0$ by the definition of w_0.

(b) Choose any point $v_0 \in E$ and trivialisation ϕ which covers this point. Then $(g \circ \pi) \circ \phi(v, b) = g \circ (\pi \circ \phi(v, b)) = g(b)$. By writing g as the product of two smooth functions, we have shown that it too is smooth.

18. Triviality of the homomorphism bundle.

Let (E, B, π) and (E', B, π') be two vector bundles over a base manifold B. Consider the *homomorphism bundle* $(\mathrm{Hom}(E, E'), B, \pi'')$. People often say "Hom-bundle" for short.

(a) What is the rank of $\mathrm{Hom}(E, E')$.

(b) Show that when E and E' are trivial bundles, then so too is $\mathrm{Hom}(E, E')$.

(c) Prove or disprove: $\mathrm{Hom}(E, E')$ is trivial, then the bundles E and E' must be trivial.

Hint: Examine the Möbius bundle M from Exercise 16(c). Every homomorphism from $\mathbb{R}$ to $\mathbb{R}$ has the form $x \mapsto a\,x$, for some $a \in \mathbb{R}$. Why is the choice of a is independent from choice of trivialisation?

Solution.

(a) The rank is the dimension of the fibre. The fibres of the Hom-bundle are the homomorphisms from the fibre F of the first bundle to the fibre F' of the second. If these vector spaces are dimensions r and r' respectively then the homomorphisms can be identified with $r' \times r$ matrices. Hence they form a vector space of dimension rr'.

(b) If E and E' are trivial, we know that there exists non-vanishing sections $\{v_1, \ldots, v_r\}$ and $\{v'_1, \ldots, v'_{r'}\}$ which are every point are linearly independent. This means there is an isomorphism of vector bundles between E and $\mathbb{R}^r \times B$, and composing with the coordinate projections $\mathbb{R}^r \to \mathbb{R}$ gives us smooth functions $a_i : E \to \mathbb{R}$ such that $v = \sum_{k=1}^{r} a_k(v) v_k(\pi(v))$. These functions are linear because they are the composition of linear functions. Therefore we have bundle homomorphisms

$$s_{ij}(b) : v \mapsto a_i(v) v'_j(b) \in E' \text{ for } v \in \pi^{-1}[\{b\}].$$

The zero homomorphism is the one that maps all vectors to zero. Notice that for each of s_{ij} and for each $b \in B$ we have $s_{ij}(b)(v_i(b)) = v'_j(b) \neq 0$. Hence there is at least one vector in $\pi^{-1}[\{b\}]$ that is not mapped to zero, which shows that s_{ij} is non-vanishing.

They are also linearly independent: suppose that $0 = \sum_{k,l} c_{kl} s_{kl}(b)$. Applying this to the point $v_i(b)$ gives $0 = \sum_l c_{il} v'_l(b)$. The linear independence of the $v'_j(b)$ now forces $c_{il} = 0$ for all l. Repeating this with the other basis sections of E shows all coefficients to be zero.

We have found rr' linearly-independent non-vanishing sections of $\mathrm{Hom}(E, E')$. Therefore it is trivial.

Each of these functions is essentially the matrix with 1 in the $(i, j)^{th}$ component and 0 elsewhere because the map sends $v_i(b)$ to $v'_j(b)$ and other vectors $v_k(b)$ to zero and these are basis vectors of $\pi^{-1}[\{b\}]$ and $\pi'^{-1}[\{b\}]$ respectively. This proof was essentially the proof that matrices uniquely represent homomorphisms with respect to given bases of the vector spaces.

(c) This is false. We give as our counterexample the bundle $H := \mathrm{Hom}(M, M)$ as suggested by the hint. Because the rank of M is 1, so too is the rank of H, as discussed in part (a). Thus it is sufficient to give a non-vanishing section of H. But this is easy: the identity map id_M fits the description. When we think of it as a section of H, perhaps it is better to write it as s:

$$s(x) : v \mapsto v \in M \text{ for } v \in \pi^{-1}[\{x\}].$$

Let us give a more complete picture of H, following the second part of the hint. Every bundle homomorphism $s : M \to M$ must act as scalar multiplication on each fibre, because those are the only homomorphism $\mathbb{R} \to \mathbb{R}$. But scaling the fibre is independent of the choice of trivialisations; the trivialisations preserve the vector space structure. Therefore for each point $h \in H$ we can describe it as a pair (a, x) where $x = \pi(h)$ is the base point and a is the scalar. Conversely, given (a, x) consider the homomorphism $v \mapsto a \cdot v$ on $\pi^{-1}[\{x\}]$. This describes the correspondence between H and the trivial bundle $\mathbb{R} \times \mathbb{S}^1$.

This proof does not generalise to higher dimensional homomorphism bundles, because in general there are many more homomorphisms $\mathbb{R}^r \to \mathbb{R}^{r'}$ than just scaling, and these other homomorphisms do not need to be preserved by the trivialisations. It does not even generalise to the Hom-bundle between line bundles L and L', because while it is true that $\mathrm{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}^\times$, how to identify the fibres of L and L' with $\mathbb{R}$ depends on the trivialisations. It does generalise to the bundle $\mathrm{Hom}(L, L)$ for (L, B, π) a line bundle, because then we can use the special homomorphism id.

19. The dual bundle of a vector bundle.

Let (E, B, π) be a vector bundle over a manifold B, with fibre $F = \mathbb{K}^n$. Further, let $\mathcal{U}$ be an open cover of B so that π trivialises over every set $U \in \mathcal{U}$. Denote the cocycles of π with respect to this cover by $g_{U,V} : U \cap V \to \mathrm{GL}(\mathbb{K}^n)$.

Show that the *dual bundle* $(\tilde{E}, B, \tilde{\pi})$ to π is described over $\mathcal{U}$ by the cocycle $(\tilde{g}_{U,V})_{U,V \in \mathcal{U}}$ with

$$\tilde{g}_{U,V} : U \cap V \to \mathrm{GL}(\mathbb{K}^n),\ x \mapsto \big(g_{U,V}(x)^T\big)^{-1}.$$

Solution. The dual bundle is by definition a special type of homomorphism bundle, namely $\mathrm{Hom}(E, \mathbb{R} \times B)$. Thus we should look to Theorem 1.59. In that theorem, the cocycle of a Hom-bundle is described by the function $\Pi(A, B) : C \mapsto B \circ C \circ A^{-1} \in \mathrm{Hom}(F, F')$ where $A \in \mathrm{GL}(F)$ and $B \in \mathrm{GL}(F')$ are respectively the transition functions of the source and target bundles at a point and $C \in \mathrm{Hom}(F, F')$ is a homomorphism between the fibres F and F' and thus itself a point of the fibre of the Hom-bundle.

In this situation, we have $F = \mathbb{K}^n$ and $F' = \mathbb{K}$, and at some point $b \in U \cap V \subset B$ we have $A = g_{U,V}(b) \in \mathrm{GL}(\mathbb{K}^n)$ and $B = 1$ (the transition functions for the trivial bundle are always the identity matrix). We can also describe $C \in \mathrm{Hom}(\mathbb{K}^n, \mathbb{K})$ as a column vector that acts by $C : v \mapsto C^T v$ (we will explain why we should think of it this way shortly). Thus, for $v \in \mathbb{K}^n$ the transition acts as

$$\tilde{g}_{U,V}(b)(C) : v \mapsto B(C(A^{-1}(v))) = C^T g_{U,V}(b)^{-1} v = \left((g_{U,V}(b)^T)^{-1} C\right)^T v \in \mathbb{K}$$
$$\tilde{g}_{U,V}(b) : C \mapsto (g_{U,V}(b)^T)^{-1} C \in \mathrm{Hom}(\mathbb{K}^n, \mathbb{K})$$
$$\tilde{g}_{U,V}(b) = (g_{U,V}(b)^T)^{-1} \in \mathrm{GL}(\mathrm{Hom}(\mathbb{K}^n, \mathbb{K}))$$

We can now explain why we thought of $C \in \mathrm{Hom}(\mathbb{K}^n, \mathbb{K})$ as a column vector, because we want it to be acted on by an element of $\mathrm{GL}(\mathrm{Hom}(\mathbb{K}^n, \mathbb{K}))$ and these act on column vectors. Notice that transpose and inversion of matrices commute, so it doesn't matter which order we write those operations.

20. Classification of line bundles over $\mathbb{S}^1$.

In this exercise we want to show: every real line bundle over the circle $\mathbb{S}^1$ is either trivial or isomorphic to the Möbius bundle.

Let $(E, \mathbb{S}^1, \pi)$ be a line bundle and recall the notation of Exercise 16(c), namely the cover $\{U_N, U_S\}$ of $\mathbb{S}^1$ with $U_N \cap U_S = H_- \sqcup H_+$.

(a) With the help of 16(b), argue why we can assume that π trivialises over the cover $\{U_N, U_S\}$.

(b) Let $\phi_k : \mathbb{R} \times U_k \to \pi^{-1}[U_k]$ be trivialisations of π. Show that $f_k : U_k \to E$, $p \mapsto \phi_k(1, p)$ are non-vanishing local sections.

(c) Prove that there exists a function $\chi : U_N \cap U_S \to \mathbb{R}$ with

$$\forall p \in U_1 \cap U_2 \ : \ \phi_S(\chi(p), p) = f_N(p).$$

Explain why it is non-vanishing, why its sign is constant on H_+, and why its sign is constant on H_-.

(d) Suppose that χ has the same sign on H_+ and H_-. Show that the bundle π is trivial.

(e) Lastly we consider the case that χ has different signs on the two sets $H_\pm$; assume that $\chi|H_+ > 0$ und $\chi|H_- < 0$. Let $(M, \mathbb{S}^1, \pi_M)$ denote the Möbius bundle with trivialisations $\phi_{M,k} : \mathbb{R} \times U_k \to \pi_M^{-1}[U_k]$ compatible with the cocycle given in 16(c) (to construct these was part of the exercise, check the solution for more details).

Show that the vector bundle homomorphism $G : E \to M$ given by

$$\forall p \in U_N \ : \ G(f_N(p)) = \phi_{M,N}(1, p) \text{ and } \forall p \in U_S \ : \ G(f_S(p)) = \frac{1}{|\chi(p)|} \cdot \phi_{M,S}(1, p)$$

is well defined, and that it is in fact a vector bundle isomorphism.

Solution.

(a) We know from Exercise 16(b) that every line bundle over $\mathbb{R}$ trivialises. It follows that they also trivialise over every manifold diffeomorphic to $\mathbb{R}$, which in this case would be U_N and U_S. Consider then $E_N = \pi^{-1}[U_N]$ as a vector bundle with base U_N. It must be trivial, so there is a bundle isomorphism $\varphi_N : \mathbb{R} \times U_N \to E_N$. But this is exactly the same thing as a local trivialisation of π over U_N. The same applies to U_S.

(b) They are smooth local sections, since $\pi \circ \phi_k(p) = p$. They are non-vanishing because ϕ_k is bijective: $\phi_k^{-1} \circ f_k(p) = (1, p) \neq (0, p)$.

(c) As stated in the previous part, ϕ_S is bijective, so $(\chi(p), p) = \phi_S^{-1}(f_N(p)) = \phi_S^{-1}(\phi_N(1, p))$. We see that χ is actually the transition function $g_{U_N,U_S} : U_N \cap U_S \to \mathrm{GL}(\mathbb{R}) = \mathbb{R}^\times$, which shows it is non-vanishing. Because H_+ is connected and χ is continuous $\chi(H_+)$ is connected in $\mathbb{R}^\times$, so it has a definite sign. The same applies to H_-.

(d) Assume that the sign is positive. If not, work with $-\chi$. Let φ_k be a partition of unity subordinate to $\{U_N, U_S\}$. That means they are non-negative functions such that $\varphi_N + \varphi_S = 1$ and they have compact support within their domains. It is possible to explicitly construct these functions if you like. Let $f = \varphi_N \cdot f_N + \varphi_S \cdot f_S$. We will show that this is a non-vanishing section.

It is a well defined global section, because where f_N is not defined we know that φ_N is zero, ditto for f_S. It is also non-zero: at N, S because $f(N) = f_N(N)$ and $f(S) = f_S(S)$. At $x \in U_N \cap U_S$, in the trivialisation ϕ_S we calculate:

$$\phi_S^{-1} \circ f(x) = (\chi(x) \cdot \varphi_N(x) + \varphi_S(x), x).$$

The first component cannot be zero. Therefore f is non-vanishing. This shows that E is trivial.

(e) The first step is to check that G is well-defined. Suppose that $x \in U_N \cap U_S$, so that both formulae apply. Choose a point $v \in \pi^{-1}[\{x\}]$. In the ϕ_N trivialisation we can write this point as $v = a \cdot f_N(x)$ for $a \in \mathbb{R}$, because the fibre is one-dimensional and $f_N(x)$ is never the zero vector. Likewise we can write $v = b \cdot f_S(x)$ for $b \in \mathbb{R}$. How are a and b related? By the transition function!

$$\begin{aligned} v = a \cdot f_N(x) = a \cdot \phi_N(1, x) = a \cdot \phi_S \circ \phi_S^{-1} \circ \phi_N(1, x) = a \cdot \phi_S(g_{U_N,U_S}(x)1, x) \\ = a \cdot \phi_S(\chi(x)1, x) = a\chi(x) \cdot f_S(x) = b \cdot f_S(x). \end{aligned}$$

Another way to talk about this calculation is that the coordinates of v with respect to the ϕ_N trivialisation is (a, x). If you apply the transition function you get the coordinates with respect to ϕ_S, namely $(b,x) = (g_{U_N,U_S}(x)a, x) = (\chi(x)a, x)$. This is just a difference in styles between talking about coordinates in $\mathbb{R} \times U_k$ using trivialisations ϕ_k and talking about sections $f_k(x) \in E$.

Thus we can compare

$$\begin{aligned} G(a\, f_N(p)) &= a \cdot \phi_{M,N}(1,p) \\ G(bf_S(p)) &= \frac{b}{|\chi(p)|} \cdot \phi_{M,S}(1,p) \\ &= a\, \mathrm{sign}\chi(p) \cdot \phi_{M,S}(1,p) \end{aligned}$$

The transition function for M given in exercise 16(c) is $1\cdot$ on H_+ and $-1\cdot$ on H_-. In other words, it is $\mathrm{sign}\chi$. Thus $\phi_{M,N}(1,p) = \mathrm{sign}\chi(p) \cdot \phi_{M,S}(1,p)$, which shows that the two formula agree.

By its definition G is linear on the fibres. Also $\pi_M(G(v)) = p = \pi(v)$. So it is a bundle homomorphism. It's also easy to define the inverse

$$\forall p \in U_N \;:\; H(\phi_{M,N}(1,p)) = f_N(p) \text{ and } \forall p \in U_S \;:\; H(\phi_{M,S}(1,p)) = |\chi(p)| \cdot f_S(p).$$

This is well-defined for the same reasons G is and the other properties are similarly proved.

Perhaps the isomorphism is even clearer if we define compatible trivialisations $\tilde{\phi}_N(w,p) = \phi_N(w,p)$ and $\tilde{\phi}_S(w,p) = |\chi(p)|^{-1} \cdot \phi_S(w,p)$ on E. The (only) cocycle $\tilde{g}_{U_N,U_S}$ is exactly the same as the cocycle of M.