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Analysis III
7. Exercises

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18. Let (E, B, π) be a vector bundle.

- **(a)** **Cutting through a target.** Let $b_0 \in B$ and $v \in \pi^{-1}[\{b_0\}]$. Show: There exists a global section $f : B \to E$ with $f(b_0) = v$.
  Hint: start with a local trivialisation of π in a neighbourhood U of b_0 and see if you can construct a local section whose support is contained in U.
- **(b)** **Yet another differentiability test.** Show for every manifold Z and any arbitrary map $g : B \to Z$: if $g \circ \pi$ is smooth, so too is g.
  (Compare this to Exercise 15.)

19. **Triviality of the homomorphism bundle.**

Let (E, B, π) and (E', B, π') be two vector bundles over a base manifold B. Consider the *homomorphism bundle* $(\mathrm{Hom}(E, E'), B, \pi'')$.

- **(a)** What is the rank of $\mathrm{Hom}(E, E')$.
- **(b)** Show that when E and E' are trivial bundles, then so too is $\mathrm{Hom}(E, E')$.
- **(c)** Prove or disprove: $\mathrm{Hom}(E, E')$ is trivial, then the bundles E and E' must be trivial.
  Hint: Examine the Möbius bundle M from Exercise 16(c). Every homomorphism from $\mathbb{R}$ to $\mathbb{R}$ has the form $x \mapsto a\,x$, for some $a \in \mathbb{R}$. Why is the choice of a is independent from choice of trivialisation $H_\pm$.

20. **The dual bundle of a vector bundle.**

Let (E, B, π) be a vector bundle over a manifold B, with fibre $F = \mathbb{K}^n$. Further, let $\mathcal{U}$ be an open cover of B so that π trivialise over every set $U \in \mathcal{U}$. Denote the cocycles of π with respect to this cover by $g_{U,V} : U \cap V \to \mathrm{GL}(\mathbb{K}^n)$.

Show that the *dual bundle* $(\tilde{E}, B, \tilde{\pi})$ to π is described over $\mathcal{U}$ by the cocycle $(\tilde{g}_{U,V})_{U,V \in \mathcal{U}}$ mit

$$\tilde{g}_{U,V} : U \cap V \to \mathrm{GL}(\mathbb{K}^n),\ x \mapsto \big(g_{U,V}(x)^T\big)^{-1}.$$

21. **Classification of line bundles over $\mathbb{S}^1$.**

In this exercise we want to show: every real line bundle over the circle $\mathbb{S}^1$ is either trivial or isomorphic to the Möbius bundle.

Let $(E, \mathbb{S}^1, \pi)$ be a line bundle and recall the notation of Exercise 16(c), namely the cover $\{U_N, U_S\}$ of $\mathbb{S}^1$ with $U_N \cap U_S = H_- \sqcup H_+$.

(a) With the help of 16(b), argue why we can assume that π trivialises over the cover $\{U_N, U_S\}$.

(b) Let $\phi_k : \mathbb{R} \times U_k \to \pi^{-1}[U_k]$ be trivialisations of π. Show that $f_k : U_k \to E$, $p \mapsto \phi_k(1,p)$ are non-vanishing local sections.

(c) Prove that there exists a function $\chi : U_N \cap U_S \to \mathbb{R}$ with

$$\forall p \in U_1 \cap U_2 \;:\; \phi_S(\chi(p), p) = f_N(p).$$

Explain why it is non-vanishing, why its sign is constant on H_+, and why its sign is constant on H_-.

(d) Suppose that χ has the same sign on H_+ and H_-. Show that the bundle π is trivial.

(e) Lastly we consider the case that χ has different signs on the two sets $H_\pm$; assume that $\chi|H_+ > 0$ und $\chi|H_- < 0$. Let $(M, \mathbb{S}^1, \pi_M)$ denote the Möbius bundle with trivialisations $\phi_{M,k} : \mathbb{R} \times U_k \to \pi_M^{-1}[U_k]$ compatible with the cocycle given in 16(c) (to construct these was part of the exercise, check the solution for more details).

Show that the vector bundle homomorphism $G : E \to M$ given by

$$\forall p \in U_1 \;:\; G(f_1(p)) = \phi_{M,1}(1,p) \text{ and } \forall p \in U_2 \;:\; G(f_2(p)) = \frac{1}{|\chi(p)|} \cdot \phi_{M,2}(1,p)$$

is well defined, and that it is in fact a vector bundle isomorphism.