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Analysis III
6. Exercises

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14. Sections of vector bundles.

Let (E, B, π) be a $\mathbb{K}$-vector bundle, $f, f_1, f_2 : B \to E$ smooth sections of (E, B, π), and $g : B \to \mathbb{K}$ a smooth function. Show:

(a) The *zero section* $O : B \to E,\ b \mapsto 0_b$ is a smooth section. By 0_b we mean this: $F_b = \pi^{-1}[\{b\}]$ is a vector space, so it has a zero element $0_b \in F_b \subset E$. *(2 Points)*

(b) $f_1 + f_2$ and $g \cdot f$ are smooth sections. *(2 Points)*

(c) Interpret g as a global section of the trivial bundle $E = \mathbb{K} \times B$. *(1 Point)*

(d) The image $f[B]$ is a submanifold of E. *(3 Points)*

Solution. In parts (a) and (b) we need to show something is a smooth section. A section of a vector bundle (E, B, π) is a smooth map $s : B \to E$ with $\pi \circ s = \mathrm{id}_B$ (Definition 1.57).

(a) First we describe special charts on E. Choose neighbourhood U of b such that there is a local trivialisation of the vector bundle over U. A local trivialisation is a diffeomorphism $\phi : F \times U \to \pi^{-1}[U]$ (Definition 1.49). By decreasing U if necessary, we may assume that U belongs to a chart $\psi : U \to \mathbb{R}^n$ of B. Both ϕ and ψ together give us a chart $\chi = (\mathrm{id} \times \psi) \circ \phi^{-1}$ of E, namely

$$\chi : E \to F \times \mathbb{R}^n, \;\; e \mapsto (f, b) = \phi^{-1}(e) \mapsto (f, \psi(b))$$

This is a special type of chart of E that shows the bundle structure.

We use the chart ψ on B and χ on E to write O in local coordinates $\chi \circ O \circ \psi^{-1}$, which will allow us to see if it is a smooth map. In the local trivialisation, the point 0_b splits as $\phi^{-1}(0_b) = (0, b)$ because its definition is that it is the zero element of the fibre F. So $\chi \circ O(b) = (0, \psi(b))$. Hence $\chi \circ O \circ \psi^{-1}(x) = (0, \psi(\psi^{-1}(x))) = (0, x)$. This is a smooth function (in the Euclidean sense), so O is smooth (in the manifold sense).

The second property that we must show, namely that $\pi \circ O = \mathrm{id}_B$, follows from the property of local trivialisations that they respect the projection π; explicitly $\pi \circ \phi(f, b) = b$. Then

$$\pi \circ O(b) = \pi \circ \phi \circ \phi^{-1} \circ O(b) = \pi \circ \phi(0, b) = b.$$

We can turn this into a lemma: a function $f : B \to E$ is a smooth section exactly when for every trivialisation $\phi : F \times U \to \pi^{-1}[U]$ we have $\phi^{-1} \circ (f|_U)(b) = (\tilde{f}(b), b)$ for a smooth function $\tilde{f} : U \to F$.

Proof: $\chi \circ f \circ \psi^{-1}(x) = (\tilde{f} \circ \phi^{-1}(x), x)$, which is smooth exactly when $\tilde{f} : U \subset B \to F$ is smooth (F is a vector space, so we use the chart id_F). And we have seen that the $\pi \circ f$ is equal to the second component of $\phi^{-1} \circ f$, so this must be b.

(b) This question reduces to the fact that the operations are *fibre-wise*. This means that if we use a local trivialisation ϕ to write the value of the section in a pair (f, b) the operation acts on the first of the pair. Then the sum is a smooth section because the sum of smooth functions to F is a smooth function to F and apply (a):

$$\phi^{-1} \circ (f_1 + f_2)(b) := (\tilde{f}_1(b) + \tilde{f}_2(b), b).$$

Likewise

$$\phi^{-1} \circ (g \cdot f)(b) := (g(b) \cdot \tilde{f}(b), b)$$

is a smooth section.

(c) More generally from a global function $\tilde{h} : B \to F$ we get a section $b \mapsto (\tilde{h}(b), b)$ of the trivial bundle $F \times B$. Conversely, a section of this bundle gives a function through projection to the first component.

This does *not* work for any bundle, because in general we only have projection to the second component and this just gives back the point in B. Projection to the first component depends on the trivialisation ϕ. For this reason, sections are a generalisation of functions from a manifold B to a vector space.

(d) First observe that $\pi \circ f = \mathrm{id}_B$ shows that f is a homeomorphism. It remains to show that it an immersion. But we have already seen that a section in local coordinates has the form $\chi \circ f \circ \psi^{-1}(x) = (\tilde{f} \circ \psi^{-1}(x), x)$, so the Jacobian is $(J(\tilde{f} \circ \psi^{-1}) \,|\, \mathbb{1})$. The identity matrix block shows that it is injective.

15. The tangent bundles of low dimensional spheres.

In this exercise we will examine the tangent bundle of the n-sphere

$$\mathbb{S}^n := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \ldots + x_{n+1}^2 = 1 \},$$

for $n \leq 3$.

(a) We know that $\mathbb{S}^n$ is an n dimensional submanifold of $\mathbb{R}^{n+1}$ and so the embedding map ι is an immersion. Let v be a tangent vector in $T_x\mathbb{S}^n$. Show that $w := T_x(\iota)v \in \mathbb{R}^{n+1}$ is perpendicular to x.

Conversely, choose any $w \in \mathbb{R}^{n+1}$ with $\langle w, x \rangle = 0$ and set $\alpha(t) = (\cos |w|t)x + (\sin |w|t)\widehat{w}$. Show that $w = T_x(\iota)[\alpha]$. *(2 Points)*

Hence we make the identification

$$T_x\mathbb{S}^n = \{ w \in \mathbb{R}^{n+1} \mid \langle w, x \rangle = 0 \} .$$

This means that we can describe a section of $T\mathbb{S}^n$ as a smooth function $s : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $\langle s(x), x \rangle = 0$ for all $x \in \mathbb{S}^n$.

(b) Finde a non-vanishing section of the tangent bundle $T\mathbb{S}^1$ (a section that never takes the value 0).

Hence $T\mathbb{S}^1$ is trivial. *(3 Points)*

(c) Show that the vector bundle $T\mathbb{S}^3$ is trivial. *(2 Points)*

Hint. Use Lemma 1.58 and consider the following sections

$$f_1(x_1, x_2, x_3, x_4) := (-x_2, x_1, x_4, -x_3)\ , \quad f_2(x_1, x_2, x_3, x_4) := (-x_3, -x_4, x_1, x_2)$$
$$\text{and} \quad f_3(x_1, x_2, x_3, x_4) := (-x_4, x_3, -x_2, x_1)$$

Remark. We can identify $\mathbb{S}^3$ with the unit sphere in the Quaternions $\mathbb{H}$. Then $f_1 = ix$, $f_2 = jx$ and $f_3 = kx$.

(d) Let $x_N := (0, 0, 1) \in \mathbb{S}^2$ and $x_S := (0, 0, -1) \in \mathbb{S}^2$. With the aid of stereographic projection N and S, write down local trivialisations of $T\mathbb{S}^2$ over $U_N := \mathbb{S}^2 \setminus \{x_N\}$ and $U_S := \mathbb{S}^2 \setminus \{x_S\}$ (compare Example 1.56), and calculate the transition function $g_{U_N,U_S} : \mathbb{S}^2 \setminus \{N, S\} \to \mathrm{GL}(\mathbb{R}^2)$. *(8 Points)*

Remark. $T\mathbb{S}^2$ is not trivial, but this require some more theory to prove. It is a consequence of the "hairy ball theorem": every section of $T\mathbb{S}^2$ has a zero.

Solution.

(a) Let α be a path in $\mathbb{S}^n$ representing v, that is $x = \alpha(0)$ and $v = [\alpha]$. Then $T_x(\iota)v = [\iota \circ \alpha]$. But ι is just the identity map considered as a map between manifolds so $\iota \circ \alpha$ is just $t \mapsto \alpha(t) \in \mathbb{R}^{n+1}$ and $w = \alpha'(0)$.

To show that w is perpendicular to x, note that $|\alpha(t)|^2 = 1$ because it lies in the sphere. Differentiating gives $2\alpha(t) \cdot \alpha'(t) = 0$. At $t = 0$ this gives $x \cdot w = 0$.

Conversely, suppose $w \in \mathbb{R}^{n+1}$ is perpendicular to x. We need to find a path in $\mathbb{S}^n$ with this as its tangent vector: $\alpha(t) = (\cos |w|t)x + (\sin |w|t)\widehat{w}$ works.

(b) In $\mathbb{R}^2$ there is the rotation operator $R(x, y) = (-y, x)$. This creates an equal-length perpendicular vector, ie $|x| = |R(x)|$ and $x \cdot R(x) = 0$. The section $x \mapsto (R(x), x)$ is a section of the tangent bundle and non-vanishing.

(c) First, note the value of these functions are perpendicular to x, eg $(x_1, x_2, x_3, x_4) \cdot (-x_2, x_1, x_4, -x_3) = -x_1x_2+x_2x_1+x_3x_4-x_4x_3 = 0$, and unit length $|(-x_2, x_1, x_4, -x_3)| = |x| = 1$. Hence they are non-vanishing sections of $T\mathbb{S}^3$. It remains to show they are linearly independent. But this follows from the fact that they are all perpendicular, eg

$$f_1 \cdot f_2 = (-x_2, x_1, x_4, -x_3) \cdot (-x_3, -x_4, x_1, x_2) = x_2x_3 - x_1x_4 + x_4x_1 - x_3x_2 = 0.$$

Hence by Lemma 1.58 it follows that $T\mathbb{S}^3$ is trivial.

(d) This question relates to Example 1.56(i) which shows that a chart of a manifold gives a local trivialisation of its tangent bundle via the tangent map of the chart. So we should begin with the chart

$$N(x_1, x_2, x_3) = \frac{1}{1-x_3}(x_1, x_2) = \frac{1}{1-\langle x, e_3\rangle}(x - e_3) + e_3$$
$$N^{-1}(y_1, y_2) = \frac{1}{1+|y|^2}(2y_1, 2y_2, |y|^2 - 1) = \frac{1}{1+|y|^2}(2y + (|y|^2 - 1)e_3).$$

Take a vector $w \in T_{N(x)}\mathbb{R}^2$ for $x \in U_N$. We want to compute the push-forward of this vector $T_{N(x)}(N^{-1})w$. We can write a path representing w easily, $\beta(t) = wt + N(x)$. So compute $T_{N(x)}(N^{-1})w = (N^{-1} \circ \beta)'(0)$:

$$N^{-1} \circ \beta(t) = \frac{1}{1+|\beta(t)|^2}(2\beta(t) + (|\beta(t)|^2 - 1)e_3)$$
$$(N^{-1} \circ \beta)'(t) = -\frac{2\beta(t)\cdot\beta'(t)}{(1+|\beta(t)|^2)^2}(2\beta(t) + (|\beta(t)|^2 - 1)e_3)$$
$$+ \frac{1}{1+|\beta(t)|^2}(2\beta'(t) + 2\beta(t)\cdot\beta'(t)\, e_3)$$
$$= -\frac{2\beta(t)\cdot\beta'(t)}{1+|\beta(t)|^2}N^{-1}(\beta(t)) + \frac{1}{1+|\beta(t)|^2}(2\beta'(t) + 2\beta(t)\cdot\beta'(t)\, e_3)$$
$$(N^{-1} \circ \beta)'(0) = \frac{2}{1+|N(x)|^2}\,(w + \langle N(x), w\rangle\;(e_3 - x))$$
$$= (1 - x_3)\,(w + \langle N(x), w\rangle\;(e_3 - x))$$

Since $w \cdot x = w \cdot (x_1, x_2) = w \cdot (1 - x_3)N(x)$ and $|x|^2 = 1$ one can see that this vector above is perpendicular to x as expected. Thus

$$\phi_{U_N} : (w, x) \in \mathbb{R}^2 \times U_N \mapsto (1 - x_3)\,(w + \langle N(x), w\rangle\;(e_3 - x)) \in T_x\mathbb{S}^2$$

defines a trivialisation of $T\mathbb{S}^2$ over U_N.

If we repeat this calculation for

$$S(x_1, x_2, x_3) = \frac{1}{1+x_3}(x_1, x_2) = \frac{1}{1+\langle x, e_3\rangle}(x + e_3) - e_3$$
$$S^{-1}(y_1, y_2) = \frac{1}{1+|y|^2}(2y_1, 2y_2, 1 - |y|^2) = \frac{1}{1+|y|^2}(2y + (1 - |y|^2)e_3),$$

then we get the trivialisation

$$\phi_{U_S} : (w, x) \in \mathbb{R}^2 \times U_S \mapsto (1 + x_3)\,(w - \langle S(x), w\rangle\;(e_3 + x)) \in T_x\mathbb{S}^2$$

Next we want to find the transition between these two trivialisations, one approach is to compose them (where they are both defined on $U_N \cap U_S = Q$) to get a function $\phi_{U_S}^{-1} \circ \phi_{U_N} : \mathbb{R}^2 \times Q \to \mathbb{R}^2 \times Q$ and then the part that maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

gives $g_{U_N,U_S}(x)$. This method will work for any trivialisations; in the case that the trivialisations come from charts we give an easier method below.

$$\phi_{U_S}^{-1}(v) = \left(\frac{1}{1+x_3}v - \frac{v_3}{(1+x_3)^2}(e_3+x),\ x\right).$$

third component of $\phi_{U_N}(w,x) = (1-x_3)(0 + \langle N(x), w\rangle (1-x_3))$.

$$\begin{aligned}
&\phi_{U_S}^{-1}(\phi_{U_N}(w,x)) \\
&\quad = \frac{1-x_3}{1+x_3}\left(w + \langle N(x), w\rangle (e_3 - x) - \frac{\langle N(x), w\rangle (1-x_3)}{1+x_3}(e_3+x)\right) \\
&\quad = \frac{1-x_3}{1+x_3}\left(w + \frac{2\langle N(x), w\rangle}{1+x_3}(x_3e_3 - x)\right) \\
&\quad = \frac{1}{(1+x_3)^2}\left((1-x_3^2)w + 2(x_1w_1 + x_2w_2)(x_3e_3 - x)\right).
\end{aligned}$$

Note that $x_3e_3 - x$ is a vector in $\mathbb{R}^2$ so this is well defined. Further it is linear in w. So for any $x \in Q$ we have a linear map $g_{U_N,U_S}(x)$ that maps w to the above vector:

$$g_{U_N,U_S}(x)w = \frac{1}{(1+x_3)^2}\begin{pmatrix} 1 - x_3^2 - 2x_1^2 & -2x_2x_1 \\ -2x_1x_2 & 1 - x_3^2 + 2x_2^2 \end{pmatrix} w$$

This is an invertible linear transformation because the determinant $-(1+x_3)^{-2}(1-x_3)^2(x_1^2+x_2^2)$ is never zero on Q.

Another approach to finding the transition is to use the fact that the local trivialisations are $T(\phi_{U_N}^{-1})$ and $T(\phi_{U_S}^{-1})$ so that

$$g_{U_N,U_S} = (T(\phi_{U_S}^{-1}))^{-1} \circ T(\phi_{U_N}^{-1}) = T(\phi_{U_S}) \circ T(\phi_{U_N}^{-1}) = T(\phi_{U_S} \circ \phi_{U_N}^{-1}).$$

The transition between charts $\phi_{U_S} \circ \phi_{U_N}^{-1}$ is simply $y \mapsto \|y\|^{-2}\, y$. And this is a map between Euclidean spaces, so the tangent map is just Jacobian and we calculate as normal:

$$J(\phi_{U_S} \circ \phi_{U_N}^{-1}(y)) = \frac{1}{\|y\|^4}\begin{pmatrix} y_2^2 - y_1^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix}.$$

We should write it not in local coordinates $y \in \mathbb{R}^2$ but rather in terms of $x \in \mathbb{S}^2$, with $y = N(x)$. Then

$$g_{U_N,U_S}(x) = \frac{1}{(1+x_3)^2}\begin{pmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix},$$

which is the same as the first method because of the relation $1 = x_1^2 + x_2^2 + x_3^2$.

16. Trivial and non-trivial bundles.

(a) The tangent bundle of a vector space is trivial. Let V be a finite dimensional $\mathbb{K}$-vector space. Show that the tangent bundle TV is trivial. *(2 Points)*

(b) Line bundles over $\mathbb{R}$ are trivial. Prove that ever line bundle (a vector bundle whose fibre dimension is 1) over $\mathbb{R}$ is trivial. *(8 Points)*

Hint. Let $(E, \mathbb{R}, \pi)$ be a line bundle. Choose a point x_0 and show that there is an interval $(x_0 - \epsilon, x_0 + \epsilon)$ with a non-vanishing section s. Then consider

$$J := \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{There exists an extension } s_x \text{ of } s \text{ to } (x, x_0 + \epsilon) \text{ or } (x_0 - \epsilon, x), \\ \text{such that } s_x \text{ is non-vanishing} \end{array} \right\},$$

where the choice of $(x, x_0 + \epsilon)$ or $(x_0 - \epsilon, x)$ depends whether $x \leq x_0$ or $x \geq x_0$ Show that J is non-empty, open. Argue further that $J = \mathbb{R}$.

(c) A non-trivial line bundle over $\mathbb{S}^1$. On the circle $\mathbb{S}^1 \subset \mathbb{R}^2$ choose the poles $x_N = (0, 1)$ and $x_S = (0, -1)$. Then set

$$U_N := \mathbb{S}^2 \setminus \{x_N\} \quad \text{and} \quad U_S := \mathbb{S}^2 \setminus \{x_S\}.$$

The intersection $Q = U_N \cap U_S = \mathbb{S}^1 \setminus \{x_N, x_S\}$ consists of two connected components H_+ und H_-.

Work through the construction following Beispiel 1.51 of cocycles, that there is a line bundle determined by the cover (U_N, U_S),$F := \mathbb{R}$ and the function

$$g_{U_N, U_S} : U_N \cap U_S \to \mathrm{GL}(\mathbb{R}),\; x \mapsto \begin{cases} \mathrm{id}_{\mathbb{R}} & \text{for } x \in H_+ \\ -\mathrm{id}_{\mathbb{R}} & \text{for } x \in H_- \end{cases}.$$

Prove that this bundle is non-trivial.

It is called the *Möbius band* or *Möbius bundle.* *(8 Points)*

Hint about the non-triviality: Suppose you had a non-vanishing section and examine it in the local trivialisations.

Solution.

(a) Let $\{e_k\}$ be the standard basis vectors of V. Recall in a vector space we have an identification of $T_x V$ with V given by usual derivative. So then $f_k : x \mapsto e_k$ is a section of TV. These sections are non-vanishing, linearly independent, and there are $\dim V$ of them, so they trivialise TV.

(b) We might as well take $x_0 = 0$. There is a trivialisation ϕ_U over $U \ni 0$. U is open, so contains an interval of the form $(-\epsilon, \epsilon)$. We may restrict to this interval, so we may assume that $U = (-\epsilon, \epsilon)$. Consider the local section s over U given by $x \mapsto \phi_U(1, x) \in E$. This is non-vanishing.

With s in hand, we can now define J. Immediately $(-\epsilon, \epsilon) \subset J$ so it is non-empty. Choose $x \in J$ with $x > 0$. Let V be a trivialisation containing x. Again we can assume that $V = (x-\eta, x+\eta)$ with $x - \eta > 0$. In this trivialisation, for $y \in (x-\eta, x)$ the local expression $\phi_V^{-1} \circ s_x(y) = (f(y), y)$. We know that $f(y)$ is a non-vanishing smooth function. This extends to a non-vanishing function $\tilde{f}(y)$ on all of $(x-\eta, x+\eta)$. Then

$$s_{x+\eta}(y) = \begin{cases} s_x(y) & \text{for } y \in (-\epsilon, x) \\ \phi_V(\tilde{f}(y), y) & \text{for } y \in (x-\eta, x+\eta) \end{cases}$$

is a smooth non-vanishing section on $(-\epsilon, x+\eta)$. There is a similar proof for $x < 0$. This shows that J is open.

Suppose that there were a point $x \notin J$ with $x > 0$. Because J is open $[0, x+1] \cap (\mathbb{R}^+ \setminus J)$ is compact. Thus there is a minimum point $x \notin J$ with $x > 0$. Choose an interval $(x-\eta, x+\eta)$ over which E trivialises. Since x is minimal there must exist a non-vanishing section s_t over $(-\epsilon, t)$ for $t \in (x-\eta, x)$. But then we can extend s_t to $(-\epsilon, x+\eta)$ in the same way as above. Therefore $x \in J$, which is a contradiction. A similar argument shows that all negative points belong to J as well. This completes the proof that $J = \mathbb{R}$.

(c) Consider the two spaces $M_N = \mathbb{R} \times U_N$ and $M_S = \mathbb{R} \times U_S$ with the relation $(v, x) \in M_N \sim (w, x) \in M_S$ if $x \in Q$ and $w = g_{U_N, U_S}(x) v$. Let M be the set of equivalence classes. There are inclusions maps $M_1, M_2 \hookrightarrow M$. A set is open in M if its restriction to both M_N and M_S are open. This gives M a topology. In particular, $M_N \cup M_S$ is an open cover of M. The functions id $\times N$ and id $\times S$ are charts of M. This makes M a manifold. For any point of M its x value is well defined because it is the same under the equivalence relation. $\pi(m) = x$ then makes $(M, \mathbb{S}^1, \pi)$ a vector bundle with the local trivialisations $M_N \to \pi^{-1}[U_N]$ and $M_S \to \pi^{-1}[U_S]$.

Suppose this bundle were trivial. Then there would be a non-vanishing section f. Over M_N, it would have the form $U_N \to M_N$, $x \mapsto (f_N(x), x)$ with $f_N : U_N \to \mathbb{R}$. Since f is non-vanishing and U_N is connected , f_N has a constant sign; it is either entirely positive or entirely negative. Without loss of generality assume that it is positive (or consider $-f$). But now consider f over M_S namely $U_S \to M_S$, $x \mapsto (f_S(x), x)$. By the same reasoning, f_S must have a constant sign, but we can show that it does not. What is the relation between f_N and f_S? It is $f_S(x) = g_{U_N, U_S}(x) f_N(x)$. This tells us that f_S is also positive on H_+ and negative on H_-. This we have a contradiction, and there cannot be a non-vanishing section.

An alternative way to make this argument would be to suppose there was a section of M that was non-vanishing on U_N. Then because f_S would have one sign on H_+ and a different sign on H_- is must vanish at x_S by the intermediate value theorem (Zwischenwertsatz).

Terminology

Schnitt = section

nullstellenfreien = non-vanishing

Geradenbündel = line bundle

American spelling is fiber, British spelling is fibre.