Martin Schmidt
Ross Ogilvie

Analysis III
6. Exercises

19 April 2022

14. Sections of vector bundles.

Let (E, B, π) be a $\mathbb{K}$-vector bundle, $f, f_1, f_2 : B \to E$ smooth sections of (E, B, π), and $g : B \to \mathbb{K}$ a smooth function. Show:

(a) The *zero section* $O : B \to E,\ b \mapsto 0_b$ is a smooth section. By 0_b we mean this: $F_b = \pi^{-1}[\{b\}]$ is a vector space, so it has a zero element $0_b \in F_b \subset E$. *(2 Points)*

(b) $f_1 + f_2$ and $g \cdot f$ are smooth sections. *(2 Points)*

(c) Interpret g as a global section of the trivial bundle $E = \mathbb{K} \times B$. *(1 Point)*

(d) The image $f[B]$ is a submanifold of E. *(3 Points)*

15. The tangent bundles of low dimensional spheres.

In this exercise we will examine the tangent bundle of the n-sphere

$$\mathbb{S}^n := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \ldots + x_{n+1}^2 = 1 \},$$

for $n \leq 3$.

(a) We know that $\mathbb{S}^n$ is an n dimensional submanifold of $\mathbb{R}^{n+1}$ and so the embedding map ι is an immersion. Let v be a tangent vector in $T_x\mathbb{S}^n$. Show that $w := T_x(\iota)v \in \mathbb{R}^{n+1}$ is perpendicular to x.

Conversely, choose any $w \in \mathbb{R}^{n+1}$ with $\langle w, x\rangle = 0$ and set $\alpha(t) = (\cos|w|t)x + (\sin|w|t)\widehat{w}$. Show that $w = T_x(\iota)[\alpha]$. *(2 Points)*

Hence we make the identification

$$T_x\mathbb{S}^n = \{ w \in \mathbb{R}^{n+1} \mid \langle w, x\rangle = 0 \} .$$

This means that we can describe a section of $T\mathbb{S}^n$ as a smooth function $s : \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $\langle s(x), x\rangle = 0$ for all $x \in \mathbb{S}^n$.

(b) Finde a non-vanishing section of the tangent bundle $T\mathbb{S}^1$ (a section that never takes the value 0).

Hence $T\mathbb{S}^1$ is trivial. *(3 Points)*

(c) Show that the vector bundle $T\mathbb{S}^3$ is trivial. *(2 Points)*

Hint. Use Lemma 1.58 and consider the following sections

$$f_1(x_1, x_2, x_3, x_4) := (-x_2, x_1, x_4, -x_3)\ , \quad f_2(x_1, x_2, x_3, x_4) := (-x_3, -x_4, x_1, x_2)$$
$$\text{and} \quad f_3(x_1, x_2, x_3, x_4) := (-x_4, x_3, -x_2, x_1)$$

Remark. We can identify $\mathbb{S}^3$ with the unit sphere in the Quaternions $\mathbb{H}$. Then $f_1 = ix$, $f_2 = jx$ and $f_3 = kx$.

(d) Let $x_N := (0,0,1) \in \mathbb{S}^2$ and $x_S := (0,0,-1) \in \mathbb{S}^2$. With the aid of stereographic projection N and S, write down local trivialisations of $T\mathbb{S}^2$ over $U_N := \mathbb{S}^2 \setminus \{x_N\}$ and $U_S := \mathbb{S}^2 \setminus \{x_S\}$ (compare Example 1.56), and calculate the transition function $g_{U_N,U_S} : \mathbb{S}^2 \setminus \{N,S\} \to \mathrm{GL}(\mathbb{R}^2)$. *(8 Points)*

Remark. $T\mathbb{S}^2$ is not trivial, but this require some more theory to prove. It is a consequence of the "hairy ball theorem": every section of $T\mathbb{S}^2$ has a zero.

16. Trivial and non-trivial bundles.

(a) **The tangent bundle of a vector space is trivial.** Let V be a finite dimensional $\mathbb{K}$-vector space. Show that the tangent bundle TV is trivial. *(2 Points)*

(b) **Line bundles over $\mathbb{R}$ are trivial.** Prove that ever line bundle (a vector bundle whose fibre dimension is 1) over $\mathbb{R}$ is trivial. *(8 Points)*

Hint. Let $(E, \mathbb{R}, \pi)$ be a line bundle. Choose a point x_0 and show that there is an interval $(x_0 - \epsilon, x_0 + \epsilon)$ with a non-vanishing section s. Then consider

$$J := \left\{ x \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{There exists an extension } s_x \text{ of } s \text{ to } (x, x_0+\epsilon) \text{ or } (x_0-\epsilon, x), \\ \text{such that } s_x \text{ is non-vanishing} \end{array} \right\},$$

where the choice of $(x, x_0+\epsilon)$ or $(x_0-\epsilon, x)$ depends whether $x \le x_0$ or $x \ge x_0$ Show that J is non-empty, open. Argue further that $J = \mathbb{R}$.

(c) **A non-trivial line bundle over $\mathbb{S}^1$.** On the circle $\mathbb{S}^1 \subset \mathbb{R}^2$ choose the poles $x_N = (0,1)$ and $x_S = (0,-1)$. Then set

$$U_N := \mathbb{S}^2 \setminus \{x_N\} \quad \text{and} \quad U_S := \mathbb{S}^2 \setminus \{x_S\}.$$

The intersection $Q = U_N \cap U_S = \mathbb{S}^1 \setminus \{x_N, x_S\}$ consists of two connected components H_+ und H_-.

Work through the construction following Beispiel 1.51 of cocycles, that there is a line bundle determined by the cover (U_N, U_S),$F := \mathbb{R}$ and the function

$$g_{U_N,U_S} : U_N \cap U_S \to \mathrm{GL}(\mathbb{R}),\ x \mapsto \begin{cases} \mathrm{id}_{\mathbb{R}} & \text{for } x \in H_+ \\ -\mathrm{id}_{\mathbb{R}} & \text{for } x \in H_- \end{cases}.$$

Prove that this bundle is non-trivial.

It is called the *Möbius band* or *Möbius bundle.* *(8 Points)*

Hint about the non-triviality: Suppose you had a non-vanishing section and examine it in the local trivialisations.

Terminology

Schnitt = section

nullstellenfreien = non-vanishing

Geradenbündel = line bundle

American spelling is fiber, British spelling is fibre.