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Analysis III
5. Exercises

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14. Alternative version of the Constant Rank Theorem.

Let M and N be manifolds of dimensions m and n respectively, and $f : M \to N$ a map of constant rank r. Consider the standard set-up: at every point $p \in X$ there exists charts $\phi : U \to \mathbb{R}^m$ of M and $\psi : V \to \mathbb{R}^n$ of N with $p \in U$, $\phi(p) = 0$, $f[U] \subset V$ and $\psi(f(p)) = 0$.

Show: That there exists such charts ϕ and ψ with the further property that $\psi \circ f \circ \phi^{-1}$ has the form

$$\psi \circ f \circ \phi^{-1} : \phi[U] \to \mathbb{R}^n,\ (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{n-r})\,. \qquad \textit{(3 Points)}$$

[Hint. Apply Theorem 1.44.]

Remark. This theorem shows that maps of constant rank can be written as a composition of a submersion and an immersion in a neighbourhood of every point (i.e. locally). For this reason such maps are sometimes called *Subimmersions*. Can you see the connection between this result and the first isomorphism theorem of linear algebra?

Solution. We apply Theorem 1.44 as suggested. Thus we know that $\psi \circ f \circ \phi^{-1}$ is equal to some linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ restricted to $\phi[U]$. The rest is just linear algebra.

We know that there is a basis of $\mathbb{R}^m$ such that A has reduced row eschleon form. Further choose the basis $\{Ae_j | 1 \leq j \leq r\}$ for the image of A and extend it to a basis of $\mathbb{R}^n$. In these bases A has the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Let L and R be the change of base matrices of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. We can view these as invertible linear operators $\mathbb{R}^m \to \mathbb{R}^m$ and $\mathbb{R}^n \to \mathbb{R}^n$. Finally, notice that $L^{-1} \circ \phi$ is a chart of M and $\mathbb{R} \circ \psi$ is a chart of N, and in these charts f has the desired expression in local coordinates.

Clearly a map which has this form has locally constant rank. Hence this exercise is equivalent to the constant rank theorem. In fact, many books call this the constant rank theorem.

15. A differentiability test.

Let X, Y, and Z be manifolds, $g : Y \to Z$ an immersion, and $f : X \to Y$ a continuous function. Show that, if $g \circ f$ is smooth, then f must be smooth. *(2 Points)*

[Hint. Exercise 14.]

Solution. Choose a point $a \in X$ and let $b := f(a) \in Y$ and $c := g(b) \in Z$. Since g is an immersion, its rank is $n = \dim Y$, we apply Exercise 16 to choose charts ψ on Y and χ on Z with the special form

$$\chi \circ g \circ \psi^{-1}(y_1, \dots, y_n) = (y_1, \dots, y_n, 0, \dots, 0).$$

Choose any chart ϕ on X containing a. The assumption is that $\chi \circ g \circ f \circ \phi^{-1}$ is smooth and we must show that $\psi \circ f \circ \phi^{-1}$ is smooth. But

$$\begin{aligned} \chi \circ g \circ f \circ \phi^{-1}(x) &= (\chi \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})(x) \\ &= \Big((\psi \circ f \circ \phi^{-1})_1(x), \dots, (\psi \circ f \circ \phi^{-1})_n(x), 0, \dots, 0\Big) \end{aligned}$$

shows that every component of $\psi \circ f \circ \phi^{-1}$ is smooth. Hence we are done.

This shows that the version of the constant rank theorem in exercise 14 is very useful in applications.

16. The Cartesian product of manifolds.

Let M and N be manifolds with dimensions m and n respectively. Verify the claim from Section 1.6 that "the map $\phi \times \psi$ is a chart of $M \times N$ and the collection of all such maps is an atlas." This shows that $M \times N$ is a manifold with dimension $m + n$.

You may assume that the map is homeomorphism if you don't want to try to understand the topology of the product. In that case, you must still show each map is bijective and they are compatible. *(4 Points)*

Solution. First, let us recall what we mean by $P := M \times N$. It is the set of pairs (x, y) with $x \in M$ and $y \in N$. This set can be given a topology: $S \subset P$ is open if and only if for every point $(x, y) \in P$ there are neighbourhoods $U \subset M$ of x and $V \subset N$ of y such that $U \times V \subset S$. This is called the *product topology*.

This way of defining a topology is similar to using balls to define the topology of $\mathbb{R}^n$; a ball is a special type of neighbourhood of $\mathbb{R}^n$. Indeed, if we consider $\mathbb{R} \times \mathbb{R}$ as an example, then the special neighbourhoods are open rectangles, or balls in the 1-norm. Since we know from Analysis II that every norm on $\mathbb{R}^n$ gives the same topology, the product topology of $\mathbb{R}^2$ is the same as its usual topology.

The lecture notes address the question of whether the product is Hausdorff and Lindelöf: it is. So we proceed to examine the charts.

Let $\phi : U \to \mathbb{R}^m$ be a chart of M and $\psi : V \to \mathbb{R}^n$ be a chart of N. We claim that

$$\phi \times \psi : U \times V \to \mathbb{R}^{m+n}, \;\; \phi \times \psi(x, y) := (\phi(x), \psi(y))$$

is a chart of P. It is a bijection because $\phi^{-1} \times \psi^{-1}(v, w) := (\phi^{-1}(v), \psi^{-1}(w))$ is its inverse. Let $f = \phi \times \psi$.

You might want to skip this step: We next show that the function is a homeomorphism. Note that $U \times V$ is an open set of P by definition of the product topology. Choose any open set $S \subset f(U \times V) \subset \mathbb{R}^{m+n}$ and consider $R := f^{-1}[S]$. Choose any point of (x, y) of R. Because S is open, there is an open rectangle $I \times J \subset S$ containing $f(x, y)$. Then $f^{-1}[I \times J] = \phi^{-1}[I] \times \psi^{-1}[J]$ is an open neighbourhood of (x, y) since ϕ and ψ are continuous. Since this holds for every point of S, this set is open. The proof that f^{-1} is continuous is the same.

Hence f is a chart of P. Finally, we must show that all such charts are compatible. But consider the transition functions, they have the form:

$$\tilde{f} \circ f^{-1}(v, w) = (\tilde{\phi} \times \tilde{\psi})(\phi^{-1}(v), \psi^{-1}(w)) = (\tilde{\phi}(\phi^{-1}(v)), \tilde{\psi}(\psi^{-1}(w))).$$

The components of this map are just transitions function of M and N, which are smooth. And since these are maps between open subsets of Euclidean space, they are smooth if and only if the components are smooth. Therefore all the transition functions are smooth, i.e. the charts are all compatible.

17. More examples of submanifolds.

This question is an application of Exercise 13(b).

(a) Let $a > 0$ and $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = \left(a - \sqrt{x^2 + y^2}\right)^2 + z^2.$$

Show that the preimage $f^{-1}[\{b^2\}]$ is a submanifold of $\mathbb{R}^3$ for every b with $0 < b < a$. What is this space?

(b) Investigate for which values $t \in \mathbb{R}$

$$A_t := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = t \}$$

is a submanifold of $\mathbb{R}^{n+1}$. What is this space?

(4 Points)

Solution. If the gradient does not vanish on the preimage, then Example 1.18(iv) shows that they are manifolds and Exercise 13(b) shows that they are submanifolds. In (a), the gradient vanishes only at the origin, which does not belong to the preimage because $b < a$. In (b) the gradient also only vanishes at the origin, and this belongs to the level set A_0. The spaces in (a) are tori (doughnut/donut shape), and in (b) for $t > 0$ it is one-sheeted hyperboloid and for $t < 0$ it is a two-sheet hyperboloid.

18. The Veronese map

Let $S := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 3 \}$ and the Veronese map $f : S \to \mathbb{R}^5$ be defined by

$$f(x, y, z) = \left(xy, xz, yz, \tfrac{1}{2}(x^2 - y^2), \tfrac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2) \right) .$$

(a) Show that f is an immersion. *(3 Points)*

(b) Consider two points in $(x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}) \in S$ whose image under f is equal, that is, $f(\widetilde{x}, \widetilde{y}, \widetilde{z}) = f(x, y, z)$. Show that this occurs if and only if $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \pm(x, y, z)$. Thus $f : S \to \mathbb{R}^5$ is not an embedding. *(3 Points)*

[Hint. The solution can be simplified by using complex numbers. Set $w := x + iy$ and $\widetilde{w} := \widetilde{x} + i\widetilde{y}$. Rewrite the first, fourth, and fifth components of f using $\Im w^2$, $\Re w^2$, and $|w|^2$]

(c) Show that $f[S] \subset \mathbb{R}^5$ is a submanifold.

This part is very difficult. I've included it here so you can understand how difficult it can be to show that the image of an immersion is a submanifold.

Remark. The manifold $f[S]$ is called the real projective plane $\mathbb{RP}^2$. There is a real projective space $\mathbb{RP}^n$ for every dimension n. These spaces have beautiful geometry. For example, every pair of lines in the real projective plane intersect exactly once. The natural way to define these spaces is not as a submanifold. I will try to talk about these spaces at some point, but it doesn't really fit with the themes of this course. The Veronese map shows however that $\mathbb{RP}^n$ can be embedded in Euclidean space. This raises an important question: can every manifold be embedded in Euclidean space?

Solution.

(a) We have already practised with immersion in Exercise 12, but this is more difficult. Let $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^5$ be given by the same formula. The differential is

$$dF = \begin{pmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \\ x & -y & 0 \\ \frac{1}{\sqrt{3}}x & \frac{1}{\sqrt{3}}y & -\frac{2}{\sqrt{3}}z \end{pmatrix}$$

If x is not zero, the first, second, and fourth rows are linearly independent. If y is not zero, the first, third, and fourth row are linearly independent. If both $x = y = 0$ then it must be that z is not zero, in which case the second, third, and fifth rows are independent. In every case the rank is 3, so F is an immersion at every point of $\mathbb{R}^3$ except the origin. Because S is a submanifold of $\mathbb{R}^3$, the identity map is an embedding $\iota : S \to \mathbb{R}^3$. Since $f = F \circ \iota : S \to \mathbb{R}^3 \to \mathbb{R}^5$ and the composition of two immersions is an immersion, this shows that f is an immersion.

(b) We follow the hint. The first component of $f(x, y, z)$ is $\frac{1}{2}\Im w^2$ and the fourth component is $\frac{1}{2}\Re w^2$. If $f(\widetilde{x}, \widetilde{y}, \widetilde{z}) = f(x, y, z)$ then we have $w^2 = \widetilde{w}^2$, which has two solutions $w = \pm\tilde{w}$. In either case, the fifth component shows us that $|w|^2 - 2z^2 = |\tilde{w}|^2 - 2\tilde{z}^2 \Rightarrow z = \pm\tilde{z}$. This gives four possibilities. But checking the second component shows that there are in fact only two: $(x, y, z) = \pm(\tilde{x}, \tilde{y}, \tilde{z})$.

(c) Because f is not injective, it cannot be a homeomorphism, and therefore we cannot apply Definition 1.42 directly. However with Theorem 1.45 we can still show the result. That theorem says that being a submanifold is a local property of the subset: $X \subset Y$ is a submanifold exactly when for all points $x \in X$ there is a neighbourhood U_x in Y such that $X \cap U_x$ is a submanifold.

Notice that we can write the last component of f as $\frac{\sqrt{3}}{2}(1 - z^2)$. Choose a point $w \in f[S]$.

Case 1: $w_5 \neq \frac{\sqrt{3}}{2}$. This case corresponds to $z \neq 0$. In this case, choose a neighbourhood of w of the form $U = V \times (w_5 - \epsilon, w_5 + \epsilon) \subset \mathbb{R}^5$ for V an open set of $\mathbb{R}^4$ and $\epsilon = \frac{1}{4}(\sqrt{3} - 2w_5)$. This choice means that $f^{-1}[f[S] \cap U]$ contains no points of S on the equator $z = 0$. Let $W := f^{-1}[f[S] \cap U] \cap \{z > 0\}$, which is an open subset of S and so a manifold. Hence by (ii) we know that $f|_W : W \to f[S] \cap U$ is a bijection. It is also the restriction of the continuous map F, and so continuous.

We only need to show that $(f|_W)^{-1}$ is continuous also. We can solve the fifth component to find z, namely $z = g(w_5) := +\sqrt{1 - \frac{2}{\sqrt{3}}w_5}$. Because of the restriction of w_5, g is a continuous and non-zero function on U. Then we see $x = w_2/g(w_5)$ and $y = w_3/g(w_5)$. Let

$$G : U \to \mathbb{R}^3,\ x \mapsto (w_2/g(w_5),\ w_3/g(w_5),\ g(w_5)).$$

This is a continuous map, so the restriction $G|_{f[S]\cap U}$ is continuous, but this is exactly $(f|_W)^{-1}$. Therefore $f|_W$ is an embedding and we have shown that $f[S] \cap U$ is a submanifold.

Case 2: $w_5 = \frac{\sqrt{3}}{2}$, $w_1 \neq 0$. In this case we have $z = 0$ but $xy \neq 0$ at w. Let $\epsilon = \frac{1}{2}|w_1|$ and choose a neighbourhood of w of the form $(w_1 - \epsilon, w_1 + \epsilon) \times V \subset \mathbb{R}^5$. This time, none of the points in the preimage can have $x = 0$, so setting $W := f^{-1}[f[S] \cap U] \cap \{x > 0\}$ gives that $f|_W : W \to f[S] \cap U$ is a bijection. It is also continuous.

On U, the function g from case 1 is not well defined. By assumption, x is non-zero, so

$$y = \frac{w_1}{x} \;\Rightarrow\; 2w_4 = x^2 - \frac{w_1^2}{x^2} \;\Rightarrow\; x = h(w) := +\sqrt{w_4^2 + \sqrt{w_4^2 + w_1^2}}.$$

Then $H(w) = (h(w),\, w_1/h(w),\, w_2/h(w))$ is a continuous function from U to $\mathbb{R}^3$, which shows that $(f|_W)^{-1} = H|_{f[S]\cap U}$ is also continuous. Hence $f[S]$ is a submanifold at these points too.

Case 3a: $w_5 = \frac{\sqrt{3}}{2}$, $w_1 = 0$. Assume that z and y vanish at w, but x does not. Consider the fourth component of f, $f_4 = \frac{1}{2}(x^2 - y^2)$. Thus $w_4 > 0$, which allows us to set $\epsilon = \frac{1}{2}w_4$. As is now a familiar tactic, let U be a set such that the range of its fourth component is contained in $(w_4 - \epsilon, w_4 + \epsilon)$, so that $f^{-1}[U]$ contains no points of S with $x = 0$. Then $f|_{f^{-1}[U]\cap\{x>0\}}$ is bijection. The function $H|_{f[S]\cap U}$ is its inverse.

Case 3b: $z = x = 0$, but $y \neq 0$. Similar to case 3a.

In summary, we have shown that every point of $f[S]$ has a neighbourhood (in the subspace topology) that is a submanifold, so it is a submanifold.

There is another approach to this question; you can try to describe $f[S]$ as the level set of a function. One way to do this is to start with the equations

$$a = xy,\; b = xz,\; c = yz,\; d = \frac{1}{2}(x^2 - y^2),\; e = \frac{\sqrt{3}}{2}(1 - z^2),\; x^2 + y^2 + z^2 = 3$$

and eliminate x, y, and z. However, you must be careful to find an equivalent system of equations; you must not introduce any new solutions. This is very difficult, and you probably need to split this into cases as well. Then you can apply the implicit function theorem and show the Jacobian has full rank, as you would normally do.