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Analysis III
5. Exercises

12 April 2022

14. Alternative version of the Constant Rank Theorem.

Let M and N be manifolds of dimensions m and n respectively, and $f : M \to N$ a map of constant rank r. Consider the standard set-up: at every point $p \in X$ there exists charts $\phi : U \to \mathbb{R}^m$ of M and $\psi : V \to \mathbb{R}^n$ of N with $p \in U$, $\phi(p) = 0$, $f[U] \subset V$ and $\psi(f(p)) = 0$.

Show: That there exists such charts ϕ and ψ with the further property that $\psi \circ f \circ \phi^{-1}$ has the form

$$\psi \circ f \circ \phi^{-1} : \phi[U] \to \mathbb{R}^n,\ (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{n-r}) . \qquad \textit{(3 Points)}$$

[Hint. Apply Theorem 1.44.]

Remark. This theorem shows that maps of constant rank can be written as a composition of a submersion and an immersion in a neighbourhood of every point (i.e. locally). For this reason such maps are sometimes called *Subimmersions*. Can you see the connection between this result and the first isomorphism theorem of linear algebra?

15. A differentiability test.

Let X, Y, and Z be manifolds, $g : Y \to Z$ an immersion, and $f : X \to Y$ a continuous function. Show that, if $g \circ f$ is smooth, then f must be smooth. *(2 Points)*

[Hint. Exercise 14.]

16. The Cartesian product of manifolds.

Let M and N be manifolds with dimensions m and n respectively. Verify the claim from Section 1.6 that "the map $\phi \times \psi$ is a chart of $M \times N$ and the collection of all such maps is an atlas." This shows that $M \times N$ is a manifold with dimension $m + n$.

You may assume that the map is homeomorphism if you don't want to try to understand the topology of the product. In that case, you must still show each map is bijective and they are compatible. *(4 Points)*

17. More examples of submanifolds.

This question is an application of Exercise 13(b).

(a) Let $a > 0$ and $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = \left(a - \sqrt{x^2 + y^2}\right)^2 + z^2.$$

Show that the preimage $f^{-1}[\{b^2\}]$ is a submanifold of $\mathbb{R}^3$ for every b with $0 < b < a$. What is this space?

(b) Investigate for which values $t \in \mathbb{R}$

$$A_t := \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = t \}$$

is a submanifold of $\mathbb{R}^{n+1}$. What is this space?

(4 Points)

18. The Veronese map

Let $S := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 3 \}$ and the Veronese map $f : S \to \mathbb{R}^5$ be defined by

$$f(x, y, z) = \left(xy, xz, yz, \tfrac{1}{2}(x^2 - y^2), \tfrac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)\right) .$$

(a) Show that f is an immersion. *(3 Points)*

(b) Consider two points in $(x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}) \in S$ whose image under f is equal, that is, $f(\widetilde{x}, \widetilde{y}, \widetilde{z}) = f(x, y, z)$. Show that this occurs if and only if $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \pm(x, y, z)$. Thus $f : S \to \mathbb{R}^5$ is not an embedding. *(3 Points)*

[Hint. The solution can be simplified by using complex numbers. Set $w := x + iy$ and $\widetilde{w} := \widetilde{x} + i\widetilde{y}$. Rewrite the first, fourth, and fifth components of f using $\Im w^2$, $\Re w^2$, and $|w|^2$]

(c) Show that $f[S] \subset \mathbb{R}^5$ is a submanifold.

This part is very difficult. I've included it here so you can understand how difficult it can be to show that the image of an immersion is a submanifold.

Remark. The manifold $f[S]$ is called the real projective plane $\mathbb{RP}^2$. There is a real projective space $\mathbb{RP}^n$ for every dimension n. These spaces have beautiful geometry. For example, every pair of lines in the real projective plane intersect exactly once. The natural way to define these spaces is not as a submanifold. I will try to talk about these spaces at some point, but it doesn't really fit with the themes of this course. The Veronese map shows however that $\mathbb{RP}^n$ can be embedded in Euclidean space. This raises an important question: can every manifold be embedded in Euclidean space?