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Analysis III
4. Exercises

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11. The tangent space.

Let X, Y be manifolds and $f : X \to Y$. The vector space T_pX is called the tangent space of X at p and the map $T_p(f)$ is called the tangent map. It is also called the push-forward map or the differential.

(a) Let $\alpha, \beta : (-\varepsilon, \varepsilon) \to X$ be two smooth paths, with $p = \alpha(0) = \beta(0)$. Let ϕ be a chart that contains p. Show that these paths are tangential at p (Definition 1.32), or equivalently give the same tangent vector at p (Definition 1.33), if and only if $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0) \in \mathbb{R}^n$. *(2 Points)*

(b) Consider the cylinder Z. Do the paths $\alpha(\lambda) = (0, \cos\lambda, \sin\lambda)$ and $\beta(\lambda) = (\lambda^2, \sqrt{1-\lambda^2}, \lambda)$ give the same tangent vector at $p = (0, 1, 0)$? *(2 Points)*

(c) Consider the map G from Exercise 8(b) and v the vector given by α from above. What is $T_{(0,1,0)}(G)(v)$?
Hint. $T_p(G)$ is a map between tangent spaces, so your answer should be a tangent vector of $\mathbb{S}^2$, ie a path in $\mathbb{S}^2$. *(2 Points)*

(d) Prove directly that the vector space structure on the tangent space does not depend on the choice of chart (Theorem 1.36(i)). *(Just to think about.)*

(e) Let X be connected. Show that f is constant if and only if $T_x(f) = 0$ for all $x \in X$. *(4 Points)*

(f) In the lectures we prove an equivalence between tangent vectors and derivations. Consider the three coordinate functions $\Pi_k : Z \to \mathbb{R}$ defined by $\Pi_k(x_1, x_2, x_3) := x_k$ for $k = 1, 2, 3$. Again let v the vector given by α from above, and D_v the corresponding derivation. How does $D_v : C^1(Z, \mathbb{R}) \to \mathbb{R}$ act on the coordinate functions? *(1 Point)*

Solution.

(a) In terms of Definition 1.32, we must compare α and β in the charts id on $\mathbb{R}$ and ϕ on X. That is, we must compare the derivatives of $\phi \circ \alpha \circ \mathrm{id}^{-1} = \phi \circ \alpha$ and $\phi \circ \beta$ at 0 as linear maps $\mathbb{R} \to \mathbb{R}^n$. Recall that the derivative of a function $F = (F_1, \ldots, F_n) : \mathbb{R}^m \to \mathbb{R}^n$ as a linear map at a point a is an $n \times m$ matrix, also called the Jacobian:

$$JF_a := \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a) & \ldots & \frac{\partial F_1}{\partial x_m}(a) \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(a) & \ldots & \frac{\partial F_n}{\partial x_m}(a) \end{pmatrix}.$$

In our situation, this matrix is a single column given by $\frac{d}{dt}(\phi \circ \alpha)\big|_{t=0} = (\phi \circ \alpha)'(0)$. Because it is only a single column, we identify this matrix with a vector in $\mathbb{R}^n$. The two matrices are the same if and only if the two vectors are the same.

(b) We need to choose a chart containing p. The chart $\phi_{-\pi}(x_1, x_2, x_3) = (x_1, \arcsin x_3)$ is suitable. Using the previous criterion,

$$\frac{d}{d\lambda}(\phi \circ \alpha)\bigg|_{\lambda=0} = \frac{d}{d\lambda}(0, \lambda)\bigg|_{\lambda=0} = (0, 1),$$

and

$$\frac{d}{d\lambda}(\phi \circ \beta)\bigg|_{\lambda=0} = \frac{d}{d\lambda}(\lambda^2, \arcsin \lambda)\bigg|_{\lambda=0} = \left(2\lambda, \frac{1}{\sqrt{1-\lambda^2}}\right)\bigg|_{\lambda=0} = (0, 1).$$

So these curves are tangential at p.

(c) A vector on a manifold at a point is an equivalence class of paths. The map $T_p(G)$ maps the path α to the path $G \circ \alpha$. Hence we compute

$$G \circ \alpha(\lambda) = G((0, \cos \lambda, \sin \lambda)) = (\tanh 0, \cos \lambda \operatorname{sech} 0, \sin \lambda \operatorname{sech} 0) = (0, \cos \lambda, \sin \lambda).$$

This is a path in $\mathbb{S}^2$ with $G \circ \alpha(0) = (0, 1, 0) = G((0, 1, 0))$. Its equivalent class defines a tangent vector of $\mathbb{S}^2$ at $(0, 1, 0)$.

Notice that the derivative of this path as a function in $\mathbb{R}^3$ is $(G \circ \alpha)'(0) = (0, 0, 1)$ which is tangent to the sphere in the usual sense of $\mathbb{R}^3$. Can you explain this?

(d) Let p be a point in X and ϕ a chart containing p. Without loss of generality, assume that $\phi(p) = 0$ (you can add a translation of $\mathbb{R}^n$ to achieve this). Then because ϕ is a homeomorphism, the induced map $\Phi := T_p(\phi)$ is a bijection from $V := T_pX$ to $T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n$. We give V the structure of a vector space which makes Φ an isomorphism. The question is, does the vector space structure depend on the choice of chart? If we add two vectors according to one chart, so we get the same answer to when we add them according to another chart?

We begin by describing the inverse $\Phi^{-1} : \mathbb{R}^n \to V$ more carefully. Given any vector $a \in \mathbb{R}^n$ consider the path $\tilde{\gamma}_a(t) = at$ in $\mathbb{R}^n$. Then $\gamma_a = \phi^{-1} \circ \tilde{\gamma}_a$ is a path in X with $\gamma_a(0) = p$. Note that $\Phi(\gamma_a)$ is defined to be the path $\phi \circ \gamma_a \circ \mathrm{id}^{-1} = \tilde{\gamma}_a$, which shows $\Phi^{-1}(a) = \gamma_a$. So, part (a) shows us how to identify a tangent vector in X with a vector in $\mathbb{R}^n$ and this construction shows us how to start with a vector in $\mathbb{R}^n$ and build a path in X. The vector space structure is $\alpha + \beta = \gamma_{\Phi(\alpha)+\Phi(\beta)}$ and $C\alpha = \gamma_{C\Phi(\alpha)}$.

Let ψ be another such chart, and $\Psi = T_{\psi(p)}(\psi)$ and $\delta_a(t) = \Psi^{-1}(at)$ be the construction in this chart. We need to show that $\gamma_{\Phi(\alpha)+\Phi(\beta)}$ and $\delta_{\Psi(\alpha)+\Psi(\beta)}$ are tangential at p and that so too are $\gamma_{C\Phi(\alpha)}$ and $\delta_{C\Psi(\alpha)}$. In the ϕ chart,

$$(\phi \circ \gamma_{\Phi(\alpha)+\Phi(\beta)})'(0) = \left(\phi \circ \phi^{-1}\Big([\Phi(\alpha) + \Phi(\beta)]t\Big)\right)'(0) = \Phi(\alpha) + \Phi(\beta)$$

and

$$\begin{aligned}
\left(\phi \circ \delta_{\Psi(\alpha)+\Psi(\beta)}\right)'(0) &= \left(\phi \circ \psi^{-1}\Big([\Psi(\alpha)+\Psi(\beta)]t\Big)\right)'(0) \\
&= J(\phi \circ \psi^{-1})_0\,[\Psi(\alpha)+\Psi(\beta)] \\
&= J(\phi \circ \psi^{-1})_0\,[(\psi \circ \alpha)'(0) + (\psi \circ \beta)'(0)] \\
&= J(\phi \circ \psi^{-1})_0(\psi \circ \alpha)'(0) + J(\phi \circ \psi^{-1})_0(\psi \circ \beta)'(0) \\
&= J(\phi \circ \psi^{-1} \circ \psi \circ \alpha)_0 + J(\phi \circ \psi^{-1} \circ \psi \circ \beta)_0 \\
&= J(\phi \circ \alpha)_0 + J(\phi\psi \circ \beta)_0 \\
&= \Phi(\alpha) + \Phi(\beta),
\end{aligned}$$

where we used the chain rule $J(F \circ G) = JF \cdot FG$ in the second line, and in reverse in the 5th line. Because these vectors are equal, by part (a) we know that the paths are tangential. The proof for vector scaling is similar. This shows that the vector space structure does not depend on the choice of chart.

(e) Suppose that f is constant. Then $f(x) = q$ for some $q \in Y$ and all $x \in X$. Choose any point $x \in X$ and path α through x. Then the push-forward $f \circ \alpha$ is the constant map $t \to q$. This is the zero element of the tangent space.

Conversely, suppose that $T_x(f) = 0$. Then its rank is everywhere 0. By Corollary 1.46, for every point $y \in T[X]$ the preimage $f^{-1}[\{y\}]$ is a submanifold of dimension $\dim X - 0 = \dim X$. Submanifolds of the same dimension must be open, and because it is the preimage of a point it is also closed. Because X is connected, the submanifold must therefore be all of X. In other words, $X = f^{-1}[\{y\}]$, so $f[X] = y$, which shows f is constant.

(f) Just before Theorem 1.40 in the script we have the definition of this correspondence

$$D_v(\Pi_k) = \left.\frac{d}{dt}\right|_{t=0} \Pi_k(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} \Pi_k((0, \cos t, \sin t)) = \begin{cases} 0 & \text{if } k = 1, \\ 0 & \text{if } k = 2, \\ 1 & \text{if } k = 3. \end{cases}$$

12. Immersions.

(a) Investigate: at which points are the following maps immersive?

- **(i)** $f : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto f(t) = (\cos(2t), \sin(2t), t)$. *(1 Point)*
- **(ii)** $g : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto g(t) = (t^2, t^3)$. Is g injective? *(2 Points)*
- **(iii)** $h : (1, \infty) \to \mathbb{R}^2,\ t \mapsto h(t) = \left(\frac{t+1}{2t}\cos(2t), \frac{t+1}{2t}\sin(2t)\right)$. *(2 Points)*

(b) Let M be an n-dimensional compact manifold and $f : M \to \mathbb{R}^n$ a smooth map. Show that f cannot be an immersion. *(3 Points)*

Hint. Investigate the topological properties of $f[M]$.

Solution.

(a) A map is immersive at a point if the tangent map is injective. This is the case if it is *full-rank*. The rank is the number of linearly independent columns. We have also seen for subset of $\mathbb{R}^n$ that the tangent map is just the Jacobian.

(i)

$$T_t(f) = (-2\sin(2t), 2\cos(2t), 1).$$

The last column is never zero, so the rank is 1. Thus f is an immersion.

(ii)

$$T_t(g) = (2t, 3t^2)$$

This has rank 1, except where both columns vanish simultaneously, namely $t = 0$. This map is an injection however, since $t^3 = s^3 \Rightarrow t = s$.

(iii)

$$T_t(h) = \left(-\frac{1}{2t^2}\cos(2t) - \frac{1+t}{t}\sin(2t),\ -\frac{1}{2t^2}\sin(2t) + \frac{1+t}{t}\cos(2t)\right).$$

This is rank 1, except if both columns vanish simultaneously. That occurs when

$$\begin{aligned} 0 &= (\cos(2t) + 2t(1+t)\sin(2t))^2 + (\sin(2t) - 2t(1+t)\cos(2t))^2 \\ &= 1 + 4t^2(1+t)^2. \end{aligned}$$

We see that in fact the two columns are never simultaneously zero, so h is an immersion.

(b) Immediately we can say that $N = f[M]$ is a compact subset of $\mathbb{R}^n$, therefore closed. Suppose that f is an immersion. That means that $T(f)$ is rank n at every point. But then f is also a submersion. This make f a local diffeomorphism and so the image N must also be open. But the only closed and open sets in $\mathbb{R}^n$ are the empty set and $\mathbb{R}^n$ itself. N cannot be empty and $\mathbb{R}^n$ is not compact. Therefore we have a contradiction: f cannot be an immersion.

Can you generalise this result to for $f : M \to \mathbb{R}^m$, or provide counter-examples?

13. Submanifolds

(a) In previous examples in the lectures and exercises, we have defined manifold structures on $\mathbb{S}^n$ and Z by giving an atlas. Show that these spaces are submanifolds of $\mathbb{R}^n$. That is, show $\mathrm{id}_{\mathbb{R}^{n+1}}|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ and $\mathrm{id}_{\mathbb{R}^3}|_Z : Z \to \mathbb{R}^3$ are embeddings. *(4 Points)*

(b) Let's generalise this. Consider Example 1.18(iv) from the lecture script. It says: Let $M = f^{-1}[\{0\}]$ be the preimage of 0 of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ whose gradient ∇f has no common zeroes with f. Then M is a manifold.
Show that $\mathrm{id}_{\mathbb{R}^{n+1}}|_M : M \to \mathbb{R}^{n+1}$ is an embedding. *(3 Points)*

(c) What is the connection between the previous exercise and the constant rank theorem (Theorem 1.44/Corollary 1.46)? *(2 Points)*

Solution.

(a) To show a map is an embedding, you must show that it is an immersion and a homeomorphism onto its image.

By a comment in the lecture script, an injective immersion of a compact manifold is always an embedding. Clearly the identity function is injective. Since $\mathbb{S}^n$ is compact, it remains to show that it is an immersion. We will use the coordinate projection charts. Choose any point $x \in S^n$. Without loss of generality, assume that $x \in H_0^+$. Let $h(y) = \sqrt{1 - \|y\|^2}$. Then the tangent map in local coordinates is

$$J(\mathrm{id} \circ \mathrm{id} \circ \pi_0^{-1}) = \begin{pmatrix} \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} & \dots & \frac{\partial h}{\partial y_n} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

which is clearly rank n. Hence this is an immersion.

We can check that id is an immersion for Z in a similar way

$$J(\mathrm{id} \circ \mathrm{id} \circ \phi_a^{-1}) = \begin{pmatrix} \frac{\partial t}{\partial t} & \frac{\partial \cos s}{\partial t} & \frac{\partial \sin s}{\partial t} \\ \frac{\partial t}{\partial s} & \frac{\partial \cos s}{\partial s} & \frac{\partial \sin s}{\partial s} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sin s & \cos s \end{pmatrix}$$

This is rank 2 for all s, t, so id is an immersion. Again, of course id is injective, and so a bijection onto the image. The final property to verify, is that $\mathrm{id} : (Z, \tau) \to (Z, \tau')$ is a homeomorphism, where τ is the topology from the manifold structure (the topology that makes the charts are homeomorphisms) and τ' is the topology as a subspace of $\mathbb{R}^3$. In exercise 6(c)(i) we verifies that the charts were homeomorphisms with respect to the subspace topology, so these two topologies are the same.

(b) This is really a generalisation of exercise 5(a), because the coordinate projection charts are exactly the charts one gets by using the implicit function theorem to construct an atlas on a level set. We recall this result now. Take any point $p \in M$. By the assumption that the gradient does not vanish, there is a coordinate x_k such that $\partial_k f(p) \neq 0$. Without loss of generality, assume $k = 0$. The implicit function theorem says that there is a smooth height function h such that $\tilde{h} : y \mapsto (h(y), y)$ is an inverse to $\phi := \pi_0|_{M \cap U}$ for some neighbourhoods U of $p \in M \subset \mathbb{R}^{n+1}$ and $\pi_0[U]$ of $\pi_0(p) \in \mathbb{R}^n$. π_0 is a continuous function on $\mathbb{R}^{n+1}$, so if M is given the subspace topology, then the restriction of π_0 to M is also continuous. The function $\tilde{h} : \pi_0[U] \to \mathbb{R}^{n+1}$ has continuous components, and so is also continuous. Thus π_0 is a chart of M near p. We have previously shown that different coordinate projections are compatible in the sense of charts. So this makes M a manifold.

Let $\iota = \mathrm{id}|_M : M \to \mathbb{R}^{n+1}$ be the inclusion map, which is the identity map restricted to M but considered as a map between two manifolds. Clearly it is a bijection between M and $\iota[M]$. As we saw in the previous part, whether ι is a homeomorphism is exactly the same question as to whether the charts are continuous in the subspace topology.

Here is another way to see that ι^{-1} is continuous that shows an useful tactic. π_0 is a continuous function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Then $\tilde{h}$ is a continuous from a subset of $\mathbb{R}^n$ to a neighbourhood in M. The final piece is that $\iota^{-1}|_{M \cap U} = \tilde{h} \circ \pi_0|_{M \cap U}$, which shows ι^{-1} locally as the composition of two continuous functions.

It remains to show that ι is an immersion. This is clear from the form of the inverse of $\phi = \pi_0|_{M \cap U}$ given by the implicit function theorem:

$$J(\mathrm{id} \circ \iota \circ \phi^{-1}) = J\tilde{h} = \begin{pmatrix} \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} & \cdots & \frac{\partial h}{\partial y_n} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

(c) The result in Example 1.18(iv) is a special case of the constant rank theorem (or Corollary 1.46): $\nabla f \neq 0$ implies that the rank is always 1 so the preimage of a point is a submanifold of $\mathbb{R}^{n+1}$.

Terminology

f und g berühren = f and g are tangential.

Umkehrsatz = Inverse Function Theorem.

Rangsatz = Constant Rank Theorem.

Rang = rank (symbol is rk).

Exercise for Easter break: Here is a challenging question for you to attempt over the Easter break. Feel free to ask me questions about it, but we won't discuss it in the tutorials.

Classification of 1-dimensional connected manfiolds.

In this exercise we will prove the following theorem: Every 1-dimensional connected manifold M is diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$.

There are two parts. In the first part we construct an special atlas. In the second part we use this atlas to construct either a single chart for all of M, or show it must be a circle. Part (b) is essentially constructing a metric on M, something that we will cover later in the course, so feel free to assume (b) for now and attempt it later.

Let $(M, \mathcal{A})$ be a 1-dimensional connected manifold. We will construct a compatible atlas $\tilde{A}$ with the following two properties:

$$\begin{aligned}&\text{Every chart } \tilde{\varphi} \in \widetilde{\mathcal{A}} \text{ maps to an open interval } J_{\tilde{\varphi}} \text{ i.e. } \tilde{\varphi} : U_{\tilde{\varphi}} \to J_{\tilde{\varphi}} \text{ and}\\ &\text{For every two charts } \tilde{\varphi}, \tilde{\psi} \text{ with } U_{\tilde{\varphi}} \cap U_{\tilde{\psi}} \neq \emptyset \text{ we have } |(\tilde{\psi} \circ \tilde{\varphi}^{-1})|' = 1.\end{aligned} \qquad (*)$$

(a) Show that for any chart φ and point $p \in U_\varphi$ there is a compatible chart containing p that maps to an open interval I_φ. Thus we can assume that all charts in $\mathcal{A}$ map to an open interval.

(b) With the help of a partition of unity, show that for every chart ϕ there exists a smooth function $f_\varphi : U_\varphi \to \mathbb{R}^+$, so that for every pair of overlapping charts φ, ψ with $U_\varphi \cap U_\psi \neq \emptyset$,

$$\frac{f_\varphi(x)}{f_\psi(x)} = |(\psi \circ \varphi^{-1})|'(\varphi(x)) \quad \text{for all } \ x \in U_\varphi \cap U_\psi.$$

(c) Show that for every chart φ there is an interval J_φ and a diffeomorphism $\Phi_\varphi : I_\varphi \to J_\varphi$ such that

$$\Phi'_\varphi(\varphi(x)) = f_\varphi(x) \quad \text{for all } x \in U_\varphi.$$

Define $\widetilde{\varphi} := \Psi_\varphi \circ \varphi : U_\varphi \to J_\varphi$. It follows that the atlas $\widetilde{\mathcal{A}}$ containing the charts $\widetilde{\varphi}$ also fulfils the second property.

With this atlas we can now describe *arc-length parameterisations* (ALP): $f : I \to M$ is an ALP when it is a homeomorphism and for any chart ϕ of M, we have $|(\phi \circ f)|'(x) = 1$ for all x for which the composition is defined. This does not depend on the choice of chart, because of (b).

(d) Let $f : I \to M$ and $g : J \to M$ be two ALP. Show that $f[I] \cap g[J]$ contains at most two connected components.
Hint. Consider the graph $\{(s, t) \in I \times J \mid f(s) = g(t)\}$.

(e) Suppose that $f[I] \cap g[J]$ contains two connected components. Show that M is diffeomorphic to $\mathbb{S}^1$.

(f) Suppose that $f[I] \cap g[J]$ is connected. How can f be extended to an ALP of $f[I] \cup g[J]$?

(g) Complete the proof of the theorem.

Solution. Parts (d)-(g) are proved in the Appendix of Milnor's *Topology from a differentiable viewpoint*. It can be found online here `https://math.uchicago.edu/~may/REU2017/MilnorDiff.pdf`. He uses arc-length parameterisations, which requires a metric on M. We will cover metrics later in the course. All manifolds can be given a metric, and the proof does not depend on the choice of metric, so there is no loss of generality. Indeed, it is often a good strategy to use additional structure. There are purely topological proofs, eg Gale 1987 `https://www.jstor.org/stable/2322421`. This uses the order relation in place of the arc-length.

A completely different approach is given by the notion of covering spaces. There is a theorem that every manifold is the image of the local diffeomorphism of a simply connected manifold. One then proves that the only simply connected 1-dimensional manifold is $\mathbb{R}$ and the only local diffeomorphisms map to $\mathbb{R}$ or $\mathbb{S}^1$ (up to diffeomorphism).