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Analysis III
4. Exercises

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11. The tangent space.

Let X, Y be manifolds and $f : X \to Y$. The vector space T_pX is called the tangent space of X at p and the map $T_p(f)$ is called the tangent map. It is also called the push-forward map or the differential.

(a) Let $\alpha, \beta : (-\varepsilon, \varepsilon) \to X$ be two smooth paths, with $p = \alpha(0) = \beta(0)$. Let ϕ be a chart that contains p. Show that these paths are tangential at p (Definition 1.32), or equivalently give the same tangent vector at p (Definition 1.33), if and only if $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0) \in \mathbb{R}^n$. *(2 Points)*

(b) Consider the cylinder Z. Do the paths $\alpha(\lambda) = (0, \cos\lambda, \sin\lambda)$ and $\beta(\lambda) = (\lambda^2, \sqrt{1-\lambda^2}, \lambda)$ give the same tangent vector at $p = (0, 1, 0)$? *(2 Points)*

(c) Consider the map G from Exercise 8(b) and v the vector given by α from above. What is $T_{(0,1,0)}(G)(v)$?
Hint. $T_p(G)$ is a map between tangent spaces, so your answer should be a tangent vector of $\mathbb{S}^2$, ie a path in $\mathbb{S}^2$. *(2 Points)*

(d) Prove directly that the vector space structure on the tangent space does not depend on the choice of chart (Theorem 1.36(i)). *(Just to think about.)*

(e) Let X be connected. Show that f is constant if and only if $T_x(f) = 0$ for all $x \in X$. *(4 Points)*

(f) In the lectures we prove an equivalence between tangent vectors and derivations. Consider the three coordinate functions $\Pi_k : Z \to \mathbb{R}$ defined by $\Pi_k(x_1, x_2, x_3) := x_k$ for $k = 1, 2, 3$. Again let v the vector given by α from above, and D_v the corresponding derivation. How does $D_v : C^1(Z, \mathbb{R}) \to \mathbb{R}$ act on the coordinate functions? *(1 Point)*

12. Immersions.

(a) Investigate: at which points are the following maps immersive?

(i) $f : \mathbb{R} \to \mathbb{R}^3,\ t \mapsto f(t) = (\cos(2t), \sin(2t), t)$. *(1 Point)*

(ii) $g : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto g(t) = (t^2, t^3)$. Is g injective? *(2 Points)*

(iii) $h : (1, \infty) \to \mathbb{R}^2,\ t \mapsto h(t) = \left(\frac{t+1}{2t}\cos(2t), \frac{t+1}{2t}\sin(2t)\right)$. *(2 Points)*

(b) Let M be an n-dimensional compact manifold and $f : M \to \mathbb{R}^n$ a smooth map. Show that f cannot be an immersion. *(3 Points)*

Hint. Investigate the topological properties of $f[M]$.

13. Submanifolds

(a) In previous examples in the lectures and exercises, we have defined manifold structures on $\mathbb{S}^n$ and Z by giving an atlas. Show that these spaces are submanifolds of $\mathbb{R}^n$. That is, show $\mathrm{id}_{\mathbb{R}^{n+1}}|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ and $\mathrm{id}_{\mathbb{R}^3}|_Z : Z \to \mathbb{R}^3$ are embeddings. *(4 Points)*

(b) Let's generalise this. Consider Example 1.18(iv) from the lecture script. It says: Let $M = f^{-1}[\{0\}]$ be the preimage of 0 of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ whose gradient ∇f has no common zeroes with f. Then M is a manifold.
Show that $\mathrm{id}_{\mathbb{R}^{n+1}}|_M : M \to \mathbb{R}^{n+1}$ is an embedding. *(3 Points)*

(c) What is the connection between the previous exercise and the constant rank theorem (Theorem 1.44/Corollary 1.46)? *(2 Points)*

Terminology

f und g berühren = f and g are tangential.
Umkehrsatz = Inverse Function Theorem.
Rangsatz = Constant Rank Theorem.
Rang = rank (symbol is rk).

Exercise for Easter break: Here is a challenging question for you to attempt over the Easter break. Feel free to ask me questions about it, but we won't discuss it in the tutorials.

Classification of 1-dimensional connected manfiolds.

In this exercise we will prove the following theorem: Every 1-dimensional connected manifold M is diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$.

There are two parts. In the first part we construct an special atlas. In the second part we use this atlas to construct either a single chart for all of M, or show it must be a circle. Part (b) is essentially constructing a metric on M, something that we will cover later in the course, so feel free to assume (b) for now and attempt it later.

Let $(M, \mathcal{A})$ be a 1-dimensional connected manifold. We will construct a compatible atlas $\tilde{A}$ with the following two properties:

$$\begin{aligned}&\text{Every chart } \tilde{\varphi} \in \widetilde{\mathcal{A}} \text{ maps to an open interval } J_{\tilde{\varphi}} \text{ i.e. } \tilde{\varphi} : U_{\tilde{\varphi}} \to J_{\tilde{\varphi}} \text{ and}\\ &\text{For every two charts } \tilde{\varphi}, \tilde{\psi} \text{ with } U_{\tilde{\varphi}} \cap U_{\tilde{\psi}} \neq \emptyset \text{ we have } |(\tilde{\psi} \circ \tilde{\varphi}^{-1})|' = 1.\end{aligned} \tag{$*$}$$

(a) Show that for any chart φ and point $p \in U_\varphi$ there is a compatible chart containing p that maps to an open interval I_φ. Thus we can assume that all charts in $\mathcal{A}$ map to an open interval.

(b) With the help of a partition of unity, show that for every chart ϕ there exists a smooth function $f_\varphi : U_\varphi \to \mathbb{R}^+$, so that for every pair of overlapping charts φ, ψ with $U_\varphi \cap U_\psi \neq \emptyset$,

$$\frac{f_\varphi(x)}{f_\psi(x)} = |(\psi \circ \varphi^{-1})|'(\varphi(x)) \quad \text{for all } \; x \in U_\varphi \cap U_\psi.$$

(c) Show that for every chart φ there is an interval J_φ and a diffeomorphism $\Phi_\varphi : I_\varphi \to J_\varphi$ such that

$$\Phi'_\varphi(\varphi(x)) = f_\varphi(x) \quad \text{for all } x \in U_\varphi.$$

Define $\widetilde{\varphi} := \Psi_\varphi \circ \varphi : U_\varphi \to J_\varphi$. It follows that the atlas $\widetilde{\mathcal{A}}$ containing the charts $\widetilde{\varphi}$ also fulfils the second property.

With this atlas we can now describe *arc-length parameterisations* (ALP): $f : I \to M$ is an ALP when it is a homeomorphism and for any chart ϕ of M, we have $|(\phi \circ f)|'(x) = 1$ for all x for which the composition is defined. This does not depend on the choice of chart, because of (b).

(d) Let $f : I \to M$ and $g : J \to M$ be two ALP. Show that $f[I] \cap g[J]$ contains at most two connected components.
Hint. Consider the graph $\{(s, t) \in I \times J \mid f(s) = g(t)\}$.

(e) Suppose that $f[I] \cap g[J]$ contains two connected components. Show that M is diffeomorphic to $\mathbb{S}^1$.

(f) Suppose that $f[I] \cap g[J]$ is connected. How can f be extended to an ALP of $f[I] \cup g[J]$?

(g) Complete the proof of the theorem.