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Analysis III
3. Exercises

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8. Examples of smooth maps.

(a) Show that a map $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth in the sense of Definition 1.22, exactly if it is smooth in the usual sense. *(2 Points)*

(b) On the cylinder Z from Exercise 6(c). Show that the map $G : Z \to \mathbb{S}^2$

$$G(x_1, x_2, x_3) = (\tanh x_1,\ x_2 \operatorname{sech} x_1,\ x_3 \operatorname{sech} x_1)$$

is smooth. *(2 Points)*

(c) On the sphere $\mathbb{S}^n$: Show that the following maps are smooth (in the sense of Definition 1.22).

(i) The antipodal map of the sphere $\alpha : \mathbb{S}^n \to \mathbb{S}^n,\ x \mapsto -x$. *(2 Points)*

(ii) The projections to a coordinate plane $\pi_k : \mathbb{S}^n \to \mathbb{R},\ (x_0, \dots, x_n) \mapsto (x_0, \dots, \hat{x}_k, \dots, x_n)$ for $k \in \{0, \dots, n\}$.
(1 Point)

(iii) The projections to a coordinate axis $\Pi_k : \mathbb{S}^n \to \mathbb{R},\ (x_0, \dots, x_n) \mapsto x_k$ for $k \in \{0, \dots, n\}$.
(1 Point)

(iv) The Hopf map

$$\beta : \mathbb{S}^3 \to \mathbb{S}^2,\ (w, x, y, z) \mapsto (2(wy + xz), 2(xy - wz), w^2 + x^2 - y^2 - z^2)\ .$$
(3 Points)

Solution.

(a) As we have seen, the standard atlas on $\mathbb{R}^n$ has simply one chart, the identity map $\mathrm{id}_{\mathbb{R}^n}$. This covers all of $\mathbb{R}^n$, so in Definition 1.22 we can take $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$. The 'function in local coordinates' is then just $\mathrm{id}_{\mathbb{R}^m} \circ F \circ (\mathrm{id}_{\mathbb{R}^n})^{-1} = F$. A function is smooth in the sense of manifold if and only if it is smooth in local coordinates, but here they are exactly the same.

(b) First note, this is well defined map to $\mathbb{S}^2$ because

$$\tanh^2 x_1 + (x_2^2 + x_3^2) \operatorname{sech}^2 x_1 = \tanh^2 x_1 + \operatorname{sech}^2 x_1 = 1.$$

Choose any point $p \in Z$ on the cylinder. Note that $-1 < \tanh x_1 < 1$ so $G(p) \notin \{e_0, -e_0\}$. We take in Definition 1.22 then $\psi = N$ and $V = \mathbb{S}^2 \setminus \{e_0\}$. Since $G^{-1}[V] = Z$ we can take $\phi = \phi_a$ (for some a) and $U = \{(x_1, x_2, x_3) \in \mathbb{Z} \mid (x_2, x_3) \neq$

$(\cos a, \sin a)\} \subset G^{-1}[V]$. Then the 'function in local coordinates' is $N \circ G \circ (\phi_a)^{-1}$ from $\phi_a(U) = \mathbb{R} \times (a, a+2\pi)$ to $N(V) = \mathbb{R}^2$.

$$\begin{aligned} N \circ G \circ (\phi_a)^{-1}(t,s) &= N \circ G(t, \cos s, \sin s) \\ &= N(\tanh t, \cos s \operatorname{sech} t, \sin s \operatorname{sech} t) \\ &= e_0 + \frac{(\tanh t - 1, \cos s \operatorname{sech} t, \sin s \operatorname{sech} t)}{1 - \tanh t} \\ &= \left(\frac{\cos s \operatorname{sech} t}{1 - \tanh t}, \frac{\sin s \operatorname{sech} t}{1 - \tanh t} \right). \end{aligned}$$

This is smooth at $\phi_a(p)$, so G is smooth at p.

(c) **(i)** For this question, we will use stereographic projection as the chart, but the calculation is easier with the $\pi_k^{\pm}$ charts.

We must show that α is smooth at every point $p \in \mathbb{S}^n$. There are two cases to consider: $p = -e_0$ and all remaining points. Suppose that $p \neq -e_0$, then $\alpha(p) \neq e_0$. These points belong to the domain of the chart S and N respectively. We take in Definition 1.22 then $\psi = N$ and $V = \mathbb{S}^n \setminus \{e_0\}$. Since $\alpha^{-1}[V] = \mathbb{S}^n \setminus \{-e_0\}$ we can take $\phi = S$ and $U = \mathbb{S}^n \setminus \{-e_0\}$. We compute, for $y \in \mathbb{R}^n$,

$$\begin{aligned} N \circ \alpha \circ S^{-1}(y) &= N \circ \alpha \left(\frac{(1 - \|y\|^2)e_0 + 2y}{1 + \|y\|^2} \right) \\ &= N \left(\frac{-(1 - \|y\|^2)e_0 - 2y}{1 + \|y\|^2} \right) \\ &= e_0 + \frac{\frac{-(1-\|y\|^2)e_0 - 2y}{1+\|y\|^2} - e_0}{1 - \left\langle \frac{-(1-\|y\|^2)e_0 - 2y}{1+\|y\|^2}, e_0 \right\rangle} \\ &= e_0 + \frac{\frac{-2e_0 - 2y}{1+\|y\|^2}}{1 + \frac{1-\|y\|^2}{1+\|y\|^2}} \\ &= -y. \end{aligned}$$

Since $y \mapsto -y$ is a smooth map $\mathbb{R}^n \to \mathbb{R}^n$, then α is a smooth map also for all points $p \neq -e_0$.

For the case $p = -e_0$ we have $\alpha(p) = e_0$. These points belong to the domains of N and S respectively. Hence we have to compute $S \circ \alpha N^{-1}$, but this turns out to be $y \mapsto -y$ also.

(ii) These maps are very similar to the charts $\pi_k^{\pm}$ in exercise 5. What is the difference? The π_k are defined on the whole sphere and they are not injective. The charts $\pi_k^{\pm}$ are restricted to hemispheres and they are then injective. Because they are so similar, it makes the calculation easier to use these charts.

Fix some k. Let $p \in \mathbb{S}^n$ be any point on the sphere. This point belongs to a hemisphere $H_l^{\pm}$. Note that l does not need to be equal to k. We need to compute π_k in the chart $\pi_l^\pm$ on $U = H_l^\pm \subset \mathbb{S}^n$ and $\mathrm{id}_\mathbb{R}$ on $V = \mathbb{R} \subset \mathbb{R}$. We use hat $\hat{\cdot}$ to mean that term is left out and let $h(y) = \pm\sqrt{1 - \|y\|^2}$ be the height of the sphere.

$$\mathrm{id}_\mathbb{R} \circ \pi_k \circ (\pi_l^+)^{-1}(y) = \pi_k(y_1, \ldots, h(y), \ldots, y_n)$$
$$= \begin{cases} y & \text{if } k = l, \\ (y_1, \ldots, h(y), \ldots, \hat{y}_k, \ldots, y_n) & \text{if } k \neq l. \end{cases}$$

In either case, the result is smooth, so π_k is smooth.

(iii) These maps are similar to the ones in the previous question, so we use a similar technique. Fix some k. Let $p \in \mathbb{S}^n$ be any point on the sphere. This point belongs to a hemisphere $H_l^\pm$. We need to compute Π_k in the chart $\pi_l^\pm$ on $U = H_l^\pm \subset \mathbb{S}^n$ and $\mathrm{id}_\mathbb{R}$ on $V = \mathbb{R} \subset \mathbb{R}$. Again let $h(y) = \pm\sqrt{1 - \|y\|^2}$ be the height of the sphere.

$$\mathrm{id}_\mathbb{R} \circ \Pi_k \circ (\pi_l^+)^{-1}(y) = \Pi_k(y_1, \ldots, h(y), \ldots, y_n)$$
$$= \begin{cases} y_{k+1} & \text{if } k < l, \\ h(y) & \text{if } k = l, \\ y_k & \text{if } k > l. \end{cases}$$

All three possibilities are smooth, so Π_k is smooth.

(iv) Let's try to avoid calculation and instead prove a more general statement. Choose any point $p \in \mathbb{S}^3$. It belongs to some hemisphere $H \subset \mathbb{S}^3$ with chart π_k and its image $\beta(p)$ belongs to some hemisphere $\tilde{H} \subset \mathbb{S}^2$ with chart $\tilde{\pi}_l$. Then

$$\pi_k \circ \beta \circ (\tilde{\pi}_l)^{-1}(y) = \pi_k \circ \beta(y_1, \ldots, h(y), \ldots, y_n)$$
$$= \pi_k\Big(\beta_0(y_1, \ldots, h(y), \ldots, y_n), \ldots, \beta_3(y_1, \ldots, h(y), \ldots, y_n)\Big)$$
$$= \Big(\ldots, \widehat{\beta_0}(y_1, \ldots, h(y), \ldots, y_n), \ldots\Big).$$

Because h is smooth for any hemisphere, and the components of β are all smooth, the result is smooth.

9. Diffeomorphism.

Let X, Y be differential manifolds. Show that X and Y are diffeomorphic (Def 1.21) exactly when there is a bijective smooth map $F : X \to Y$ whose inverse is also smooth.

(4 Points)

Solution. Suppose that there is a bijective smooth map $F : X \to Y$ whose inverse is also smooth. Then F is a homeomorphism. It remains to show that for any chart ψ of Y the composition $\psi \circ F$ is a chart of X compatible with the atlas of X. It is compatible when for any chart ϕ of X, the compositions $(\psi \circ F) \circ \phi^{-1} = \psi \circ F \circ \phi^{-1}$ and $\phi \circ (\psi \circ F)^{-1} = \phi \circ F^{-1} \circ \psi^{-1}$ are smooth. But this is exactly the condition that F and F^{-1} are smooth.

Conversely, if F is a diffeomorphism then it is bijective, and further the condition that $\psi \circ F$ is compatible with the atlas on X for every chart of Y is the same as the condition that F and F^{-1} are smooth.

10. A partition of unity for the interval $(0, 4)$.

We consider the open interval $M = (0, 4)$ as a 1-dimensional manifold. Take an open cover of M:

$$U_1 := (0, 2)\ , \quad U_2 := (1, 3)\ , \quad \text{and} \quad U_3 := (2, 4).$$

(a) Give an example of three functions $f_1, f_2, f_3 \in C^\infty(M)$ with these properties:

$$0 \leq f_k \leq 1, \quad \operatorname{supp}(f_k) \subset U_k, \quad f_1 + f_2 + f_3 = 1.$$

(The support of a function is defined to be the closure of the points on which it is non-zero, $\operatorname{supp}(f_k) := \overline{\{ x \in M \mid f_k(x) \neq 0 \}} \subset M$.) These functions form a partition of unity for M (Definition 1.26). *(3 Points)*

(b) Theorem 1.27 is even stronger! What additional property does the partition of unity given by Theorem 1.27 have, which our example does not have? *(1 Point)*

(c) Is it possible to have a partition of unity of M with this additional property and which has only finitely many functions (f_k)? *(2 Points)*

Solution.

(a) Suppose that we could find a smooth function g that was constant 1 on $(2-a, 2+a)$ for $0 < a < 0.5$, constant 0 outside $(1 + a, 3 - a)$ and valued in $[0, 1]$. Then $f_1 = 1 - g, f_2 = g, f_3 = 1 - g$ would be an example of the sort we want.

There is a few standard ways to construct such smooth functions, often called bump, hat, cut-off, or plateau functions by various authors. In "Beweis der Existenz der Zerlegung der Eins" Martin gives one. Here is another. We begin with the basic example of a non-analytic function $A(x) = \exp(-x^{-1})$ for $x > 0$ and $A(x) = 0$ for $x \leq 0$. The function $A(x)A(1 - x)$ is then smooth and zero outside $(0, 1)$. For many purposes this function is already useful. Let $I(x) = \int_0^x A(t)A(1 - t)\ dt$ and $B(x) = I(x)/I(1)$. Then $B(x)$ is a smooth function that is constant 0 for $x \leq 0$

and constantly 1 for $x \geq 1$. In other words, it is a smooth function that 'joins' two constant functions.

We can take then

$$g(x) = B\left(\frac{x-(1+a)}{(2-a)-(1+a)}\right) + B\left(\frac{x-(3-a)}{(2+a)-(3-a)}\right).$$

https://www.desmos.com/calculator/fdhywa4enf.

(b) The stronger property that the partition in Theorem 1.27 has is that the supports of the f_k are compact, not just closed. For our example, $\text{supp}(f_1) = \overline{(0, 2-a)} = (0, 2-a]$ is closed in M (note, the closure is taken in the manifold M). This is not compact, because the open cover $\{(n^{-1}, 2)\}_{n\in\mathbb{N}}$ has no finite subcover.

(c) No. By the definition of partition of unity $M = \bigcup \text{supp}(f_k)$. If there were only finitely many f_k and their supports were compact, then M would be the finite union of compact sets, thus compact. But M is not compact.

Terminology

glatt = smooth.
Zerlegung der Eins = partition of unity.
Träger = support (symbol is supp).