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Analysis III
3. Exercises

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8. Examples of smooth maps.

(a) Show that a map $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth in the sense of Definition 1.22, exactly if it is smooth in the usual sense. *(2 Points)*

(b) On the cylinder Z from Exercise 6(c). Show that the map $G : Z \to \mathbb{S}^2$

$$G(x_1, x_2, x_3) = (\tanh x_1,\ x_2 \operatorname{sech} x_1,\ x_3 \operatorname{sech} x_1)$$

is smooth. *(2 Points)*

(c) On the sphere $\mathbb{S}^n$: Show that the following maps are smooth (in the sense of Definition 1.22).

(i) The antipodal map of the sphere $\alpha : \mathbb{S}^n \to \mathbb{S}^n,\ x \mapsto -x$. *(2 Points)*

(ii) The projections to a coordinate plane $\pi_k : \mathbb{S}^n \to \mathbb{R},\ (x_0, \dots, x_n) \mapsto (x_0, \dots, \hat{x}_k, \dots, x_n)$ for $k \in \{0, \dots, n\}$.

(1 Point)

(iii) The projections to a coordinate axis $\Pi_k : \mathbb{S}^n \to \mathbb{R},\ (x_0, \dots, x_n) \mapsto x_k$ for $k \in \{0, \dots, n\}$.

(1 Point)

(iv) The Hopf map

$$\beta : \mathbb{S}^3 \to \mathbb{S}^2,\ (w, x, y, z) \mapsto (2(wy + xz), 2(xy - wz), w^2 + x^2 - y^2 - z^2)\ .$$

(3 Points)

9. Diffeomorphism.

Let X, Y be differential manifolds. Show that X and Y are diffeomorphic (Def 1.21) exactly when there is a bijective smooth map $F : X \to Y$ whose inverse is also smooth. *(4 Points)*

10. A partition of unity for the interval $(0, 4)$.

We consider the open interval $M = (0, 4)$ as a 1-dimensional manifold. Take an open cover of M:

$$U_1 := (0, 2)\ , \quad U_2 := (1, 3)\ , \quad \text{and} \quad U_3 := (2, 4).$$

(a) Give an example of three functions $f_1, f_2, f_3 \in C^\infty(M)$ with these properties:

$$0 \le f_k \le 1, \quad \operatorname{supp}(f_k) \subset U_k, \quad f_1 + f_2 + f_3 = 1.$$

(The support of a function is defined to be the closure of the points on which it is non-zero, $\operatorname{supp}(f_k) := \overline{\{ x \in M \mid f_k(x) \neq 0 \}} \subset M$.) These functions form a partition of unity for M (Definition 1.26). *(3 Points)*

(b) Theorem 1.27 is even stronger! What additional property does the partition of unity given by Theorem 1.27 have, which our example does not have? *(1 Point)*

(c) Is it possible to have a partition of unity of M with this additional property and which has only finitely many functions (f_k)? *(2 Points)*

Terminology

glatt = smooth.

Zerlegung der Eins = partition of unity.

Träger = support (symbol is supp).