Martin Schmidt
Ross Ogilvie

Analysis III
2. Exercises

8 March 2021

5. Compatibility of charts.

Consider the sphere from Example 1.18(iii) in the lecture script. There we defined two charts $N : \mathbb{S}^2 \setminus \{e_0\} \to \mathbb{R}^2$ and $S : \mathbb{S}^2 \setminus \{-e_0\} \to \mathbb{R}^2$ with formulae

$$N(x) = e_0 + \frac{x - e_0}{1 - \langle x, e_0 \rangle}, \quad S(x) = -e_0 + \frac{x + e_0}{1 + \langle x, e_0 \rangle}.$$

Let us name six hemispheres $H_i^\pm = \{x \in \mathbb{S}^2 \mid \pm x_i > 0\}$ and the corresponding coordinate projections $\pi_i^\pm : H_i^\pm \to \mathbb{R}^2$

$$\pi_0^\pm(x_0, x_1, x_2) = (x_1, x_2), \quad \pi_1^\pm(x_0, x_1, x_2) = (x_0, x_2), \quad \pi_2^\pm(x_0, x_1, x_2) = (x_0, x_1).$$

(a) Show that π_0^+ is a chart of $\mathbb{S}^2$. *(2 Points)*

(b) Show that π_0^+ is compatible with π_1^+. *(1 Point)*

(c) Is π_0^+ compatible with π_0^-? *(Just to think about.)*

(d) Show that π_0^+ is compatible with S. Why does it immediately follow that it is compatible with N? *(2 Points)*

(e) Prove the following: If $\phi : U \to \mathbb{R}^n$ is a chart of X and V is an open subset of U, then $\psi := \phi|_V : V \to \mathbb{R}^n$ is a chart of X that is compatible with ϕ. *(1 Point)*

Solution.

(a) The inverse of $\pi_0^+ : H_0^+ \to B(0,1)$ is given by $(\pi_0^+)^{-1} : B(0,1) \to H_0^+$,

$$(\pi_0^+)^{-1}(x_1, x_2) = (\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2).$$

This shows that it is bijective. π_0^+ is continuous because it is the restriction of the continuous map $\mathbb{R}^3 \to \mathbb{R}^2$, $(x_0, x_1, x_2) \mapsto (x_1, x_2)$. $(\pi_0^+)^{-1}$ is continuous because the square root is continuous for positive inputs, and $x_1^2 + x_2^2 < 1$. This shows π_0^+ is a chart.

(b) The domains of the two charts are not equal, so according to the definition on page 11, we consider the restriction to $Q := H_0^+ \cap H_1^+ = \{x \in \mathbb{S}^2 \mid x_0 > 0, x_1 > 0\}$. This is a quarter of the sphere. We must consider the functions $\pi_1^+ \circ (\pi_0^+)^{-1} : \pi_0^+$ and $\pi_0^+ \circ (\pi_1^+)^{-1}$. Because the two charts are homeomorphisms, so are the compositions. The question is: are these smooth functions? We compute the first function:

$$\pi_1^+ \circ (\pi_0^+)^{-1}(x_1, x_2) = \pi_1^+\left(\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right) = \left(\sqrt{1 - x_1^2 - x_2^2}, x_2\right).$$

Yes, this is a smooth function. The calculation for the other function is similar.

(c) This is somewhat a question about terminology. The definition requires us to restrict the charts to the intersection $H_0^+ \cap H_0^- = \emptyset$. What does it mean to restrict a function to the empty set? In this situation, we say that the charts are compatible. Below is the reason that we say that the charts are compatible. It is not so important for this course, but perhaps you will find it interesting.

In the definition of a function, a function $f : X \to Y$ is a subset $F \subset X \times Y$ such that for all $x \in X$ there is exactly one $y \in Y$ with $(x, y) \in F$. When $X = \emptyset$, the product $\emptyset \times Y = \emptyset$ and so there is only one function $e_Y : \emptyset \to Y$ defined by $E_Y = \emptyset$. "$\forall x \in \emptyset\ \exists! y \in Y : (x, y) \in E_Y$" is vacuously true. e_Y is called the empty function to Y.

It therefore follows that both functions we must consider are the empty function $\emptyset \to \emptyset$. These functions are smooth because they are smooth at every point (again, vacuously true).

(d) The intersection of the two domains is $H_0^+ \cap (\mathbb{S}^2 \setminus \{-e_0\} = H_0^+$. We compute the composition $S \circ (\pi_0^+)^{-1} : B(0,1) \to B(0,1)$,

$$
\begin{aligned}
S \circ (\pi_0^+)^{-1}(x_1, x_2) &= S\Big(\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\Big) \\
&= (-1, 0, 0) + \frac{1}{1 + \sqrt{1 - x_1^2 - x_2^2}}\Big(1 + \sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\Big) \\
&= \frac{1}{1 + \sqrt{1 - x_1^2 - x_2^2}}(0, x_1, x_2) \\
&= \frac{1}{1 + \sqrt{1 - x_1^2 - x_2^2}}(x_1, x_2).
\end{aligned}
$$

This is a smooth function. The other composition is similar and also smooth. Therefore the charts S and π_0^+ are compatible.

To answer the second part, notice that the intersection of the domains of N and π_0^+, namely $H_0^+ \cap (\mathbb{S}^2 \setminus \{e_0\} = H_0^+ \setminus \{e_0\}$ are a subset of above. Therefore the following equality holds at every point:

$$N \circ (\pi_0^+)^{-1} = (N \circ S^{-1}) \circ (S \circ (\pi_0^+)^{-1}).$$

We know the two functions on in brackets on the left hand side are diffeomorphisms, and so it their composition.

(e) The intersection of domains is simply V. Therefore we must consider $\phi \circ \psi^{-1}$ from $\psi(V) = \phi(V)$ to $\phi(V)$. But $\psi^{-1} = \phi^{-1}|_{\phi(V)}$, so the composition is simply the identity function. The identity function is smooth.

6. More examples of manifolds.

In the lectures, a manifold was defined as a Hausdorff and Lindelöf topological space together with an atlas. Here are facts that make it easy to check the topological properties:
(1) Every subset of a metric space is a metric space.
(2) Every metric space is Hausdorff.
(3) Definition 1.28: A topological space is called *locally compact* when every point has a neighbourhood U so that $\overline{U}$ is compact.
(4) Every open subset and every closed subset of a locally compact space is a locally compact space.
(5) Theorem 1.29(ii,iii): A locally compact Hausdorff space is Lindelöf if and only if it can be written as the countable union of compact sets.

(a) Why does it follow that every closed subset of $\mathbb{R}^n$ is Hausdorff and Lindelöf.
[Hint. Let $K_n = \overline{B(0,n)}$ and notice $\mathbb{R}^n = \cup_{n\in\mathbb{N}} K_n$.] *(2 Points)*

(b) Let $B \subset \mathbb{R}^2$ be the set defined in Exercise 3. Define an atlas $\mathcal{A}$ for B so that $(B, \mathcal{A})$ is a 1-dimensional manifold. *(3 Points)*

(c) Consider the cylinder

$$Z := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2^2 + x_3^2 = 1 \} \subset \mathbb{R}^3,$$

the subsets

$$U_\alpha := \{ (x_1, x_2, x_3) \in Z \mid (x_2, x_3) \neq (\cos\alpha, \sin\alpha) \},$$

and the maps

$$\psi_\alpha : \mathbb{R} \times (\alpha, \alpha + 2\pi) \to U_\alpha, \quad (t, s) \mapsto (t, \cos(s), \sin(s))$$

defined for every $\alpha \in \mathbb{R}$.

(i) Show for each $\alpha \in \mathbb{R}$ that ψ_α is a homeomorphism *(2 Points)*

(ii) Show that inverse map $\phi_\alpha := \psi_\alpha^{-1}$ is a chart for Z. *(3 Points)*

(iii) Show that any two charts ϕ_α, ϕ_β are compatible. *(2 Points)*

(iv) Show that $\{ \phi_\alpha \mid \alpha \in \mathbb{R} \}$ is an atlas for Z. This shows that Z is a 2-dimensional manifold. *(2 Points)*

Solution.

(a) First observe that $\mathbb{R}^n$ is locally compact: for any point $x \in \mathbb{R}^n$ take the closed ball $\overline{B(x,1)}$. From point (4), we know then that every closed subsets of $\mathbb{R}^n$ is also locally compact. Because $\mathbb{R}^n$ is a metric space, by points (1) and (2) we know that every closed subset of $\mathbb{R}^n$ is Hausdorff.

It remains to show, due to point (5), that every closed subset of $\mathbb{R}^n$ can be written as the union of countably many compact sets. Let the closed subset be called A. Following the hint, define $A_n := K_n \cap A$. These sets are compact because they are the intersection of a compact set and a closed set. Finally

$$\bigcup A_n = \bigcup (A \cap K_n) = A \cap \bigcup K_n = A \cap \mathbb{R}^n = A$$

demonstrates A can written as the union of countably many compact sets.

(b) Let's address first the topological properties. B is not the closed subset of $\mathbb{R}^2$, so the previous exercise does not completely apply. However, we can conclude from (a) that it is a locally compact Hausdorff space. It is Lindelöf: let $f(x) = \sin(x^{-1})$, observe B is the union of the compact sets $f[[n^{-1}, n]]$ for $n \in \mathbb{N}$, and apply (5).

We now must define an atlas for B. Consider the chart $\pi_1 : B \to \mathbb{R}^+$ given by $\pi_1(x, y) = x$. You should check this is a chart. The domain of π_1 covers all of B by itself, so $\mathcal{A} := \{\pi_1\}$ is an atlas.

(c) **(i)** We first note that ψ_α is continuous. It is injective, because if $\psi_\alpha(t, s) = \psi_\alpha(t', s')$ then $t = t'$ and $s - s' \in 2\pi\mathbb{Z}$. However, $s, s' \in (\alpha, \alpha + 2\pi)$ so it follows that $s = s'$. Hence it is a bijection onto its image.

To show the inverse is continuous, there are two methods. One method is to apply the inverse function theorem. The matrix of first derivatives is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sin(s) & \cos(s) \end{pmatrix}$$

, which always has rank 2. Hence the inverse exists and is a smooth function.

The other method is to directly write the inverse function. This is a little tricky, because we have to make sure the inverse has the correct codomain. This is a easier with complex numbers. Let $\mathrm{Arg} : \{z \in \mathbb{C} \mid |z| = 1\} \to (-\pi, \pi]$ be an angle function on the circle, also called the principal argument. This is a smooth function, so

$$\phi_\alpha(x_1, x_2, x_3) = (x_1, \pi + \alpha + \mathrm{Arg}(-e^{-i\alpha} e^{x_2 + ix_3}))$$

is also a smooth function (and in particular it is continuous) from U_α to $\mathbb{R} \times (\alpha, \alpha + 2\pi)$. I leave it to you to check this is actually the inverse.

(ii) The inverse of a homeomorphism is a homeomorphism.

(iii) It is enough to show that $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth for all α and β.

We consider two cases. Case 1: $\beta = \alpha + 2\pi n$ for $n \in \mathbb{Z}$. In this case the two domains are equal. The composition is $\phi_\beta \circ \phi_\alpha^{-1}(t, s) = (t, s + 2\pi n)$. This is clearly smooth.

Case 2: $\alpha + 2\pi n < \beta < \alpha + 2\pi(n+1)$ for $n \in \mathbb{Z}$. In this case, the intersection of the domains $U_\alpha \cap U_\beta$ has two components. Where the angle $s \in (\alpha, \beta - 2\pi n)$, we have $\phi_\beta \circ \phi_\alpha^{-1}(t, s) = (t, s + 2\pi(n+1))$. Where the angle $s \in (\beta - 2\pi n, \alpha + 2\pi)$, we have $\phi_\beta \circ \phi_\alpha^{-1}(t, s) = (t, s + 2\pi n)$. The function is clearly smooth on both components.

(iv) We have shown that the charts are all compatible. Since $U_0 \cup U_\pi = Z$, they cover the cylinder. And the cylinder is a closed subset of $\mathbb{R}^3$, so it is Hausdorff and Lindelöf.

7. Non-compatible differentiable atlases.

Let $\mathcal{A}$ be the natural atlas of $\mathbb{R}$, namely $\mathcal{A} = \{\mathrm{id}_\mathbb{R}\}$. Find another atlas $\widetilde{\mathcal{A}}$ of $\mathbb{R}$ that is not compatible with $\mathcal{A}$. (Compare to Exercise 1.20 in the script.)

[Hint. It is possible to find such an atlas $\widetilde{\mathcal{A}} = \{f\}$ that contains only one chart.]

(2 Points)

Solution. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq 0, \\ 2x & \text{if } x > 0. \end{cases}$$

This function is a homeomorphism, but it is not smooth (it is not differentiable at $x = 0$). Therefore the transition function $f \circ \mathrm{id}^{-1} = f$ is not smooth. This shows the two charts are not compatible.

Terminology

Definitionsbreich = domain.
dicht = dense.
Homöomorphismus = homeomorphism.
Karte = chart.
Umkehrabbildung = inverse map.
verträglich = compatible.