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Analysis III
2. Exercises

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5. Compatibility of charts.

Consider the sphere from Example 1.18(iii) in the lecture script. There we defined two charts $N : \mathbb{S}^2 \setminus \{e_0\} \to \mathbb{R}^2$ and $S : \mathbb{S}^2 \setminus \{-e_0\} \to \mathbb{R}^2$ with formulae

$$N(x) = e_0 + \frac{x - e_0}{1 - \langle x, e_0 \rangle}, \quad S(x) = -e_0 + \frac{x + e_0}{1 + \langle x, e_0 \rangle}.$$

Let us name six hemispheres $H_i^\pm = \{x \in \mathbb{S}^2 \mid \pm x_i > 0\}$ and the corresponding projections $\pi_i^\pm : H_i^\pm \to \mathbb{R}^2$

$$\pi_0^\pm(x_0, x_1, x_2) = (x_1, x_2), \quad \pi_1^\pm(x_0, x_1, x_2) = (x_0, x_2), \quad \pi_2^\pm(x_0, x_1, x_2) = (x_0, x_1).$$

(a) Show that π_0^+ is a chart of $\mathbb{S}^2$. *(2 Points)*

(b) Show that π_0^+ is compatible with π_1^+. *(1 Point)*

(c) Is π_0^+ compatible with π_0^-? *(Just to think about.)*

(d) Show that π_0^+ is compatible with S. Why does it immediately follow that it is compatible with N? *(2 Points)*

(e) Prove the following: If $\phi : U \to \mathbb{R}^n$ is a chart of X and V is an open subset of U, then $\psi := \phi|_V : V \to \mathbb{R}^n$ is a chart of X that is compatible with ϕ. *(1 Point)*

6. More examples of manifolds.

In the lectures, a manifold was defined as a Hausdorff and Lindelöf topological space together with an atlas. Here are facts that make it easy to check the topological properties:
(1) Every subset of a metric space is a metric space.
(2) Every metric space is Hausdorff.
(3) Definition 1.28: A topological space is called *locally compact* when every point has a neighbourhood U so that $\overline{U}$ is compact.
(4) Every open subset and every closed subset of a locally compact space is a locally compact space.
(5) Theorem 1.29(ii,iii): A locally compact Hausdorff space is Lindelöf if and only if it can be written as the countable union of compact sets.

(a) Why does it follow that every closed subset of $\mathbb{R}^n$ is Hausdorff and Lindelöf.
[Hint. Let $K_n = \overline{B(0, n)}$ and notice $\mathbb{R}^n = \cup_{n\in\mathbb{N}} K_n$.] *(2 Points)*

(b) Let $B \subset \mathbb{R}^2$ be the set defined in Exercise 3. Define an atlas $\mathcal{A}$ for B so that $(B, \mathcal{A})$ is a 1-dimensional manifold. *(3 Points)*

(c) Consider the cylinder

$$Z := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_2^2 + x_3^2 = 1 \} \subset \mathbb{R}^3,$$

the subsets

$$U_\alpha := \{ (x_1, x_2, x_3) \in Z \,|\, (x_2, x_3) \neq (\cos\alpha, \sin\alpha) \},$$

and the maps

$$\psi_\alpha : \mathbb{R} \times (\alpha, \alpha + 2\pi) \to U_\alpha, \quad (t, s) \mapsto (t, \cos(s), \sin(s))$$

defined for every $\alpha \in \mathbb{R}$.

(i) Show for each $\alpha \in \mathbb{R}$ that ψ_α is a homeomorphism *(2 Points)*

(ii) Show that inverse map $\phi_\alpha := \psi_\alpha^{-1}$ is a chart for Z. *(3 Points)*

(iii) Show that any two charts ϕ_α, ϕ_β are compatible. *(2 Points)*

(iv) Show that $\{ \phi_\alpha \,|\, \alpha \in \mathbb{R} \}$ is an atlas for Z. This shows that Z is a 2-dimensional manifold. *(2 Points)*

7. Non-compatible differentiable atlases.

Let $\mathcal{A}$ be the natural atlas of $\mathbb{R}$, namely $\mathcal{A} = \{\mathrm{id}_\mathbb{R}\}$. Find another atlas $\widetilde{\mathcal{A}}$ of $\mathbb{R}$ that is not compatible with $\mathcal{A}$. (Compare to Exercise 1.20 in the script.)

[Hint. It is possible to find such an atlas $\widetilde{\mathcal{A}} = \{f\}$ that contains only one chart.]

(2 Points)

Terminology

Definitionsbreich = domain.
dicht = dense.
Homöomorphismus = homeomorphism.
Karte = chart.
Umkehrabbildung = inverse map.
verträglich = compatible.