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Analysis III
1. Exercises

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1. Continuity in metric spaces.

Exercise 1.7 in the skript.

In this question we show that the ε-δ-definition of continuity in metric spaces agrees with the definition of continuity in topological spaces.

Let (X, d) and (X', d') be two metric spaces, and $f : X \to X'$ a map between them. Demonstrate the following are equivalent:

(1) For every open subset O' of X', the pre-image $f^{-1}[O']$ is open in X.

(2) For every point $p \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ so that for every point $q \in X$ with $d(p, q) < \delta$ it holds that $d'(f(p), f(q)) < \varepsilon$.

(4 Points)

Solution. It is useful to restate (2) in terms of balls. It is equivalent to
(2') For every point $p \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ so that $f[B(p, \delta)] \subseteq B(f(p), \varepsilon)$.

Suppose that (1) is true. Choose any point $p \in X$ and any $\varepsilon > 0$. Consider the ball $O' = B(f(p), \varepsilon)$ in X'. By (1), we know that $O = f^{-1}[O']$ is an open set of X that contains p. Therefore there is a ball $B(p, \delta) \subseteq O$ for some $\delta > 0$. $f[B(p, \delta)] \subseteq f[O] = O' = B(f(p), \varepsilon)$.

Suppose that (2') is true. Choose any open set O' of X' and let $O = f^{-1}[O']$. Choose any point $p \in O$. We need to show that there is a ball $B(p, \delta) \subseteq O$. But O' is open and contains $f(p)$, so we know that there exists a ball $B(f(p), \varepsilon) \subseteq O'$. (2') now guarantees the existence of such a $B(p, \delta)$, because $f[B(p, \delta)] \subseteq B(f(p), \varepsilon) \subseteq O'$ implies $B(p, \delta) \subseteq f^{-1}[O']$.

2. A Characterisation of connected spaces.

Let X be a metric space. Show that the following properties are equivalent:

(1) X ist connected (Definition 1.8).

(2) There does not exists two non-empty open subsets U, V of X with $U \cup V = X$ and $U \cap V = \varnothing$.

(2 Points)

Solution. Suppose that there are two open sets with $U \cup V = X$ and $U \cap V = \varnothing$. It follows that $U = X \setminus V$. Because V is open, U is closed. Hence U and V are both open and closed. If X is connected as per Definition 1.8, then one must be the empty set, which shows (2). On the other hand, if (1) is not true then take V to be an open and closed set that is not $\varnothing$ and not X. This shows (2) is not true either.

3. An example for connected but not path-connected space.

We consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} A &:= \{ (x,y) \in \mathbb{R}^2 \mid x = 0 \text{ and } y \in [-1,1] \} \\ B &:= \{ (x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}_+ \text{ and } y = \sin\big(\tfrac{1}{x}\big) \} \\ M &:= A \cup B . \end{aligned}$$

M is called the topologist's sine curve.

(a) Show that B is connected. [Hint. Theorem 1.10(iv).] *(2 Point)*

(b) Show that $\overline{B} = M$ and so explain why it follows that M is also connected. [Hint. Theorem 1.10(i).] *(3 Points)*

(c) Let $p = (0,1) \in A$. Consider the open rectangle $S := (-1, (2\pi)^{-1}) \times (0,2)$. What are the connected components of $M \cap S$? *(3 Points)*

(d) Prove, from (c) that M is not locally connected. [Hint. Theorems 1.9 and 1.10(iv).] *(2 Points)*

(e) We say that a space M is *path-connected*, when for every pair of points $p, q \in M$ there is a continuous function $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Continuous functions from an interval to a space are called paths.

Show that there is no path $\gamma : [0,1] \to M$ with initial point $\gamma(0) \in A$ and end point $\gamma(1) \in B$. Hence M is not path-connected.
[Hint. Modify the previous argument.] *(2 Points)*

Solution.

(a) For $x \neq 0$, we know that $1/x$ and $\sin(1/x)$ are continuous functions. B is the image of the continuous function $\mathbb{R}_+ \to \mathbb{R}^2 : x \mapsto (x, \sin(1/x))$.

(b) Let $p_n(x_n, y_n)$ be a sequence in B that converges to some point $p = (x,y)$ in $\mathbb{R}^2$. If $x > 0$ then

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sin(\tfrac{1}{x_n}) = \sin(\tfrac{1}{x}),$$

and hence $p \in B$. However, if $x = 0$ then $|y| = \lim |y_n| \leq \lim 1 = 1$. Together this shows that $\bar{B} \subseteq M$.

Conversely, choose any point $(0, y) \in A$ and let $\theta_n = \arcsin y + 2\pi n$ for $n \in \mathbb{N}$. That is, $\sin\theta_n = y$. The sequence of points $p_n := (\theta_n^{-1}, y) \in B$ converges to $(0, y)$. Therefore $\bar{B} = M$.

Since B is connected, a special case of Theorem 1.10(i) shows that $\bar{B}$ is connected.

(c) The connected components are the half phases of the sine curve: $M \cap \{2\pi n < x^{-1} < \pi + 2\pi n\}$ for $n > 1$.

(d) Consider the point p from (c) and the open neighbourhood $M \cap S$. We must show that there is no connected neighbourhood of p contained in $M \cap S$. For a contradiction, suppose $U \subset M \cap S$ were a connected neighbourhood of p.

Consider the projection $P : (x, y) \mapsto x$. It is continuous. Because U is connected $P[U]$ is connected (Theorem 1.10) and hence it is a closed interval (Theorem 1.9). We have see in (b) that there must be a point of $b \in U \cap B$. It follows that

$$[0, P(b)] \subseteq P[U] \subseteq P[S] = \{0\} \cup \bigcup_n \left[\frac{1}{\pi + 2\pi n}, \frac{1}{2\pi n}\right].$$

But this is a contradiction, the left hand side is not contained in the right hand side.

(e) Suppose that there did exist a continuous path γ. We can modify the argument in the previous part to show another contradiction. We wil prove this for paths with $\gamma(0) = (0, 1)$ and $\gamma(t) \in B$ for $t > 0$; the general case is similar. Since γ is continuous at 0, for all $\varepsilon > 0$ there exists $\delta > 0$ so that $\|\gamma(t) - (0, 1)\| < \varepsilon$ for all $t < \delta$. Choosing $\varepsilon < 1$ shows that $\gamma([0, \delta]) \subseteq S$. But then we encounter the same contradiction as before: $P \circ \gamma$ is continuous so $P \circ \gamma([0, \delta])$ must contain the closed interval $[0, P(\gamma(\delta))]$ but also be a subset of $P(M \cap S)$.

4. (Not) Hausdorff and Lindelöf Manifolds, the type of spaces we study in this course, are defined to be both Hausdorff and Lindelöf. In this question we give two examples: The 'line with two origins' is not Hausdorff and the 'long ray' is not Lindelöf. This is extra material to help you understand these properties.

(a) Let $D = \mathbb{R} \cup \{0'\}$. A set U is open in D if U is a subset of $\mathbb{R}$ and is open in $\mathbb{R}$, or if U contains the new point $0'$ and $U \cup \{0\} \setminus \{0'\}$ is open in $\mathbb{R}$. Show that the sequence $(n^{-1})_{n\in\mathbb{N}^+}$ has both 0 and $0'$ as limit points (the definition of convergence in a topological space is after Definition 1.6). The space D is called the 'line with two origins'.

(b) Consider the topological space $R := \mathbb{N} \times [0, 1)$ with the ordering $(m, x) < (n, y)$ if $m < n$, or $m = n$ and $x < y$. Give a function $f : R \to [0, \infty)$ that preserves the order relation.

(c) There exists a set Ω, called the first uncountable ordinal, with the following properties:

(1) it is uncountable

(2) it is *well-ordered.* A set is well-ordered when there is an order relation $<$ in which every non-empty subset has a minimum, a smallest element. $\mathbb{R}$ with the normal order is not well-ordered, for example $(0, 1)$ does not contain a minimum. $\mathbb{N}$ with the usual order is well-ordered.

(3) for every $a \in \Omega$, the subset $H(a) := \{b \in \Omega \mid b < a\}$ is countable.

Let $R' := \Omega \times [0, 1)$ with the ordering $(a, x) < (b, y)$ if $a < b$, or $a = b$ and $x \leq y$. Let 0_Ω be the minimum of Ω so that $O = (0_\Omega, 0)$ is the minimum of R'. An open interval in R' has the form $I(\alpha, \beta) := \{\phi \in R' \mid \alpha < \phi < \beta\}$ for $\alpha, \beta \in R'$ or $J(\beta) = \{O\} \cup I(O, \beta) = \{\phi \in R' \mid \phi < \beta\}$. Find an uncountable collection of open intervals such that no intervals intersect. Why is R' not Lindelöf?
R' is called the 'long ray' (R is called a ray, or half-line).

Solution.

(a) x is a limit point of a sequence (x_n) if and only if every open neighbourhood of x contains all but finitely many elements of (x_n).

Take any neighbourhood U of 0. It contains an interval of the form $(0, \varepsilon)$ for some $\varepsilon > 0$. By Archimedes' principle, there is a natural number $N > \varepsilon^{-1}$ and only the finitely many elements (n^{-1}) with $n < N$ do not lie in $(0, \varepsilon) \subset U$. Therefore 0 is a limit point of the sequence.

Take any neighbourhood U' of $0'$. By definition $U = U' \cup \{0\} \setminus \{0'\}$ is an open set of $\mathbb{R}$ containing 0. By the previous paragraph, it contains all but finitely many elements of the sequence. Therefore U' does too. This shows that 0 is also a limit point of the sequence.

(b) $f((m, x)) = m + x$. This preserves the order relation because if $m < n$ then $m + x < n + y$ for all $x, y \in [0, 1)$ and if $m = n$ and $x < y$ then clearly $m + x < n + y$. It is also a bijection, with inverse given by $t \mapsto (\lfloor t \rfloor, t - \lfloor t \rfloor)$.

(c) It is easy to find such a collection: choose any uncountable subset $A \subset \Omega$ and consider the intervals $I((a, 0), (a, 0.5)) = \{(a, x) \mid 0 < x < 0.5\}$.

Consider the collection $\{J((a, 0))\}_{a \in \Omega}$. This is a cover of R', but there is no countable subcover: Take any countable subset $I \subset \Omega$. By property (3), for every $a \in I$, $H(a)$ is countable. Then $H = \bigcup_{a \in I} H(a)$ is a countable union of countable sets, and so itself countable. Therefore $H \neq \Omega$, because Ω is uncountable. But if $(b, x) \in \bigcup_{a \in I} \{J((a, 0))\}$ then $b \in H$. Because there are elements of Ω not in H, this shows the collection $\{J((a, 0))\}_{a \in I}$ is not a cover.

If you are interested in these strange topological spaces, the famous reference is Steen and Seebach's Counterexamples in Topology. An online reference is the database website π-Base https://topology.jdabbs.com/.

Terminology

Umgebung = neighbourhood.
zusammenhängend = connected.
wegzusammenhängend = path-connected.
unabzählbar = uncountable.

Your solutions are due on Monday 8.03.2021 at noon. Please make a pdf of your solutions and email them to r.ogilvie@math.uni-mannheim.de .