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Analysis III
1. Exercises

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1. Continuity in metric spaces.

Exercise 1.7 in the skript.
In this question we show that the ε-δ-definition of continuity in metric spaces agrees with the definition of continuity in topological spaces.

Let (X, d) and (X', d') be two metric spaces, and $f : X \to X'$ a map between them. Demonstrate the following are equivalent:

(1) For every open subset O' of X', the pre-image $f^{-1}[O']$ is open in X.

(2) For every point $p \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ so that for every point $q \in X$ with $d(p, q) < \delta$ it holds that $d'(f(p), f(q)) < \varepsilon$.

(4 Points)

2. A Characterisation of connected spaces.

Let X be a metric space. Show that the following properties are equivalent:

(1) X ist connected (Definition 1.8).

(2) There does not exists two non-empty open subsets U, V of X with $U \cup V = X$ and $U \cap V = \varnothing$.

(2 Points)

3. An example for connected but not path-connected space.

We consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} A &:= \{ (x, y) \in \mathbb{R}^2 \,|\, x = 0 \text{ and } y \in [-1, 1] \} \\ B &:= \{ (x, y) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}_+ \text{ and } y = \sin\left(\tfrac{1}{x}\right) \} \\ M &:= A \cup B\,. \end{aligned}$$

M is called the topologist's sine curve.

(a) Show that B is connected. [Hint. Theorem 1.10(iv).] *(2 Point)*

(b) Show that $\overline{B} = M$ and so explain why it follows that M is also connected. [Hint. Theorem 1.10(i).] *(3 Points)*

(c) Let $p = (0, 1) \in A$. Consider the open rectangle $S := (-1, (2\pi)^{-1}) \times (0, 2)$. What are the connected components of $M \cap S$? *(3 Points)*

(d) Prove, from (c) that M is not locally connected. [Hint. Theorems 1.9 and 1.10(iv).] *(2 Points)*

(e) We say that a space M is *path-connected*, when for every pair of points $p, q \in M$ there is a continuous function $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Continuous functions from an interval to a space are called paths.

Show that there is no path $\gamma : [0,1] \to M$ with initial point $\gamma(0) \in A$ and end point $\gamma(1) \in B$. Hence M is not path-connected.
[Hint. Modify the previous argument.] *(2 Points)*

4. (Not) Hausdorff and Lindelöf Manifolds, the type of spaces we study in this course, are defined to be both Hausdorff and Lindelöf. In this question we give two examples: The 'line with two origins' is not Hausdorff and the 'long ray' is not Lindelöf. This is extra material to help you understand these properties.

(a) Let $D = \mathbb{R} \cup \{0'\}$. A set U is open in D if U is a subset of $\mathbb{R}$ and is open in $\mathbb{R}$, or if U contains the new point $0'$ and $U \cup \{0\} \setminus \{0'\}$ is open in $\mathbb{R}$. Show that the sequence $(n^{-1})_{n \in \mathbb{N}^+}$ has both 0 and $0'$ as limit points (the definition of convergence in a topological space is after Definition 1.6). The space D is called the 'line with two origins'.

(b) Consider the topological space $R := \mathbb{N} \times [0,1)$ with the ordering $(m,x) < (n,y)$ if $m < n$, or $m = n$ and $x < y$. Give a function $f : R \to [0,\infty)$ that preserves the order relation.

(c) There exists a set Ω, called the first uncountable ordinal, with the following properties:

(1) it is uncountable

(2) it is *well-ordered*. A set is well-ordered when there is an order relation $<$ in which every non-empty subset has a minimum, a smallest element. $\mathbb{R}$ with the normal order is not well-ordered, for example $(0,1)$ does not contain a minimum. $\mathbb{N}$ with the usual order is well-ordered.

(3) for every $a \in \Omega$, the subset $H(a) := \{b \in \Omega \mid b < a\}$ is countable.

Let $R' := \Omega \times [0,1)$ with the ordering $(a,x) < (b,y)$ if $a < b$, or $a = b$ and $x \leq y$. Let 0_Ω be the minimum of Ω so that $O = (0_\Omega, 0)$ is the minimum of R'. An open interval in R' has the form $I(\alpha,\beta) := \{\phi \in R' \mid \alpha < \phi < \beta\}$ for $\alpha, \beta \in R'$ or $J(\beta) = \{O\} \cup I(O,\beta) = \{\phi \in R' \mid \phi < \beta\}$. Find an uncountable collection of open intervals such that no intervals intersect. Why is R' not Lindelöf?
R' is called the 'long ray' (R is called a ray, or half-line).

If you are interested in these strange topological spaces, the famous reference is Steen and Seebach's Counterexamples in Topology. An online reference is the database website π-Base `https://topology.jdabbs.com/`.

Terminology

Umgebung = neighbourhood.

zusammenhängend = connected.

wegzusammenhängend = path-connected.

unabzählbar = uncountable.

Your solutions are due on Monday 8.03.2021 at noon. Please make a pdf of your solutions and email them to `r.ogilvie@math.uni-mannheim.de` .