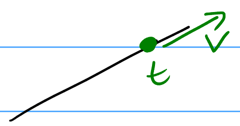
Tutorial 13

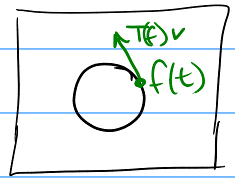
Pullbacks, exterior derivative

Integration, orientation, differential topology

$$X=\mathbb{R} \xrightarrow{\ f(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\ } Y=\mathbb{R}^2 \qquad \alpha = x\,dy - y\,dx$$

formal symbols





$$f^*\alpha := \Lambda T'(f) \circ \alpha \circ f. \qquad T(f) = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$$

$$(f^*\alpha)(t) : T_tX \to \mathbb{R}$$

$t\in X$

$x=\cos t$
$y=\sin t$

$$(f^*\alpha)(t)(v) = T'(f)\left[(\alpha\circ f)(t)\right](v) = \alpha\big(f(t)\big)\big(T(f)v\big)$$

$$= (\cos t\ dy - \sin t\ dx)\begin{pmatrix}-\sin t\ v\\ \cos t\ v\end{pmatrix}$$

$$= (\cos t)(\cos t\ v) - (\sin t)(-\sin t\ v) = v.$$

β is a 2-form on Y

$$(f^*\beta)(t)(v,w) = \beta(f(t))\big(T(f)v, T(f)w\big)$$

$f: X \to \mathbb{R}$

$T_p(f): T_pX \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ a 1-linear map on T_pX

$df(p)(v) := T_p(f)v$ exterior derivative for functions.

Stereo S^2

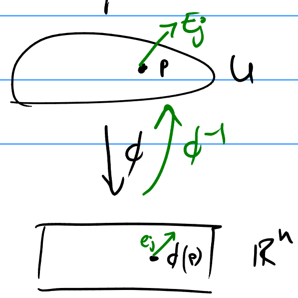
$(u,v,w) \mapsto \left(\frac{u}{1-w}, \frac{v}{1-w}\right)$, $\frac{u}{1-w} = x$, $\frac{v}{1-w} = y$

Consider a manifold chart $\phi: U \to \mathbb{R}^n$

$$\phi_i = x_i(p) = \pi_i(\phi(p))$$

$$x_i : U \to \mathbb{R}$$

We also know that $E_j(p) = T_{\phi(p)}(\phi^{-1})\, e_j$



What is $d(x_i)(p)(E_j) \overset{\text{ext der}}{=} T_p(x_i)\, E_j \overset{\text{coord vector field}}{=} T_p(x_i)\, T_{\phi(p)}(\phi^{-1}) e_j$

$$= T_p(\pi_i \circ \phi)\, T_{\phi(p)}(\phi^{-1})\, e_j$$

$$= T_{\phi(p)}(\pi_i \circ \cancel{\phi} \circ \cancel{\phi^{-1}})\, e_j$$

$$= \mathrm{Jac}(\pi_i)\, e_j = \delta_{ij}$$

$f : X \to Y$ $\quad g : Y \to \mathbb{R}$. What is $f^*(dg) \overset{\text{using the prop.}}{=} d(f^* g) = d(\underbrace{g \circ f}_{: X \to \mathbb{R}})$

$$(f^* dg)(x)(\underbrace{v}_{\in T_x X}) = dg(f(x))(T_x(f) v)$$

$$= T_{f(x)}(g)\, T_x(f)\, v$$

$$= T_x(g \circ f)\, v$$

$$= d(g \circ f)(x)(v)$$

Proof of Proposition in this case.

$\Rightarrow \quad f^* dg = d(g \circ f)$

Higher degree d.

We know $d(\overset{k\text{-degree}}{f_I\, dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}}) = (df_I) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$

If $\underset{\underbrace{\Lambda^n T_x' X}_{\dim n}}{V}$ is p dimensional the only k-multilinear map for $k > p$ is zero.

X is orientable $\Longleftrightarrow \exists$ $(\dim X)$-form without zeroes.

Oriented Manifolds

S^2 $\quad \psi : (x,y,z) \longmapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \qquad (x,y,z) \neq (0,0,1)$

$\phi : (x,y,z) \longmapsto \left(\frac{-x}{1+z}, \frac{y}{1+z}\right) \qquad (x,y,z) \neq (0,0,-1)$

$\{\psi, \phi\}$ is an atlas for S^2

This atlas is oriented $\quad \det(\phi\circ\psi^{-1})' > 0$ everywhere

$$\phi\circ\psi^{-1}(u,v) = \|(u,v)\|^{-2}\begin{pmatrix}-u\\ v\end{pmatrix} = \underbrace{\frac{1}{(u^2+v^2)}}_{\mathbb{R}}\underbrace{\begin{pmatrix}-u\\ v\end{pmatrix}}_{\mathbb{R}^2}$$

$|(\phi\circ\psi^{-1})'| =$ It's an exercise for you to check this is everywhere positive

Oriented manifold. You have chosen an oriented atlas.

Orientable manifold. there existed an oriented atlas

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \qquad \int_{S^1}\omega = \int_{S^1\setminus\{(1,0)\}}\omega$$

$f : (0,2\pi) \to S^1 \qquad t \mapsto (\cos t, \sin t)$

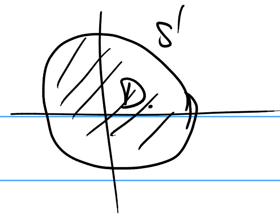
$$\int_{(0,2\pi)} f^*\omega = \int_{f[(0,2\pi)]}\omega = \int_{S^1\setminus(1,0)}\omega = \int_{S^1}\omega.$$

$$f^*\omega = -\frac{\sin t}{1}f^*dx + \frac{\cos t}{1}f^*dy = -\sin t\, d(x\circ f) + \cos t\, d(y\circ f)$$

$$= -\sin t\, d(\cos t) + \cos t\, d(\sin t)$$
$$= -\sin t\,(-\sin t\, dt) + \cos t\,(\cos t\, dt)$$
$$= dt.$$

$$\int_{(0,2\pi)} dt = 2\pi.$$

Stokes theorem $\displaystyle\int_{\partial M} \omega = \int_M d\omega.$



$\partial M = S^1 \qquad M = D.$

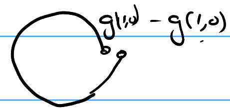
ω exact $\Rightarrow \omega = d\eta \qquad \omega$ closed $\Rightarrow d\omega = 0$

$d(d\eta) = 0 \Rightarrow$ all exact forms are closed

Why is this ω not exact?

If $\omega = dg$ for some $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ $\qquad S^1 \setminus \{(1,0)\}$

$$\int_{\partial S^1} g \overset{\text{Stokes}}{=} \int_{S^1} dg = \int_{S^1} \omega$$



$$\| \qquad dg = \underset{\|}{\frac{\partial g}{\partial x}} dx + \underset{\|}{\frac{\partial g}{\partial y}} dy$$

$$\int_{\emptyset} g \qquad = \underline{\qquad} dx + \underline{\qquad} dy$$

$\|$

$0.$

therefore ω is not exact.

$$y(t) = 2t^2 + t.$$

$$y' = 4t + 1.$$

the derivative as a linear map at $t = 0$

$$y'(0) : w \longmapsto (4 \cdot 0 + 1)\, w.$$

Regularity $C^\omega \subset C^\infty \subset \dots \subset C^3 \subset C^2 \subset C^1 \subset C^{0,\alpha} \subset C$

$$\xleftarrow{\qquad\qquad\qquad\qquad\qquad}$$

more regularity