Tutorial 12

$V = \mathbb{R}^3$ with basis e_1, e_2, e_3 $\qquad V'$ with dual $\alpha_1\ \alpha_2\ \alpha_3$

$\alpha_i(e_j) = \delta_{ij}$

eg. $\alpha_2 \otimes \alpha_3 : (v,w) \longmapsto \alpha_2(v)\alpha_3(w)$

$$A = \overset{1}{\alpha_1} \otimes \overset{2}{\alpha_2} + 2\overset{1}{\alpha_2} \otimes \overset{2}{\alpha_3} - \overset{1}{\alpha_3} \otimes \overset{2}{\alpha_2} - 2\overset{1}{\alpha_1} \otimes \overset{2}{\alpha_3}$$

$\sigma = (1\ 2)$ $\qquad \sigma(1) = 2 \quad \sigma(2) = 1$ $\qquad \sigma \in S_2$

$$\sigma . A = \alpha_2 \otimes \alpha_1 + 2\alpha_3 \otimes \alpha_2 - \alpha_2 \otimes \alpha_3 - 2\alpha_3 \otimes \alpha_1$$

Eg the inner product on $\mathbb{R}^n$ is a symmetric 2-tensor.

$$\langle v, w \rangle = \sum_{i=1}^n v_i w_i$$

antisymmetric tensor : when you swap 2 arguments, the sign changes.

$$\Leftrightarrow \quad \sigma . A = (\text{sign}\ \sigma) A. \quad \forall \sigma \in S_k$$

$\text{sign}\ \sigma = (-1)^{\text{len}\ \sigma}$

36a) $\beta = \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1$

$\text{id}.\beta = \beta = (-1)^0 \beta$ ✓

$(12).\beta = \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_2 = (-1)\beta$ $\qquad \text{sign}(12) = -1$

Antisymmeterisation, $\mathcal{A}^k$: k-tensor $\longrightarrow$ antisymm k-tensor

$$\mathcal{A}^k(A) = \sum_{\sigma \in S_k} (\text{sign}\ \sigma)\ \sigma . A$$

(in proof of thm 3.4)

$$\mathcal{A}^1(\alpha_1) = \sum_{\sigma \in S_1} (\text{sign}\ \sigma)\ \sigma . \alpha_1 = 1\ \text{id}.\alpha_1 = \alpha_1$$

$$\mathcal{A}^2(\alpha_1 \otimes \alpha_2) = \sum_{\sigma \in S_2} \text{sign}\ \sigma\ \ \sigma.\ \alpha_1 \otimes \alpha_2 = 1\ \text{id}.\ \alpha_1 \otimes \alpha_2 - (12).\ \alpha_1 \otimes \alpha_2$$

$$= \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1 = \beta = \alpha_1 \wedge \alpha_2$$

$V = \mathbb{R}^3 \qquad \delta = \gamma$

$$\mathcal{A}^4(\alpha \otimes \beta \otimes \gamma \otimes \delta) = 0$$

$\ldots + \text{sign}\ \sigma_1 \quad \beta \otimes \gamma \otimes \alpha \otimes \delta + \ldots$

$\Updownarrow -$

$+ \text{sign}\ \sigma_2 \quad \beta \otimes \delta \otimes \alpha \otimes \gamma + \ldots$

$\Downarrow$

cancel.

$A(\alpha \otimes \alpha \otimes \dots\dots) = 0$

space of antisymm k-tensor is $\overset{\text{lamba}}{\Lambda^k} V' \subseteq \mathcal{L}(\overbrace{V, V, \dots}^{k}; \mathbb{K})$

$A^2(\overbrace{\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1}^{\beta}) = 2!\, \beta$

$A^k(\beta) = \sum_{\sigma \in S_n} (\text{sign}\, \sigma)\ \sigma \circ \beta = \sum (\text{sign}\, \sigma)(\text{sign}\, \sigma)\, \beta = \sum_{\sigma \in S_n} \beta$

$= k!\, \beta$

If $\beta \in \Lambda^p V'$ $\quad \gamma \in \Lambda^q V'$ $\quad \beta \otimes \gamma$ is a $(p+q)$-tensor but <u>not</u> antisymmetric

wedge

$\beta \wedge \gamma := \frac{1}{p!} \frac{1}{q!} A^{p+q}(\beta \otimes \gamma) \in \Lambda^{p+q} V'$.

36(a) $\underset{\text{2-tensor}}{\beta} \wedge \underset{\text{1-tensor}}{\alpha_3} = \frac{1}{2!} \frac{1}{1!} A^3(\beta \otimes \alpha_3) = \frac{1}{2} A^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3)$

$A^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \sum_{\sigma \in S_3} \text{sign}\, \sigma\ \ \sigma \circ \overbrace{\alpha_1 \otimes \alpha_2 \otimes \alpha_3}^{\omega}$

$= \text{id}.\omega - (12).\omega - (13).\omega - (23).\omega + (123).\omega + (132).\omega$

$= \alpha_1 \otimes \alpha_2 \otimes \alpha_3 - \alpha_2 \otimes \alpha_1 \otimes \alpha_3 - \alpha_3 \otimes \alpha_2 \otimes \alpha_1 - \alpha_1 \otimes \alpha_3 \otimes \alpha_2 + \alpha_2 \otimes \alpha_3 \otimes \alpha_1 + \alpha_3 \otimes \alpha_2 \otimes \alpha_1$

$A^3(\alpha_2 \otimes \alpha_1 \otimes \alpha_3) = A^3((12) \circ \alpha_1 \otimes \alpha_2 \otimes \alpha_3)$

$\sigma = \tau \circ (12)$

$= \sum_{\sigma \in S_3} \text{sign}\, \sigma\ \ \overbrace{\sigma \circ (12)}^{\tau} \circ \omega = \sum_{\tau \in S_3} \text{sign}(\tau \circ (12))\ \tau \circ \omega = -\sum \text{sign}\, \tau\ \tau \circ \omega$

$= -A^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$

$\beta \wedge \alpha_3 = \cancel{\frac{1}{2}} \cdot \cancel{2}\, A^3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$

$(\alpha_1 \wedge \alpha_2) \wedge \alpha_3$

$\beta \wedge \alpha_3(u, v, w)$

$$\alpha_1 \wedge \alpha_2 (v,w) = (\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1)(v,w)$$
$$= \alpha_1(v)\alpha_2(w) - \alpha_2(v)\alpha_1(w)$$

$$\begin{vmatrix} \alpha_1(v) & \alpha_1(w) \\ \alpha_2(v) & \alpha_2(w) \end{vmatrix} = \alpha_1(v)\alpha_2(w) - \alpha_1(w)\alpha_2(v)$$

Are these <u>all</u> of the antisymm tensors? Yes.

If $\alpha_1\ \alpha_2\ \alpha_3$ is a basis of V'

$\Lambda^1 V' = V'$ basis is $\alpha_1, \alpha_2, \alpha_3$

$\Lambda^2 V'$ basis $\alpha_1 \wedge \alpha_2,\ \alpha_2 \wedge \alpha_3,\ \alpha_1 \wedge \alpha_3$ $\qquad \alpha_2 \wedge \alpha_1$?

$\Lambda^3 V'$ basis $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$

$\Lambda^p V' = 0$ for $p > \dim V'$

$$\alpha_2 \wedge \alpha_1 = \mathcal{A}^2(\alpha_2 \otimes \alpha_1) = \mathcal{A}^2((12) \circ \alpha_1 \otimes \alpha_2) = -\mathcal{A}^2(\alpha_1 \otimes \alpha_2)$$
$$= -\alpha_1 \wedge \alpha_2 .$$

Applying this to manifolds. M and a chart (U, ϕ)

We have seen how to trivialise $TU \subset TM$

$$(v, x) \longmapsto (T_x\phi(v), \ \phi(x)) \in \underset{\uparrow \text{ vectorspace.}}{\mathbb{R}^n} \times \mathbb{R}^n$$

Let $E_k = T(\phi^{-1})(e_k)$, where $e_k = (0,0,\dots,\overset{k\text{th}}{1},\dots,0)$

non vanishing vector field on TU

Every vector field on TU $\quad F(x) = \sum \underset{\text{smooth function}}{F_k(x)} E_k(x)$

a differential p-form is a "field" (like vector field) of antisymmetric p-tensors

At every point $x \in U$ $\{E_k(x)\}$ is a basis for T_xM

so there is a dual basis $\{\underline{dx_k(x)}\}$ for $(T_xM)'$.

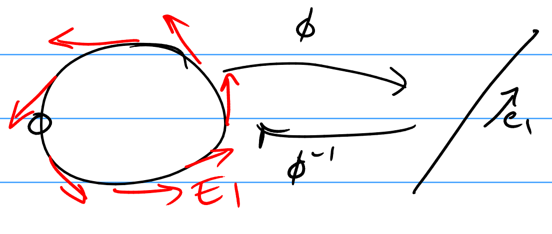
$$\underbrace{dx_k(x)}_{(T_xM)'} \ (\underbrace{E_j(x)}_{T_xM}) = \delta_{ij}$$

Example on $\mathbb{R}^3$. at every point the coord basis vector of $T_x\mathbb{R}^3$ are e_k

and so

$dx(e_1) = 1$	$dy(e_1) = 0$	$dz(e_1) = 0$
$dx(e_2) = 0$	$dy(e_2) = 1$	$dz(e_2) = 0$
$dx(e_3) = 0$	$dy(e_3) = 0$	$dz(e_3) = 1$

Example on S^1



$\theta \in (-\pi, \pi)$

$\phi^{-1}(\underset{x_1}{\theta}) = (\cos\theta, \sin\theta)$

$$E_1 = T\phi^{-1}(e_1) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} e_1$$

$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$\overset{d\theta}{dx_1}(E_1) = \delta_{11} = 1$

$$d\theta(fE_1) = f \qquad \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = x^2 e_1 + y^2 e_2 + z^2 e_3$$

back on $\mathbb{R}^3$

$$dy \wedge dz \left(\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix}, \begin{pmatrix} y+z \\ 2x \\ z+x \end{pmatrix} \right) = \begin{vmatrix} dy\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} & dy\begin{pmatrix} y+z \\ 2x \\ z+x \end{pmatrix} \\ dz\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} & dz\begin{pmatrix} y+z \\ 2x \\ z+x \end{pmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} y^2 & 2x \\ z^2 & z+x \end{vmatrix} = y^2(z+x) - 2xz^2$$

Exterior Derivative

$$df = \sum \frac{\partial f}{\partial x_j} dx_j \qquad d(f\, dg_1 \wedge dg_2 \wedge \dots dg_p) = df \wedge dg_1 \wedge \dots \wedge dg_p$$

eg

$$d(yz\, dx + xz\, dy)$$

$$= d(yz) \wedge dx + d(xz) \wedge dy$$

$$= \left(\cancel{\frac{\partial yz}{\partial x}} dx + \underset{z}{\frac{\partial yz}{\partial y}} dy + \underset{y}{\frac{\partial yz}{\partial z}} dz \right) \wedge dx + \left(\underset{z}{\frac{\partial xz}{\partial x}} dx + \cancel{\frac{\partial xz}{\partial y}} dy + \underset{x}{\frac{\partial xz}{\partial z}} dz \right) \wedge dy$$

$$= (z\, dy + y\, dz) \wedge dx + (z\, dx + x\, dz) \wedge dy$$

$$= \underline{-z\, dx \wedge dy} - y\, dx \wedge dz + \underline{z\, dx \wedge dy} - x\, dy \wedge dz$$

$$= -y\, dx \wedge dz - x\, dy \wedge dz$$